Lecture 25 Physics 342

Welcome back from break! I hope it was relaxing and fulfilling.

This week we are continuing our study of angular momentum in quantum mechanics, with an eye toward consequences for how we can label quantum particles/states in general. Before break, we had studied unitary operators that implement rotations in three dimensions, $\mathbb{U}(\vec{\theta})$, for some vector $\vec{\theta}$ of angles $\theta_x, \theta_y, \theta_z$, representing rotation about the corresponding axis. We were able to write this unitary operator as:

$$\mathbb{U}(\vec{\theta}) = e^{i(\theta_x \hat{L}_x + \theta_y \hat{L}_y + \theta_z \hat{L}_z)}$$, for angular momentum

operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$. We had identified that these angular momentum operators form a Lie Algebra and satisfy the commutation relationships:

$$[\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k.$$ Along with the other properties, this non-trivial non-linear relationship between the Hermitian angular momentum operators means that unitary 3D rotation matrices form a <u>group</u>. We had also identified that this group is not commutative, or <u>non-Abelian</u>, a consequence of the non-zero commutator of the $\hat{L}$ operators. Additionally, we had shown that rotations of 4π, not 2π, are equivalent to no rotation at all, and this implied that representations of this rotation group can be quantified by half-integer values of <u>spin</u>.

Today, we'd like to make this relationship between the value of spin and the Hilbert space spanned by eigenstates of these angular momentum operators more precise. To do this, we will use a formalism we employed when studying the infinite square well loooooong ago.

The first thing to note about $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ is that they don't commute, so they cannot be simultaneously diagonalized. Have no fear; we will just consider $\hat{L}_z$ as diagonal, and $\hat{L}_x$ and $\hat{L}_y$ whatever they are. That is, we will study the eigenspace of $\hat{L}_z$. Note that everything we will derive about $\hat{L}_z$ will hold for $\hat{L}_x$ and $\hat{L}_y$ (if but in a different basis) because there is nothing special about the z-axis; we could have labeled <u>any</u> axis the z-axis. Because there are three angular momentum operators and they all do not commute, we can only choose one of them to study its eigensystem. $\hat{L}_z$ alone is called the "maximally commuting subalgebra" or the "Cartan subalgebra", of the Lie algebra, after the French mathematician, Élie Cartan. Note that $\hat{L}_z$ does indeed commute with itself: $[\hat{L}_z, \hat{L}_z] = 0$, so eigenstates of $\hat{L}_z$ are eigenstates of, well, $\hat{L}_z$.

Now, with the remaining $\hat{L}_x$ and $\hat{L}_y$, we would like to construct another subalgebra. We would like the elements of this subalgebra to be eigenstates of the commutator with elements of the Cartan subalgebra, so that their action is as simple as possible. We'll see later why this is so nice. That is, for two operators $\hat{L}_+$ and $\hat{L}_-$, we would like:

$$[\hat{L}_+, \hat{L}_z] = \alpha \hat{L}_+, \quad [\hat{L}_-, \hat{L}_z] = \beta \hat{L}_-.$$

Note that $\hat{L}_+$ and $\hat{L}_-$ <u>cannot</u> be $\hat{L}_x$ and $\hat{L}_y$, if for no other reason that their commutators are not "eigen-commutators", as, for example:

$$[\hat{L}_x, \hat{L}_z] = -i\hbar \hat{L}_y.$$

So, we'll need to work a bit harder to determine $\hat{L}_+$ and $\hat{L}_-$.

The Lie algebra is a vector space, so we can construct $\hat{L}_+$ and $\hat{L}_-$ as a general linear combination of $\hat{L}_x$ and $\hat{L}_y$. That is, we will consider

$$\hat{L}_+ = a_x \hat{L}_x + a_y \hat{L}_y \,, \quad \hat{L}_- = b_x \hat{L}_x + b_y \hat{L}_y \,, \text{ for some constants } a_x, a_y, b_x, b_y.$$

Now, the normalization of an eigensystem is undefined, so we can, with no loss of generality just set $a_x = b_x = 1$. This fixing of the normalization of $\hat{L}_+$ and $\hat{L}_-$ does not affect the "eigenvalues" α, β in the commutators.

