Physics 342 Lecture 24

Let's finish up our construction of angular momentum in three-dimensions before spring break today. If you recall, last time we had ~~ide~~ identified some interesting properties of rotations in 3D: first, they do not commute, and second a rotation of 4π is equivalent to no rotation at all. Further, we identified the three basis operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, ~~to~~ for which an arbitrary rotation could be specified. We had identified the operators as angular momentum and their explicit form is:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\,, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z\,, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x\,.$$

Then, the unitary operator $\mathbb{U}(\vec{\theta})$ that implements a rotation by angle θ_x, θ_y, and θ_z about the x, y, and z axes is:

$$\mathbb{U}(\vec{\theta}) = \exp\left[i\left(\theta_x\hat{L}_x + \theta_y\hat{L}_y + \theta_z\hat{L}_z\right)\right], \quad \text{where}$$

$$\vec{\theta} = \begin{pmatrix}\theta_x\\ \theta_y\\ \theta_z\end{pmatrix}.$$

Just at the end of last lecture, we were asking if the set of all unitary rotation operators $\{\mathbb{U}(\vec{\theta})\}$ formed a group. We had verified three of the four group properties: existence of an identity, inverses, and associativity. (Associativity is actually a bit more subtle then we are giving it credit for, but we won't address the subtlety here.) The final property to verify is closure of the group. That is, if $\mathbb{U}(\vec{\theta})$ and $\mathbb{U}(\vec{\phi})$ are in the set of unitary rotation operators, then their product must be as well:

$$\mathbb{U}(\vec{\theta})\,\mathbb{U}(\vec{\phi}) = \mathbb{U}(\vec{\gamma}),$$ where $\vec{\gamma}$ is some other vector of angles. What is $\vec{\gamma}$? and what are the constraints on the $\hat{L}$ operators?

Well, let's just see what we find from explicit multiplication. We have the requirement that

$$\mathbb{U}(\vec{\theta})\,\mathbb{U}(\vec{\phi}) = \mathbb{U}(\vec{\gamma}) \Rightarrow e^{i\vec{\theta}\cdot\hat{\vec{L}}}\, e^{i\vec{\phi}\cdot\hat{\vec{L}}} = e^{i\vec{\gamma}\cdot\hat{\vec{L}}}$$

Now, if the factors in the exponent were just numbers, then we could simply add the exponents on the left to determine $\vec{\gamma}$, and would find that $\vec{\gamma} = \vec{\theta} + \vec{\phi}$. However, we know that this <u>cannot</u> be true because we had demonstrated that rotations are non-Abelian and do not commute. For operators that do not commute, we can't simply add exponents; we must be much more careful.

Lucky for us; this multiplication law has been worked out for non-commutative operators and it's called the Baker-Campbell-Hausdorff formula, or just BCH formula. For the case at hand it reads:

$$e^{i\vec{\theta}\cdot\hat{\vec{L}}}\, e^{i\vec{\phi}\cdot\hat{\vec{L}}} = e^{i(\vec{\theta}+\vec{\phi})\cdot\hat{\vec{L}} + \frac{1}{2}[\vec{\theta}\cdot\hat{\vec{L}},\, \vec{\phi}\cdot\hat{\vec{L}}] + \cdots} = e^{i\vec{\gamma}\cdot\hat{\vec{L}}}$$

The ellipses denote nested commutators, like $[\vec{\theta}\cdot\hat{\vec{L}}, [\vec{\theta}\cdot\hat{\vec{L}}, \vec{\phi}\cdot\hat{\vec{L}}]]$, involving three, four, or more of the angular momentum operators. This is an absolute mess! I want to emphasize again that if this product is to be a rotation matrix, then it <u>must</u> be able to be written as on the right, with the exponent as a simple real, linear combination of the $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ angular momentum operators.

For this to be true, we must require a very non-trivial non-linear relationship of the angular momentum operators. If we enforce that the commutator of two angular momenta (which is non-linear) reduces to a <u>linear</u> sum of angular momenta, then the resulting exponent is linear, and everything works out. Thus, we require:

$$\left[\vec{\theta}\cdot\hat{\vec{L}}, \vec{\phi}\cdot\hat{\vec{L}}\right] = i\vec{\alpha}\cdot\hat{\vec{L}},$$ where $\vec{\alpha}$ is some other vector of "angles."

