Physics 342 Lecture 4

Welcome to week two of quantum mechanics! HW 1 has been assigned and is due this Friday. I have office hours throughout the week, so please stop by if you have any questions. I am also available to chat whenever my door is open.

Last week we had established some common ground and properties of linear operators that we will need in the rest of this course, especially for motivation of the fundamental equation of quantum mechanics. Last week, we had established that for some linear operator $\mathcal{O}$, its matrix elements could be determined from:

$$\mathcal{O}_{ij} = \int dx\, f_i(x)\, \mathcal{O}\, f_j(x),$$ where $\{f_i(x)\}_i$ is some complete, orthonormal set of functions. Orthonormality means:

$$\text{"}\delta_{ij}\text{"} = \int dx\, f_i(x)\, f_j(x),$$ where the integral ranges over the domain of x.

"Complete" means that any other function on the domain of x can be expressed as a linear combination of the basis functions:

$$g(x) = \sum_i \alpha_i f_i(x),$$ for some constants α_i.

These matrix elements $\mathcal{O}_{ij}$ are particular to the basis $\{f_i(x)\}_i$, and will be different for a different basis. However, the basis-independent data of the operator $\mathcal{O}$ are its eigenvalues $\{\lambda_i\}_i$, which are defined by

$$\mathcal{O}\, h_\lambda(x) = \lambda\, h_\lambda(x),$$ for some eigenvalue λ and function $h_\lambda(x)$.

If you know all the eigenvalues of an operator, you know everything there is to know about it. The expression of the operator in any basis is just some function of the eigenvalues of $\mathcal{O}$.

Much of the discussion from last week was focused around explicit, finite dimensional matrices, but we demonstrated that all of this is really just a property of arbitrary linear operators. Our canonical linear operator that we've studied so far is the derivative, $\partial/\partial x$. At the end of last lecture, we had expressed its eigenvalue equation as:

$$\left(-i\frac{\partial}{\partial x}\right) h_k(x) = k\, h_k(x),$$ for some real number k, and eigenfunction $h_k(x)$.

The solution to this differential equation is:

$$h_k(x) = e^{ikx}$$

This function is very familiar from your study of Fourier transformations. That is, a general function of x, $g(x)$, can be expressed as a linear combination of these imaginary exponential functions:

$$g(x) = \sum_k \alpha_k h_k(x) \xrightarrow[\text{continuous } k]{} \int dk\, \alpha(k) e^{ikx},$$

where $\alpha(k)$ is some function of k, for all real $k \in (-\infty, \infty)$. This is nothing more than the fourier transform of $\alpha(k)$, (up to a possible sign in the exponent).

This week, we want to take these mathematical truisms, and breath more physical life into them. In particular, we want to provide physical interpretations fore everything that we've develpped for linear operators. Next week and especially in the following week, we will really go overboard and take this entire framework as a hypothesis for how the universe works. For this lecture however, let's do the slightly more mundane task of giving the derivative operator a physical interpretation.

When we introduced the derivative operator we had said that its purpose in life was to <u>translate</u> a function $f(x)$. That is, if we want to translate the argument of $f(x)$ by an amount a, we have:

$$f(x+a) = f(x) + a\frac{\partial f}{\partial x} + \frac{a^2}{2}\frac{\partial^2 f}{\partial x^2} + \dots$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{\partial^n}{\partial x^n} f(x) = e^{a\frac{\partial}{\partial x}} f(x),$$

where we have just taken the Taylor expansion of $f(x+a)$ about x. Let's express this in another way. Let's just assume we transtate $f(x)$ by $\Delta x \lll 1$, so that we can safely ignore quadratic in Δx and higher terms. Then,

$$f(x+\Delta x) \approx f(x) + \Delta x \frac{\partial}{\partial x} f(x) = \left(1 + \Delta x \frac{\partial}{\partial x}\right) f(x).$$

Now, we can move another Δx by applying the derivative again:

$$f(x+2\Delta x) \approx f(x) + 2\Delta x \frac{\partial}{\partial x} f(x)$$
$$\approx \left(1 + \Delta x \frac{\partial}{\partial x}\right)^2 f(x),$$

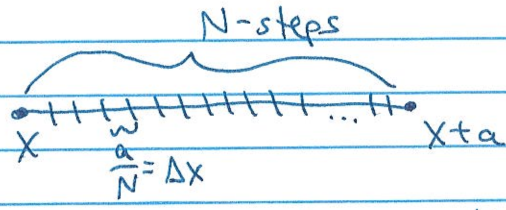
up to terms quadratic in ΔX. We can continue this process, taking N steps of size ΔX:

$$f(x+N\Delta x) \cong \left(1+\Delta x \frac{\partial}{\partial x}\right)^N f(x).$$

If we now identify $a \equiv N\Delta x$, we have

$$f(x+a) \cong \left(1+\frac{a}{N}\frac{\partial}{\partial x}\right)^N f(x),$$ so we have broken

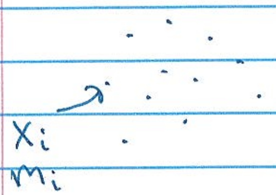
up moving a distance a from x into N steps:



Now, as $N\to\infty$, we have: $\lim_{N\to\infty}\left(1+\frac{a}{N}\frac{\partial}{\partial x}\right)^N = e^{a\frac{\partial}{\partial x}}$,

just as we found from the Taylor expansion. So, the derivative $\partial/\partial x$ translates us by an infinitesimal amount. Applying the derivative an infinite number of times enables us to move a finite distance.

