Phys 342 Lecture ~~26~~ 27

Welcome to Friday! The homework assigned before break has now matured; please turn it in.

In the week before spring break and this week, we have studied properties of rotations in three dimensions, its representation as a Lie group and Lie algebra, and the ways we can label objects that transform under rotations. This week, we had identified the representations of the rotation group. A representation is specified by an integer or half-integer l called the "spin" or generally angular momentum. For a given l, there are $2l+1$ states in the representation; hence $2l+1 = \dim_l$ is the dimension of the representation. Further, these individual states are quantified by their eigenvalue under the $\hat{z}$-component of angular momentum, $\hat{L}_z$. This eigenvalue is labeled by an integer or half-integer m that ranges in $-l$ to l : $-l \leq m \leq l$. Note that there are indeed $2l+1$ such m values. Thus, a particular ~~state~~ eigenstate of the three-dimensional rotation group can be labbled with l and m as:

$$|l,m\rangle \quad \text{such that} \quad \hat{L}_z|l,m\rangle = \hbar m|l,m\rangle.$$

Further, last lecture, we had constructed the Casimir operator C_l that measures l (or, equivalently, the dimension of the representation) independent of the particular state m. We found that

$$C_l|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle,$$

and that C_l is proportional to the identity operator to ensure that it is unaffected by rotations and thus independent of m.

By the way, these representations of the rotation group indexed by l are called "irreducible" representations, or irreps, as an irrep is completely closed under the action of $\hat{L}_x$, $\hat{L}_y$, or $\hat{L}_z$. It is "irreducible" because it is the smallest collection of states of given l for which that is closed.

In this lecture, we are going to seriously evaluate the statement that I flippantly said, that states are labeled by l and m, eigenvalues of $\hat{L}_z$ and the Casimir. We have been labeling states since we first studied an example quantum system, the infinite square well, a month ago. At that time, we basically just stated things without considering its interpretation, and its high time to correct that.

So, what is our goal here? Well, its all about naming. Shakespeare was correct to ask what's in a name, ~~but~~ and the particular word "rose" isn't special, but some name for it is important. If you say "rose", then everyone around you knows what you are talking about. Further, your name is important: the act of addressing someone personally is a very intimate action and demonstrates an almost sacred knowledge about that person. Now, you may be thinking that we're getting a bit far afield of quantum mechanics, but we have to address this issue of naming here, as well. A more mathematically precise way to think about this question is a the minimal data necessary to know completely, and precisely the state of a quantum system.

We can attack this problem by thinking about what properties we want in a quantum "name". First, for the case of addressing you by name, it must be something that we all agree on. For example, your name doesn't change if you are seated versus running to get lunch. Someone who sees you running would shout the same name as I would teaching this class to get your attention. If this were not the case, then we must have an infinite, or at least enourmous, collection of names for every possible activity we might be doing: Teaching-Andrew, Sleeping-Andrew, Chillaxing-Andrew, etc.

In the quantum analogy, we want ~~our~~ our states to be labeled in a way that is independent of basis of the Hilbert space. That way we call all agree on what the state is, irrespective of what "coordinates" we happen to use to describe our Hilbert space of states. We've also stated that the only basis independent data are eigenvalues/vectors of operators that act on the Hilbert space, thus, the basis independent way to talk about a state is as the eigenstate of some operator, or collection of operators. Thinking back to you, your name is the eigenvalue of the "name" operator, and you are the eigenstate!

Okay, so let's say we have some state $|\psi\rangle$ that is an eigenstate of a collection of operators $\{A_i\}_{i=1}^{n}$ such that

$A_i|\psi\rangle = a_i|\psi\rangle$, and a_i is the eigenvalue. In our

name analogy, we would call the collection of eigenvalues $\{a_i\}_{i=1}^{n}$ of a state its "name". If you say that collection of eigenvalues then everyone can agree on the precise state

$|\psi\rangle$ you are talking about. For example, If I tell you that $|\psi\rangle$ has eigenvalues l and m under the Casimir and $\hat{L}_z$ rotation operator, then you know that:

$$C_l|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle, \quad \hat{L}_z|\psi\rangle = m\hbar|\psi\rangle \text{ and so } |\psi\rangle = |l,m\rangle.$$

As another example, if I tell you that state $|\psi\rangle$ has eigenvalue λ under the action of the annihilation operator a, then you know that $|\psi\rangle$ ~~the~~ satisfies

$$a|\psi\rangle = \lambda|\psi\rangle \text{ and can be expressed as } |\psi\rangle = e^{\lambda a^\dagger}|\psi_0\rangle,$$

where $|\psi_0\rangle$ is the ground state of the ~~infinite square~~ quantum harmonic oscillator.

We call the collection of eigenvalues of operators of some state $|\psi\rangle$ the "quantum numbers" of a state and a sufficient collection of quantum numbers <u>uniquely</u> specifies the state. However, this still may be less than ideal for defining the state for all time, t. A general state $|\psi\rangle$ changes in time through action with the Hamiltonian $\hat{H}$, as:

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi\rangle, \text{ and } |\psi(t)\rangle \text{ might not have the}$$

same eigenvalue of some operator A that it did at time $t=0$.