Actually, this can be expressed much more compactly and in generality exclusively in terms of the angular momentum basis that we have constructed: $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$. For $\hat{L}_i, \hat{L}_j \in \{\hat{L}_x, \hat{L}_y, \hat{L}_z\}$, we must have:

$$[\hat{L}_i, \hat{L}_j] = i\left(\alpha_x \hat{L}_x + \alpha_y \hat{L}_y + \alpha_z \hat{L}_z\right),$$ for some real numbers $\alpha_x, \alpha_y, \alpha_z$.

In general, this Hermitian operator basis and this non-linear identity is called the <u>Lie algebra</u> after Sophus Lie, a Norwegian mathematician. One point to emphasize is that the Lie algebra completely specifies the multiplication law of the (Lie) group. If you demonstrate that two sets of Hermitian operators have the <u>same</u> Lie Algebra (i.e., the same constants $\vec{\alpha}$, called <u>structure constants</u>), then they are the <u>same</u> group. This will be important later...

So, with this in mind, if it actually true that our set of angular momentum operators form a Lie algebra, and hence the unitary rotation matrices are closed? Well, given our definitions of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, we can just explicitly calculate some commutators. We'll just calculte the commutator $[\hat{L}_x, \hat{L}_y]$, and then show results for the other commutators.

So, this commutator, with our definitions of these operators, is

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$$

$$= (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) - (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)$$

$$= \hat{y}\hat{p}_z\hat{z}\hat{p}_x - \cancel{\hat{y}\hat{p}_z\hat{x}\hat{p}_z} - \cancel{\hat{z}\hat{p}_y\hat{z}\hat{p}_x} + \hat{z}\hat{p}_y\hat{x}\hat{p}_z - \hat{z}\hat{p}_x\hat{y}\hat{p}_z$$

$$+ \cancel{\hat{z}\hat{p}_x\hat{z}\hat{p}_y} + \cancel{\hat{x}\hat{p}_z\hat{y}\hat{p}_z} - \hat{x}\hat{p}_z\hat{z}\hat{p}_y$$

$$= \hat{x}\hat{p}_y(\hat{z}\hat{p}_z - \hat{p}_z\hat{z}) - \hat{y}\hat{p}_x(\hat{z}\hat{p}_z - \hat{p}_z\hat{z})$$

In simplifying this expression, I used the fact that momenta commute: $[\hat{p}_i, \hat{p}_j] = 0$, positions commute: $[\hat{i}, \hat{j}] = 0$, and positions and momenta of different dimensions commute: $[\hat{i}, \hat{p}_j] = 0$ for $i \neq j$. With this in mind, I was able to group the remaining terms in the way presented above.

Now, noting that: $[\hat{z}, \hat{p}_z] = i\hbar$, the commutator of $\hat{L}_x$ and $\hat{L}_y$ reduces to:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{x}\hat{p}_y - i\hbar\hat{y}\hat{p}_x = i\hbar\hat{L}_z.$$

Amazingly, this quadratic operator commutator reduces to just a linear combination of the basis angular momentum operators! With the same techniques, you can compute the complete commutation relations of the Lie algebra:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x,$$

or, compactly, $\boxed{[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k.}$

Here, ϵ_{ijk} is called the totally antisymmetric symbol for which

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = 1$$

and $\varepsilon_{ijk}=0$ if any of the $i,j,$ or k are repeated. These commutation relations are called the Lie algebra for su(2) (lowercase su).

Again, with emphasis, the Lie algebra defines the multiplication rule of the group of unitary operators through the BCH formula. Two groups that have the same Lie algebra have the same multiplication law and are therefore the same (i.e., isomorphic) group. Further, the angular momentum operators we have considered thus far act on three-dimensional position vectors. That is, to rotate the vector $\vec{r}=(x,y,z)$, we act with:

$$\vec{r}\,' = \mathbb{U}(\vec{\theta})\,\vec{r} = \exp\left[i\left(\theta_x \hat{L}_x + \theta_y \hat{L}_y + \theta_z \hat{L}_z\right)\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We thus call this manifestation of the rotation group in three dimensions a "three dimensional representation", because it rotates three dimensional vectors.