So, let's see what these might be. We have the commutator:

$$[\hat{L}_+, \hat{L}_z] = [\hat{L}_x + a\hat{L}_y, \hat{L}_z] = [\hat{L}_x, \hat{L}_z] + a[\hat{L}_y, \hat{L}_z]$$

$$= -i\hbar \hat{L}_y + i\hbar a \hat{L}_x = \alpha \hat{L}_+ = \alpha \hat{L}_x + \alpha a \hat{L}_y.$$

So, we simply match coefficients of $\hat{L}_x$, $\hat{L}_y$ to determine a and α, appropriately. We find the two equations:

coefficient of $\hat{L}_x$: $\alpha = i\hbar a$

coefficient of $\hat{L}_y$: $\alpha a = -i\hbar$

and combining them, we find: $a^2 = -1$, so $a = \pm i$. If we define

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \text{ and } \hat{L}_- = \hat{L}_x - i\hat{L}_y, \text{ then}$$

they have the following commutators:

$$[\hat{L}_+, \hat{L}_z] = -\hbar \hat{L}_+ \quad , \quad [\hat{L}_-, \hat{L}_z] = \hbar \hat{L}_-.$$

Note also that their commutator together is:

$[\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$, which I'll leave to you to prove. :)

Now, let's see the power of this formalism. Consider the state $|\psi\rangle$, which is an eigenstate of $\hat{L}_z$:

$\hat{L}_z |\psi\rangle = c\hbar |\psi\rangle$, where c is some real number, and we put $\hbar$ there to account for the correct units ($\hbar$ has units of angular momentum).

Now, consider the state on which we act $\hat{L}_+$ on $|\psi\rangle$: $\hat{L}_+|\psi\rangle$. Is this an eigenstate of $\hat{L}_z$? Let's test it out: we have:

$$\hat{L}_z \hat{L}_+ |\psi\rangle = \left(\hat{L}_+ \hat{L}_z + [\hat{L}_z, \hat{L}_+]\right) |\psi\rangle$$

$$= \hat{L}_+ c\hbar |\psi\rangle + \hbar \hat{L}_+ |\psi\rangle = (c+1)\hbar \hat{L}_+ |\psi\rangle.$$

So, yes, $\hat{L}_+|\psi\rangle$ is an eigenstate of L_z, with eigenvalue larger than that of just $|\psi\rangle$ by $\hbar$. What about $\hat{L}_-$ on $|\psi\rangle$?

$$\hat{L}_z \hat{L}_- |\psi\rangle = \left(\hat{L}_- \hat{L}_z + [\hat{L}_z, \hat{L}_-]\right) |\psi\rangle = \hat{L}_- c\hbar |\psi\rangle - \hbar \hat{L}_- |\psi\rangle$$

$$= (c-1)\hbar \hat{L}_- |\psi\rangle.$$

So, $\hat{L}_-$ lowers the eigenvalue of $|\psi\rangle$ by one $\hbar$ unit!

I hope the "+" and "−" subscripts on these operators make sense now: $\hat{L}_+$ and $\hat{L}_-$ are exactly analogous to the raising and lowering operators we had constructed for the harmonic oscillator Hamiltonian! So, we can construct the eigenstates of $\hat{L}_z$, that is, its representation, in an exactly analogous way.

Note that acting $\hat{L}_+$ on $|\psi\rangle$ n times changes the eigenvalue of $\hat{L}_z$ by $+n\hbar$:

$$\hat{L}_z(\hat{L}_+)^n|\psi\rangle = \left((\hat{L}_+)^n\hat{L}_z + [\hat{L}_z,(\hat{L}_+)^n]\right)|\psi\rangle = c\hbar(\hat{L}_+)^n|\psi\rangle + n\hbar(\hat{L}_+)^n|\psi\rangle$$

$$= (c+n)\hbar(\hat{L}_+)^n|\psi\rangle.$$

This follows from the commutation relation:

$[(\hat{L}_+)^n,\hat{L}_z] = -n\hbar(\hat{L}_z)^n$, which can be proved by induction (but I won't do here).

So, I can keep raising and raising the eigenvalue in steps of $\hbar$ by acting with $\hat{L}_+$ on $|\psi\rangle$. Correspondingly, I can keep lowering and lowering the eigenvalue in steps of $\hbar$ by acting with $\hat{L}_-$ on $|\psi\rangle$, as

$\hat{L}_z(\hat{L}_-)^n|\psi\rangle = (c-n)\hbar(\hat{L}_-)^n|\psi\rangle$, which I leave to you to show.

