Physics 342 Lecture 26

Welcome back to more angular momentum quantum mechanics! This week, we are studying properties of angular momentum eigenstates; or, equivalently, representations of the three-dimensional rotation group. We had identified the Lie algebra of the rotation group, su(2), expressed in terms of raising, lowering, and z-component operators $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_z$. Their commutation relations were:

$$[\hat{L}_+, \hat{L}_z] = -\hbar \hat{L}_+ , \quad [\hat{L}_-, \hat{L}_z] = \hbar \hat{L}_- , \quad [\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z ,$$

and $\hat{L}_+$ and $\hat{L}_-$ are Hermitian conjugates of one another: $(\hat{L}_+)^\dagger = \hat{L}_-$. Like with the raising and lowering operators of the harmonic oscillator, we used $\hat{L}_+$ and $\hat{L}_-$ to construct all states of a given representation of the rotation group.

We've restricted our study to finite dimensional representations, for which there exist two states, $|\psi_{min}\rangle$ and $|\psi_{max}\rangle$, which are annihilated by $\hat{L}_-$ and $\hat{L}_+$, respectively, and are eigenstates of $\hat{L}_z$. We have:

$$\hat{L}_- |\psi_{min}\rangle = \hat{L}_+ |\psi_{max}\rangle = 0 \quad \text{and}$$

$$\hat{L}_z |\psi_{min}\rangle = c_{min} \hbar |\psi_{min}\rangle , \quad \hat{L}_z |\psi_{max}\rangle = c_{max} \hbar |\psi_{max}\rangle ,$$

where c_{min} and c_{max} are numbers and their difference we denoted by n: $c_{max} - c_{min} = n$. The dimension of the representation is therefore $\dim = n+1$, the number of states with eigenvalues in the range $[c_{min}, c_{max}]$, inclusive. Note that this requires n to be an integer.

At the end of the previous lecture, we had reviewed a couple representations we have seen before: the one-dimensional representation or trivial representation, and the two-dimensional or spin-1/2 representation. This lecture, we will identify properties of the representation in general, and establish an operator which quantifies a representation in general, independent of the particular state we are considering.

Recall that when we first discussed linear operators and matrices, we had identified two quantities or sets that are basis independent: the eigenvalues of that operator and its dimension (when thought of as a matrix). We've discussed eigenvalues extensively and for rotations, we can quantify states by their eigenvalue under the action of $\hat{L}_z$, and can move between states of different eigenvalues with $\hat{L}_+, \hat{L}_-$.

We've spent less time discussing the dimension of an operator as in many or even most cases we studied this semester, operators have been infinite dimensional, so this is less helpful. For example, the Hamiltonian of the infinite square well is infinite dimensional: the energy eigenvalues are unbounded from above. For the rotation group, the dimension of the representation <u>is</u> relevant as it is finite. (By the way, the rotation group has finite representations while time-translation does not, in general, because the manifold of rotations is compact, while the manifold of translations is non-compact.) So, our goal this lecture is to constuct such a "dimension counting" operator on the Lie algebra of angular momentum.

Let's enumerate properties of this dimension counting operator.

First, let's give it a name: C_R, for representation R. It is also called the "quadratic Casimir" more generally. Why quadratic will be shown soon and why Casimir; well, it had to be named after someone. Next, if this Casimir is just a measure of the dimension of a representation, then it must return the same value on any state in that representation. If a matrix always returns the same eigenvalue for any ~~vec~~ vector it acts on, then that matrix is necessarily proportional to the identity matrix. So, for an n-dimensional representation, the Casimir is:

$$C_R \equiv C_R \mathbb{1}_{n\times n},$$ where $\mathbb{1}_{n\times n}$ is the $n\times n$ identity matrix. Because it is proportional to $\mathbb{1}$, C_R is just a number, so we will typically ignore the identity matrix piece.