Now, you might think that this is the only possible representation of rotations in 3D, because what else would you rotate? However, if we find another representation with the same Lie algebra, it's the same group! With this in mind, let's consider two-dimensional matrices and see if we can construct su(2). First, these matrices must be Hermitian as we want them to exponentiate to a unitary operator on Hilbert space. In an early homework and lecture, we had identified the two-by-two matrices that form a basis for all 2x2 Hermitian matrices: the Pauli sigma matrices. Recall that they are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Okay, so what are the commutation relations of these matrices? Let's just evaluate $[\sigma_1, \sigma_2]$ explicitly here. We have:

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2i\sigma_3.$$

Fascinating! These σ-matrices indeed form a Lie algebra. It's not quite the same commutation relation as the su(2) algebra we identified earlier, but that is easily remedied. Let's call the "spin" operators

$$\hat{S}_x = \frac{\hbar}{2}\sigma_1, \quad \hat{S}_y = \frac{\hbar}{2}\sigma_2, \quad \hat{S}_z = \frac{\hbar}{2}\sigma_3.$$

Then, we have that $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ and similarly for the other commutators:

$$[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x.$$

Compactly, this is $[\hat{S}_i, \hat{S}_j] = i\hbar\varepsilon_{ijk}\hat{S}_k$, just the su(2) algebra!

We correspondingly call this representation of the rotation group the two-dimensional representation because these two-by-two matrices would act on two-component vectors (properly called "spinors"). Very strange that we can rotate a two-component object in full three dimensions, but that is what the math tells us.

Let's attempt to make more sense of these "spin" operators.

We'll use these spin operators to perform a rotation about the z-axis, and see what we find. The unitary matrix that implements a rotation by ϕ about the z-axis is:

$$\mathbb{U}(\phi) = e^{i\phi\frac{S_z}{\hbar}} = e^{i\frac{\phi}{2}\sigma_3} = \mathbb{1} + i\frac{\phi}{2}\sigma_3 - \frac{1}{2}\left(\frac{\phi}{2}\right)^2\sigma_3^2 - \frac{i}{6}\left(\frac{\phi}{2}\right)^3\sigma_3^3 + \dots$$

$$= \mathbb{1}\cos\left(\frac{\phi}{2}\right) + i\sigma_3\sin\left(\frac{\phi}{2}\right) = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}.$$

This matrix is fascinating and manifests the other property of 3D rotations we had identified last lecture. Let's perform a rotation by $2\pi = \phi$, which should, naively, just rotate the object back to where we started. However, the unitary matrix that implements this rotation is:

$$\mathbb{U}(2\pi) = \begin{pmatrix} e^{i\frac{2\pi}{2}} & 0 \\ 0 & e^{-i\frac{2\pi}{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1},$$

so this rotation by 2π negates the object! As we argued last lecture, indeed a rotation by 2π in three-dimensions is <u>not</u> equivalent to doing nothing.

By contrast, if we rotate by 4π, we do get back to where we started:

$$\mathbb{U}(4\pi) = \begin{pmatrix} e^{i\frac{4\pi}{2}} & 0 \\ 0 & e^{-i\frac{4\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1},$$

as we established with the plate trick.

This two-dimensional representation of the rotation group is also called the "spin-½ representation" because a rotation by 2π only rotates a spin-½ object by half of that, or π.

The three-dimensional representation we started with is called the "spin-1" representation because a rotation by 2π rotates a spin-1 object by 1 times that. There are representations of the rotation group indexed by spins of every non-negative integer and half integer, representing the number of rotations that the object of spin-s completes for a 2π rotation (i.e., spin-2 rotates twice for a 2π rotation). Note that there's no such thing as "spin-$\frac{1}{3}$", because <u>all</u> objects must return to their original orientation after rotation by 4π.

After break, we'll quantify these representations a bit more and provide <u>all</u> data that can be used to specify a quantum system/object.

Have a good break!