Okay, now let's consider some object, which can be visualized as a collection of point masses:



The i^{th} point mass has position x_i, say, and mass m_i. This collection of points has a center-of-mass located at the point:

$$x_{cm} = \frac{m_1x_1+m_2x_2+\dots+m_nx_n}{m_1+m_2+\dots+m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

(This indeed should be a vector equation, but for simplicity, let's stick to one-dimension for now.) Now, ~~to~~ this collection of points, let's act the derivative operator on it. That is, let's perform:

$$\left(1+\Delta x \frac{\partial}{\partial x_i}\right) x_i = x_i + \Delta x \text{, so that}$$

$$\left(1+\Delta x \frac{\partial}{\partial x}\right) \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_i m_i (x_i + \Delta x)}{\sum_i m_i}$$

$$= x_{cm} + \Delta x.$$

So, for a physical object with some well-defined center-of-mass, the derivative operator implements an infinitesimal translation of its center-of-mass! Let's keep going: if the center-of-mass of an object translates, then what quantity must that object also carry? Translation of the center-of-mass means that there is a velocity of the center-of-mass, or, that the object carries momentum.

Because the derivative translates the center-of-mass of an object and the center-of-mass translates if the object carries momentum, we therefore call the derivative operator the momentum operator. However, we have to be a bit careful with units as the object

$\frac{\partial}{\partial x}$ doesn't have correct units

of momentum. That can be corrected by multiplying by some constant c_1, which carries the necessary units.

Further, last week we had established that the operator

$$-i\frac{\partial}{\partial x}$$ has exclusively real eigenvalues.

This is very nice, because momentum that we know and love is always, indeed, a real number. So, with this physical interpretation and mathematical properties, we call the following linear, differential operator the momentum operator $\hat{p}$:

$$\boxed{\hat{p} \equiv -ic_1\frac{\partial}{\partial x}}$$

We will denote operators by the carat "hat". If this is momentum, then its units in SI are

$[p] = [mv] = MLT^{-1}$, where M, L, T are objects denoting mass, length, and time units, respectively. The derivative alone has units of inverse length:

$\left[\partial/\partial x\right] = L^{-1}$, so the units of c_1 must be

$$[c_1] = MLT^{-1} \cdot L = ML^2T^{-1} = [Et],$$

which is equivalent to Joule-seconds, in SI.

Then, the eigenvalue equation for the momentum operator $\hat{p}$ is

$\hat{p}\,h_p(x) = -ic_1\frac{\partial}{\partial x}h_p(x) = p\,h_p(x)$, which has solution

$$h_p(x) = e^{i\frac{px}{c_1}}.$$

Now, p here is just the eigenvalue of the momentum operator, which is simply some real-valued momentum.

With these eigenfunctions for momentum, we are now in a position to understand orthonormality of these complex basis functions. Going back to just considering properties of matrices, let's say that given a matrix $\mathbb{M}$, it has ~~eigen~~ complex-valued eigenvectors $\{\vec{v}_i\}$ and <u>real</u> eigenvalues $\{\lambda_i\}$. The eigenvalue equation is then

$$\mathbb{M}\vec{v}_i = \lambda_i \vec{v}_i . \qquad (*)$$

By orthonormality of the basis $\{\vec{v}_i\}$, we can further dot this expression by $\vec{v}_0$ appropriately to isolate λ_i. However, note that for complex-valued $\vec{v}_i$, its dot product with itself is in general not real:

$$(\vec{v}_i \cdot \vec{v}_i)^* = \vec{v}_i^* \cdot \vec{v}_i^* \neq \vec{v}_i \cdot \vec{v}_i$$

So, we can't just dot $(*)$ by $\vec{v}_i$ and expect to return the eigenvalue because a complex vector isn't normalized by $\vec{v}_i \cdot \vec{v}_i$. However, dotting the complex conjugate of the vector with itself <u>is</u> real:

$$(\vec{v}_i^* \cdot \vec{v}_i)^* = \vec{v}_i \cdot \vec{v}_i^* = \vec{v}_i^* \cdot \vec{v}_i$$

which can then be used to enforce normalization:

$$\vec{v}_i^* \cdot \vec{v}_j = \delta_{ij} .$$

That is, the eigenvalue λ_i is just

$\vec{v}_i^{*T} \mathbb{M} \vec{v}_i = \lambda_i$, where we both complex conjugate and transpose the vector at left.

For the derivative operator's eigenfunctions, orthogonality can be expressed in a similar manner. Because $e^{\frac{ipx}{c_1}}$ is complex, to "dot" it with another eigenfunction, we must complex conjugate appropriately. That is, for $p_1 \neq p_2$, we have the orthogonality relation:

$$0 = \int_{-\infty}^{\infty} dx\, e^{-i\frac{p_1 x}{c_1}} e^{i\frac{p_2 x}{c_1}} = \int_{-\infty}^{\infty} dx\, e^{\frac{i(p_2-p_1)x}{c_1}}$$

From a physical perspective this equation means that systems of different momenta are orthogonal in this space. We test orthogonality by seeing if one momentum plus the opposite of another is at rest; i.e., at 0 momentum. In general, for any p_1, p_2, this integral is:

$$\int_{-\infty}^{\infty} dx\, e^{\frac{i(p_2-p_1)x}{c_1}} = 2\pi\, \delta(p_2-p_1),$$

where $\delta(x)$ is the Dirac-δ function, a continuous generalization of the Kronecker-δ. $\delta(x) = 0$ if $x \neq 0$ and integrates to unity

$$\int_a^b dx\, \delta(x) = 1, \text{ assuming } a < 0 < b.$$

Taking the eigenfunctions as a starting point, we will establish more physical properties later...