Let's make this concrete. Let's assume that $|\psi\rangle$ is an eigenstate of operator A with eigenvalue a at time $t=0$:

$$A|\psi\rangle = a|\psi\rangle.$$

Further, we assume that operator A doesn't depend on time, or, we work in the Schrödinger picture inwhich all time dependence is carried by states. So, the eigenvalue equation at a later time t is:

$$e^{-\frac{i\hat{H}t}{\hbar}} A|\psi\rangle = \lambda e^{-\frac{i\hat{H}t}{\hbar}}|\psi\rangle = \left(e^{-\frac{i\hat{H}t}{\hbar}} A e^{\frac{i\hat{H}t}{\hbar}}\right) e^{-\frac{i\hat{H}t}{\hbar}}|\psi\rangle = \lambda e^{-\frac{i\hat{H}t}{\hbar}}|\psi\rangle,$$

where we just inserted "1" between A and $|\psi\rangle$ in the third equality. So, at a general time t, the state

$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi\rangle$ is an eigenstate of the time-evolved operator:

$$e^{-\frac{i\hat{H}t}{\hbar}} A e^{\frac{i\hat{H}t}{\hbar}}.$$ This is in general not just A,

so it makes our job of labeling $|\psi(t)\rangle$ at a general time more challenging. The operator underwhich $|\psi(t)\rangle$ is an eigenstate is changing in time. In general we can't ~~as~~ all agree on what this operator is because it depends on when we start our clock.

However A is a "good" operator by which to "name" the state $|\psi(t)\rangle$ for all time if time evolution does not affect A:

$$e^{-\frac{i\hat{H}t}{\hbar}} A e^{\frac{i\hat{H}t}{\hbar}} = A.$$

This is true if and only if A commutes with the Hamiltonian $\hat{H}$:

$$[\hat{H}, A] = 0 \Rightarrow e^{-\frac{i\hat{H}t}{\hbar}} A e^{\frac{i\hat{H}t}{\hbar}} = A.$$

If this is true, then we call the eigenvalues of A "good quantum numbers", in the same sense that a dog is a "good boy" because it stays when told. (I guess.)

So, to recap, those states that we all agree on and are unambiguously specified for all time are identified by a set of quantum numbers $\{a_i\}_{i=1}^n$ which are eigenvalues of the state $|\psi\rangle$ under action of operators $\{A_i\}_{i=1}^n$ that commute with the Hamiltonian: $[\hat{H}, A_i] = 0$ for all $i = 1, 2, \ldots, n$.

We've seen this commutator with the Hamiltonian before, in the time derivative of an expectation value:

$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}[\hat{H}, A]$. Apparently, if $[\hat{H}, A] = 0$, then

the expectation value of A is constant, or <u>conserved</u>, in time. Actually, if $[\hat{H}, A] = 0$, we can simultaneously diagonalize the Hamiltonian and A and so every eigenvalue of A is also constant in time. This is the statement of conservation laws in quantum mechanics. For a Hermitian operator A, the measurable quantity it corresponds to is conserved if and only if:

$$[\hat{H}, A] = 0.$$

Sometimes A is called a "charge", because it is conserved, like familiar electric charge.

In classical mechanics, you were introduced to conservation laws from a very general perspective; namely, through Noether's theorem. Noether's theorem states that a transformation that leaves the action unchanged (i.e., symmetry) has a corresponding conservation law. This vanishing commutator $[\hat{H}, A] = 0$ is the Noether's theorem of quantum mechanics. Noether's theorem really states that the laws of physics are unaffected by some transformation if and only if there is a corresponding conservation law. So far, we have only interpreted $[\hat{H}, A] = 0$ through the lense of a conservation law: that is, A corresponds to a conserved, measurable quantity. However, we could also think of this from another perspective: of $\hat{H}$ commuting with A.

If A is a Hermitian operator, then we can construct the Unitary operator through exponentiation, as we have seen many times before:

$\mathcal{U}_A = e^{iA}$. The action of $\mathcal{U}_A$ thus keeps a state in the Hilbert space, but changes it to a new state on the Hilbert space:

$$|\psi'\rangle = \mathcal{U}_A |\psi\rangle, \text{ for some } |\psi\rangle \in \mathcal{H}.$$

Thus, $\mathcal{U}_A$ is a transformation that maps the Hilbert space to itself: $\mathcal{U}_A : \mathcal{H} \to \mathcal{H}$.

Now, what is the "law" of quantum mechanics? As of now the Schrödinger equation is a close as we get to a law:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle.$$

So, what transformations can we perform on the Schrödinger equation that leave it unchanged? Well, the transformation must be unitary to maintain normalization. So, I can write it as $\mathcal{U}_A = e^{iA}$, for some Hermitian A. Now, act with $\mathcal{U}_A$ on both sides of the Schrödinger equation:

$$\mathbb{U}_A\left(i\hbar\frac{\partial|\psi\rangle}{\partial t}\right) = \mathbb{U}_A\left(\hat{H}|\psi\rangle\right) \Rightarrow i\hbar\frac{\partial}{\partial t}\left(\mathbb{U}_A|\psi\rangle\right) = \left(\mathbb{U}_A\hat{H}\mathbb{U}_A^{\dagger}\right)\mathbb{U}_A|\psi\rangle.$$

On the right, I just snuck a factor of 1 between $\hat{H}$ and $|\psi\rangle$, where

$$1 = \mathbb{U}_A\mathbb{U}_A^{\dagger}, \text{ as } \mathbb{U}_A \text{ is unitary.}$$

Now $\mathbb{U}_A|\psi\rangle = |\psi'\rangle$ is just some other state on the Hilbert space, so the schrödinger equation is unchanged as a law of physics if and only if:

$$\mathbb{U}_A\hat{H}\mathbb{U}_A^{\dagger} = e^{iA}\hat{H}e^{-iA} = \hat{H}.$$

This is only true if A and $\hat{H}$ commute: $[A,\hat{H}]=0$.

Thus, as in classical mechanics, there exists a one-to-one map between conservation laws and symmetry transformations that maintain the laws of physics.

That's it for this week; next week is the hydrogen atom!