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Physics 342 Lecture 5

Welcome to more quantum mechanics! Remember, the first homework is due Friday. While it may not yet seem like we've done much "quantum mechanics" (whatever that means), we are getting close, and today we will identify the space in which we will work.

Before we start in earnest, let's remind of where we have come. We have been discussing properties of linear operators, such as a matrix $\mathbb{M}$. We discussed how individual elements of the matrix can be identified by multiplication with vectors that are orthonormal and complete:

$\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$ for a set of vectors $\{\vec{v}_i\}_i$. Then,

the matrix element $M_{ij} = \vec{v}_i^{\,T} \mathbb{M} \vec{v}_j$, assuming that the basis vectors are real-number valued. Last lecture, we demonstrated that for complex-valued vectors, the orthonormality condition is:

$\vec{v}_i^{\,*} \cdot \vec{v}_j = \delta_{ij}$, where $*$ denotes complex conjugation: $i \to -i$.

The matrix element $M_{ij} = \vec{v}_i^{\,T*} \mathbb{M} \vec{v}_j$, in this case.

A complete, orthonormal basis $\{\vec{v}_i\}_i$ is not unique, and so individual matrix elements M_{ij} depend on the basis and are not an intrinsic property of the matrix. The <u>only</u> quantities that are intrinsic to a matrix are its size/dimension and collection of eigenvalues; i.e., those numbers λ for which

$\det(\mathbb{M}-\lambda\mathbb{I})=0$. The collection of all such λ,

$\{\lambda_i\}_i$ is called the spectrum of the matrix/linear operator. If you know the spectrum, you know everything there is to know about an operator.

With a complete basis $\{\vec{v}_i\}$, the action of a matrix $\mathbb{M}$ on this basis (or any linear combination of them) returns another linear combination of the basis vectors. For example, for an $N\times N$ matrix $\mathbb{M}$ for the basis vectors $\{\vec{v}_i\}_{i=1}^{N}$, where we choose

$$\vec{v}_i = \left.\begin{pmatrix} \left.\begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix}\right\}i \\ 0\\ \vdots\\ 0\end{pmatrix}\right\}N$$

as the basis, the action of $\mathbb{M}$ on one such basis vector is, simply,

$$\mathbb{M}\vec{v}_i = \sum_{a=1}^{N} M_{ba}(v_i)_a = \sum_{a=1}^{N} M_{ba}\delta_{ia} = \sum_{b=1}^{N} M_{bi}\vec{v}_b,$$

some linear combination of the basis vectors, $\vec{v}_b$. That is, under the action of a linear operator $\mathbb{M}$, the space that is spanned by a complete basis $\{\vec{v}_i\}$ is unchanged. We say that this vector space is therefore closed under the operation of $\mathbb{M}$. The matrix $\mathbb{M}$ scrambles the space around, but is completely described by the complete basis (hence the nomenclature "complete").

All of this holds for general linear operators, $\hat{\theta}$, like our favorite derivative, $\partial/\partial x$. Last lecture, we had found the eigenfunctions of the derivative to satisfy:

$$\hat{p}\,h_p(x) = p\,h_p(x) = -ic_1\frac{\partial}{\partial x}h_p(x),$$ which

has solution: $h_p(x) = e^{i\frac{px}{c_1}}$, for a real number p that we call <u>momentum</u>. These eigenfunctions are orthogonal for different p, and <u>any</u> linear combination of them can be defined to produce another function of x:

$$g(x) = \int_{-\infty}^{\infty} dp\, \alpha(p)\, e^{\frac{ipx}{c_1}},$$ where $\alpha(p)$ is some function of momentum p.

You recognize this as nothing more than the Fourier transform, but written this way we can demonstrate that all the action of the momentum operator does is to define a new function of x; that is, just move around in the space of functions of x. That is,

$$\hat{p}\, g(x) = -ic_1 \frac{\partial}{\partial x} g(x) = \int_{-\infty}^{\infty} dp\, \alpha(p) \left(-ic_1 \frac{\partial}{\partial x}\right) e^{\frac{ipx}{c_1}}$$

$$= \int_{-\infty}^{\infty} dp\, \alpha(p) \left(-ic_1\right)\left(i\frac{p}{c_1}\right) e^{\frac{ipx}{c_1}}$$

$$= \int_{-\infty}^{\infty} dp\, \alpha(p)\, p\, e^{\frac{ipx}{c_1}} = \int_{-\infty}^{\infty} dp\, \beta(p)\, e^{\frac{ipx}{c_1}},$$

Where now $\beta(p) = p\,\alpha(p)$, which is just some other function of p. In addition to thinking of a linear operator as some motion throughout the space of vectors/functions, we also say that the linear operator <u>transforms</u> the vector/function to a new vector/function in the space. This space clearly has some rich properties, so let's enumerate them and give it a name.

First, as a study of the derivative operator suggested, we will consider a complex vector space, spanned by some set of orthonormal vectors $\{\vec{v}_i\}_i$, where orthonormality is defined by:

$$\vec{v}_i^{T*} \vec{v}_j = \vec{v}_i^* \cdot \vec{v}_j = \delta_{ij} = \vec{v}_j^* \cdot \vec{v}_i .$$

With this complete basis, the matrix elements of a linear operator are defined to be:

$$\theta_{ij} = \vec{v}_i^{T*} \theta \vec{v}_j . \quad (\Delta)$$

Up until now, we have considered arbitrary linear combinations of the basis vectors $\{\vec{v}_i\}$ to define the resulting vector space. However, there is something special about the requirement that the basis is both orthogonal and each element of the basis has length 1. The latter requirement, for instance, enables us to say for certain that the matrix element θ_{ij} is as given by (Δ), and not some scaling of that. Typically, for some general vector space, we do not require its basis elements to satisfy any sort of normalization condition. In the case at hand, however, there is something special about the matrix elements of θ, and in particular its eigenvalues. We had seen this with the derivative operator for which, with some physical intuition, we identified the eigenvalues of $\hat{p}$ to be real (physical?) momentum. To maintain and enforce this physical interpretation, we must enforce a normalization condition on every element in the space spanned by the basis $\{\vec{v}_i\}$. In addition to the $\vec{v}_i$

being normalized, $\vec{v}_i^* \cdot \vec{v}_i = 1$, we require that the space under consideration only consists of those linear combinations of the $\vec{v}_i$ that are also normalized.