If we want to consider finite dimensional representations of the rotation group, like the two-dimensional spin-1/2 or three-dimensional spin-1 we had seen before spring break, we must enforce two properties on the set of eigenvectors of $\hat{L}_z$. First, there must be a minimal eigenvalue of $\hat{L}_z$ on some state $|\psi_{min}\rangle$ for which

$$\hat{L}_-|\psi_{min}\rangle = 0.$$

There must also be a maximal eigenvalue of $\hat{L}_z$ on some state $|\psi_{max}\rangle$ such that: $\hat{L}_+|\psi_{max}\rangle = 0$. On these two states, let's call the minimum and maximum eigenvalues c_{min} and c_{max}:

$$\hat{L}_z|\psi_{min}\rangle = c_{min}\hbar|\psi_{min}\rangle,$$
$$\hat{L}_z|\psi_{max}\rangle = c_{max}\hbar|\psi_{max}\rangle.$$

Let's also say that the difference between c_{min} and c_{max} is some integer n: $c_{min}+n = c_{max}$. Also, these states are normalized and orthogonal:

$$\langle\psi_{min}|\psi_{min}\rangle = \langle\psi_{max}|\psi_{max}\rangle = 1, \quad \langle\psi_{min}|\psi_{max}\rangle = 0.$$

We've been very abstract this lecture, and we'll continue proving general statements next lecture, but let's take a break for the rest of this lecture and make connections to the explicit representations of rotation we identified before break. First, we had studied the two-dimensional representation, or spin-1/2. What does it look like in this language? Well, "two-dimensional" means there are just two states: $|\psi_{min}\rangle$ and $|\psi_{max}\rangle$ and $c_{min}+1 = c_{max}$. We had identified $\hat{L}_z$ with a Pauli σ-matrix, namely,

$$\hat{L}_z = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this matrix basis representation, its eigenvectors are:

$|\psi_{min}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue $-\frac{\hbar}{2}$ and

$|\psi_{max}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue $\frac{\hbar}{2}$. So, indeed, these eigenvalues differ by one $\hbar$ unit of angular momentum and further, $-c_{min} = +c_{max} = +\frac{1}{2}$. In fact, the maximum/minimum eigenvalue of $\hat{L}_z$ is just the "total" angular momentum of the state; here spin-1/2. Next lecture, we'll see how to prove this in generality.

Further, the raising and lowering operators for this spin-1/2 representation are just another linear combination of the Pauli matrices. We had identified $\hat{L}_x$ and $\hat{L}_y$ as

$$\hat{L}_x = \frac{\hbar}{2}\sigma_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_2$$

so the raising and lowering operators are:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

When acting on the eigenvectors, we find:

$$\hat{L}_+|\psi_{max}\rangle = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\hat{L}_-|\psi_{max}\rangle = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar|\psi_{min}\rangle$$

and

$$\hat{L}_-|\psi_{min}\rangle = 0, \quad \hat{L}_+|\psi_{min}\rangle = \hbar|\psi_{max}\rangle,$$

exactly as expected from the general analysis.

We had also studied the spin-1, or three-dimensional representation of angular momentum, but in the short time that remains, I want to consider a representation even simpler than spin-1/2: the one-dimensional representation.

What does this mean? Well, there is only one state and it is annihilated by both $\hat{L}_+$ and $\hat{L}_-$:

$$\hat{L}_+|\psi\rangle = \hat{L}_-|\psi\rangle = 0, \text{ and thus } c_{min} = c_{max}.$$

Rather trivially, we ~~ta~~ also have

$$\hat{L}_+\hat{L}_-|\psi\rangle = 0 = \left(\hat{L}_-\hat{L}_+ + [\hat{L}_+,\hat{L}_-]\right)|\psi\rangle = 2\hbar\hat{L}_z|\psi\rangle,$$

by the commutation relation of $\hat{L}_+$ and $\hat{L}_-$. Apparently, then $\hat{L}_z$ also annihilates $|\psi\rangle$ and so $c_{min} = c_{max} = 0$. Thus, we call this representation the spin-0 representation, or even the trivial representation of the rotation group. Now, it would seem like the commutation relation of the angular momentum operators couldn't be satisfied, but, it is entirely consistent that $\hat{L}_x = \hat{L}_y = \hat{L}_z = 0$ and therefore indeed true, though but trivially so, that

$[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k$. Hence the "trivial" representation.

More general proofs and statements next time...