Another way to state this property is the following. Different states in a given representation are related to one another through the action of rotation; i.e., by acting with $\hat{L}_x$, $\hat{L}_y$, or $\hat{L}_z$. If the Casimir is to be the same for every state in a representation, then it must be unaffected by rotation; equivalently, it must commute with every angular momentum operator. This property suggests a way forward to constructing C_R. First, C_R cannot be strictly a linear combination of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$, simply because these operators don't commute with one another. If we think more broadly and consider a vector $\vec{v}$ we can ask what properties of it are rotation-invariant. First, the reason why this analogy works is that the Lie Algebra $\hat{L}_x, \hat{L}_y, \hat{L}_z$ is a vector space, just one additionally equipped with a Lie bracket $[\,,\,]$; i.e., the commutator. A vector $\vec{v}$ has a magnitude and direction, and clearly the direction changes under rotation. However, the magnitude does <u>not</u>, and, for a three-dimensional vector $\vec{v}$ this magnitude is:

$$|\vec{v}|^2 = v_x^2 + v_y^2 + v_z^2 .$$

This would suggest that the quantity:

$$L^2 \equiv C_R = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

is rotationally-invariant and could be the Casimir we want.

So, let's see if it indeed commutes with every element of the Lie Algebra. Note that we can write this as

$$\sum_{i=1}^{3} \hat{L}_i^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

and then its commutator with some $\hat{L}_j$ is:

$$\left[\hat{L}_j, \sum_{i=1}^{3} \hat{L}_i^2\right] = \sum_{i=1}^{3} \left[\hat{L}_j, \hat{L}_i^2\right] = \sum_{i=1}^{3} \left(\hat{L}_j \hat{L}_i^2 - \hat{L}_i^2 \hat{L}_j\right)$$

$$= \sum_{i=1}^{3} \left(\hat{L}_i \hat{L}_j \hat{L}_i - i\hbar\, \varepsilon_{ijk} \hat{L}_k \hat{L}_i - \hat{L}_i^2 \hat{L}_j\right)$$

$$= \sum_{\substack{i=1 \\ k=1}}^{3} \left(\hat{L}_j \hat{L}_i^2 - i\hbar\, \varepsilon_{ijk} \hat{L}_i \hat{L}_k - i\hbar\, \varepsilon_{ijk} \hat{L}_k \hat{L}_i - \hat{L}_i^2 \hat{L}_j\right)$$

$$= -i\hbar \sum_{i,k=1}^{3} \varepsilon_{ijk} \left(\hat{L}_i \hat{L}_k + \hat{L}_k \hat{L}_i\right)$$

Through this expression, I used the commutation relations of the angular momentum operators. Now, in the final form, this "manifestly" is 0. Recall that ε_{ijk} is totally antisymmetric: $\varepsilon_{ijk} = -\varepsilon_{kji}$. However, the remaining sum over angular momentum operators is symmetric:

$\hat{L}_i\hat{L}_k+\hat{L}_k\hat{L}_i$. So, when I sum over i and k, for every positive term there will be a negative term that exactly cancels it! Therefore, L^2 commutes with all angular momentum operators:

$$[L^2, \hat{L}_j] = \left[\sum_{i=1}^{3}\hat{L}_i^2, \hat{L}_j\right] = 0.$$

Again, the only operator that commutes with everything is the identity, or proportional to it. So, indeed L^2 takes the same value on any state in a given representation. Thus, it is the Casimir.

What is its value? To determine C_R, let's re-express L^2 with the raising and lowering operators $\hat{L}_+$, $\hat{L}_-$. Recall that:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y, \quad \hat{L}_- = \hat{L}_x - i\hat{L}_y, \text{ and so}$$

$$\hat{L}_x^2 + \hat{L}_y^2 = \frac{\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+}{2}.$$

Thus, the Casimir is: $L^2 = \dfrac{\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+}{2} + \hat{L}_z^2$.

Let's calculate the Casimir on the special states $|\psi_{min}\rangle$ and $|\psi_{max}\rangle$ that we defined as the "boundaries" of the representation. Starting with $|\psi_{min}\rangle$, recall that

$\hat{L}_-|\psi_{min}\rangle = 0$, so it's convenient to put all $\hat{L}_-$ operators on the right in L^2. That is,

$$L^2 = \frac{\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+}{2} + \hat{L}_z^2 = \hat{L}_+\hat{L}_- + \frac{[\hat{L}_-, \hat{L}_+]}{2} + \hat{L}_z^2 = \hat{L}_+\hat{L}_- - \hbar\hat{L}_z + \hat{L}_z^2,$$

where I used the commutator of $\hat{L}_-$ and $\hat{L}_+$.