For concreteness, consider a vector $\vec{b}$ written as a linear combination of the $\vec{v}_i$s:

$\vec{b} = \sum_i \alpha_i \vec{v}_i$, for some complex numbers α_i.

Demanding that $\vec{b}$ is normalized, we have:

$$\vec{b}^* \cdot \vec{b} = 1 = \left(\sum_j \alpha_j^* \vec{v}_j^*\right) \cdot \left(\sum_i \alpha_i \vec{v}_i\right)$$

$$= \sum_{i,j} \alpha_j^* \alpha_i \vec{v}_j^* \cdot \vec{v}_i = \sum_{i,j} \alpha_j^* \alpha_i \delta_{ij}$$

$$= \sum_i |\alpha_i|^2$$

That is, the square magnitude of the coefficients in the linear combination expansion must sum to 1.

This space is called the Hilbert Space and has the following properties:

1) Completely spanned by some orthonormal collection of ~~the~~ complex vectors $\{\vec{v}_i\}_i$

2) Only those ~~lin~~ complex linear combinations of the basis $\{\vec{v}_i\}_i$ which themselves are normalized live in the Hilbert space.

Again, this second requirement ensures that matrix elements/eigenvalues of an operator $\hat{O}$ is not scaled by the action on any state in the Hilbert space. This point will be exploited in detail in the next lecture.

Again, though we need a particular basis to talk about the Hilbert space and the vectors/states in it, it is of course independent of any basis. This feature is analogous to how we can use any arbitrary coordinate system to analyze the forces of a block on a ramp, and must find the same net acceleration, independent of coordinate choice.

All of this can be restated for continuous functions that define a Hilbert space. A complete, orthonormal set of complex functions $\{f_i(x)\}_i$ satisfying

$$\delta_{ij} = \int dx\, f_i^*(x)\, f_j(x)$$, where the integral extends over the domain of x, defines

a basis of a Hilbert space. Those functions in the Hilbert space are themselves normalized and expressed as some linear combination of the $f_i(x)$:

$$g(x) = \sum_i \alpha_i f_i(x) \quad \text{with} \quad 1 = \int dx\, g^*(x)\, g(x) = \sum_i |\alpha_i|^2.$$

Thus, the Hilbert space of complex functions consists of those functions that are L^2-normalized.

For the last few minutes, I want to study the classes of linear operators that map the Hilbert space to itself. We'll just use the familiar language of matrix algebra, but as with everything in this game, it will hold true for arbitrary Hilbert spaces, with appropriate reinterpretation, if necessary. Again, a general matrix $\mathbb{M}$ maps a vector $\vec{b}$ onto some other vector $\vec{c}$:

$\mathbb{M}\vec{b} = \vec{c}$. Let's assume that $\vec{b}$ is a vector in the Hilbert space. Then, $\vec{b}^* \cdot \vec{b} = 1$.

What is the constraint on $\mathbb{M}$ such that $\vec{c}$ is <u>also</u> in the Hilbert space?

Let's just take the dot product of $\vec{c}$ with its complex conjugate:

$$\vec{c}^* \cdot \vec{c} = \vec{c}^{T*}\vec{c} = (\mathbb{M}\vec{b})^{T*}(\mathbb{M}\vec{b}) = (\vec{b}^T \mathbb{M}^T)^* \mathbb{M}\vec{b}$$

$$= \vec{b}^{T*}\mathbb{M}^{T*}\mathbb{M}\vec{b} = 1 \text{, to be in the Hilbert space.}$$

Note that we had to transpose <u>and</u> complex conjugate the matrix $\mathbb{M}$ to calculate the inner/dot product of $\vec{c}$ with $\vec{c}^*$. This "transpose-conjugate" will come up over and over and over, so we will give it a name. We will call transpose-conjugation "Hermitian conjugation" and denote it with a dagger, $\dagger$:

$$T* \equiv \dagger.$$

So, the inner product can also be written as:

$$\vec{c}^* \cdot \vec{c} = \vec{c}^\dagger \vec{c} = \vec{b}^\dagger \mathbb{M}^\dagger \mathbb{M}\vec{b} = 1 = \vec{b}^\dagger \vec{b}.$$

If this is to be normalized for an <u>arbitrary</u> normalized vector $\vec{b}$ and linear operator $\mathbb{M}$, then there is only one possibility. The product of the matrix and its Hermitian conjugate must be the identity matrix:

$\mathbb{M}^{\dagger}\mathbb{M} = \mathbb{I}$, for which we call $\mathbb{M}$ a "unitary" matrix.

(It is also possible for $\mathbb{M}$ to first complex conjugate $\vec{b}$, and then act as a unitary matrix and maintain the normalization. Such an operator is <u>not</u> linear, and is often called "anti-unitary." While important in some contexts, we won't consider such anti-unitary operators in this class. We'll stick to good, ole' linear operators.)

That is to say that ~~these~~ unitary matrices/operators act on an element of the Hilbert space to produce another element of the Hilbert space. The Hilbert space is <u>closed</u> under the ~~operation~~ action of ~~the~~ unitary, linear operators.

We'll start to understand what these unitary operators might be next lecture.