Now, acting on $|\psi_{min}\rangle$, we find:

$$L^2|\psi_{min}\rangle = \left(\cancel{\hat{L}_+\hat{L}_-}^{\,0} - \hbar\hat{L}_z + \hat{L}_z^2\right)|\psi_{min}\rangle = \left(c_{min}^2\hbar^2 - c_{min}\hbar^2\right)|\psi_{min}\rangle$$

$$= \hbar^2 c_{min}(1-c_{min})|\psi_{min}\rangle,$$

where I have used that the eigenvalue of $|\psi_{min}\rangle$ under $\hat{L}_z$ is $c_{min}\hbar$. Next, let's evaluate L^2 on $|\psi_{max}\rangle$. Now, it is convenient to express L^2 with $\hat{L}_+$ on the right as $\hat{L}_+|\psi_{max}\rangle = 0$. That is,

$$L^2 = \frac{\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+}{2} + \hat{L}_z^2 = \hat{L}_-\hat{L}_+ + \frac{[\hat{L}_+, \hat{L}_-]}{2} + \hat{L}_z^2 = \hat{L}_-\hat{L}_+ + \hbar\hat{L}_z + \hat{L}_z^2.$$

Acting on $|\psi_{max}\rangle$, we find

$$L^2|\psi_{max}\rangle = \left(\hat{L}_-\hat{L}_+ + \hbar\hat{L}_z + \hat{L}_z^2\right)|\psi_{max}\rangle = \left(c_{max}^2\hbar^2 + c_{max}\hbar^2\right)|\psi_{max}\rangle$$

$$= \hbar^2 c_{max}(1+c_{max})|\psi_{max}\rangle.$$

This must be the same eigenvalue as that when acting on $|\psi_{min}\rangle$, so, demanding that $c_{max} > c_{min}$ forces that:

$$c_{min} = -c_{max}.$$

We had already observed this for the one- and two-dimensional representations of the rotation group; now we've proved it in general.

We also stated that the dimension of the representation is $n+1$, where $n = c_{max} - c_{min} = 2c_{max}$, using the result above. Thus, the dimension of a representation is

$\dim_R = n+1 = 2c_{max}+1$. Before break, we had established from general arguments that representations of the rotation group of all non-negative integers and half-integers, depending on how many rotations the representation went through for a total 2π rotation. We called these "spin-ℓ" for $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and note that the number of states of a given spin (i.e., its dimension) is $2\ell+1$. That is, $\ell = \frac{1}{2}$ is the $\dim = 2 \cdot \frac{1}{2} + 1 = 2$ dimensional representation, etc. Well, we had also just derived that $\dim = 2c_{max}+1 = 2\ell+1$, therefore

$$c_{max} = \ell = -c_{min}.$$

That is, states in a representation of the rotation group are indexed by their eigenvalue of $\hat{L}_z$, and the eigenvalues range over:

$$-\ell, -\ell+1, \ldots, \ell-1, \ell, \text{ a total of } 2\ell+1 \text{ values.}$$

Finally, we can evaluate the Casimir in terms of ℓ:

$$C_\ell = L^2 = \hbar^2 \ell(\ell+1), \quad \text{or with } \dim = 2\ell+1 \text{ so } \ell = \frac{\dim - 1}{2},$$

$$C_\ell = \hbar^2 \frac{\dim-1}{2}\frac{\dim+1}{2} = \hbar^2 \frac{(\dim)^2-1}{4}.$$

So indeed, the Casimir encodes information about the dimension of the representation of interest.

On Friday, we will work from this point to define the quantities that can be used to uniquely identify a state. I just want to make a couple more comments in this lecture. First, note that the Casimir is <u>not</u> in the Lie algebra. The Lie algebra is a vector space of the angular momentum operators and the Casimir is a <u>quadratic</u> function of the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$. However, the Casimir <u>is</u> a Hermitian operator as it is proportional to the identity $\mathbb{1}$, and the constant of proportionality is a <u>real</u> number. Thus the Casimir is measurable. That is, we can determine the <u>total</u> angular momentum of some system through experiment. Indeed, this makes sense as that is basis-independent data of our system.