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Physics 342 Lecture 7

Welcome back! Hope you had a relaxing weekend.

In this lecture, we'll review where we have gone so far, introduce some useful notation and work toward the greatest conceptual leap in perhaps all of physics if not all of science. In the last lectures, we introduced the Hilbert Space $\mathcal{H}$, the space of all complex vectors $\vec{b}$ for which $\vec{b}^\dagger \vec{b} = 1$:

$$\mathcal{H} = \{ \vec{b} \in \mathbb{C}^N \mid \vec{b}^\dagger \vec{b} = 1 \}$$

For continuous functions, the Hilbertspace is defined as the space of all complex-valued L^2-normalized functions:

$$\mathcal{H} = \{ g(x) \in \mathbb{C}, x \in \mathbb{R} \mid \int dx\, g(x)^* g(x) = 1 \}$$

We had identified a special class of linear operator that acts on an element of the Hilbert space to return another element of the Hilbert space. For the vector-valued Hilbert space, such linear operators can be expressed as unitary matrices. A unitary matrix $\mathbb{M}$ is one for which:

$$\mathbb{M}^\dagger \mathbb{M} = \mathbb{I}.$$

In analogy with complex numbers of unit norm, we can express a unitary matrix as the exponential of another matrix T:

$$\mathbb{M}^\dagger \mathbb{M} = \mathbb{I} \Rightarrow \mathbb{M} = e^{iT},$$

where T is a Hermitian matrix, $T = T^\dagger$. Hermitian matrices have exclusively real eigenvalues. All of this holds for a general linear operator (like the derivative) $\mathcal{O}$. If $\mathcal{O}$ is unitary, then it can always be expressed as

$$\mathcal{O} = e^{iT}, \text{ where } T \text{ is some Hermitian operator.}$$

Last lecture, we had shown that the momentum operator:

$$\hat{p} = -ic_1 \frac{\partial}{\partial x}$$

is indeed a Hermitian operator. Therefore by exponentiating it, we can define a unitary operator of translations, $D(x_0)$:

$$D(x_0) = e^{\frac{i x_0 \hat{p}}{c_1}} = e^{x_0 \frac{\partial}{\partial x}}.$$

The operator $D(x_0)$ translates (i.e., moves you) an amount x_0, just like we saw with the Taylor expansion early in this class.

Last lecture, we also noted that the only quantities that are basis independent in the Hilbert space are eigenvalues of Hermitian operators, which are all real numbers. This is exactly as would be expected/required of physical results: physics is independent of basis/coordinates (art can imitate life, but life cannot imitate art), and the results of an experiment are always real numbers. Thus, we also refer to Hermitian operators that act on a Hilbertspace as observables, as they at least have the potential to connect to experimental observables.

With this set up, we are in a position to make a profound interpretation of the vectors/~~states~~ functions in a Hilbert space. We hypothesize that the elements of a Hilbert space correspond to the possible physical states that a system can take. Thus, we say that the Hilbert space is also the state space. We will provide a more detailed interpretation later this week, but now, I want to introduce a useful, and widely-used, formalism to describe states in the Hilbert space and the action of operators on these states.

Let's first consider the discrete vector Hilbert space

$$\mathcal{H} = \{ \vec{b} \in \mathbb{C}^N \mid \vec{b}^{\dagger} \vec{b} = 1 \},$$

but we will generalize to the function-valued Hilbert space later. Paul Adrian Maurice Dirac, one of the founders of quantum mechanics, introduced a powerful notation for describing a quantum system. This is now called Dirac notation, and is especially convenient.

A vector $\vec{b}$ in the Hilbert space $\mathcal{H}$ can be expressed as a ket in Dirac notation:

$$\vec{b} \Leftrightarrow |b\rangle.$$

The ket is analogous to a column vector in usual vector notation. The Hermitian conjugate of the vector $\vec{b}$, $\vec{b}^{\dagger}$, can be expressed as a bra in Dirac notation:

$$\vec{b}^{\dagger} \Leftrightarrow \langle b|.$$

The bra is analogous to a row vector in usual vector notation. Dirac notation is also sometimes referred to as "bra-ket" notation, because of these names. As for why "bra" and "ket", well, putting them together, you get "bra(c)ket", and smashing a bra with a ket is indeed a bracket. In fact, consider two vectors $\vec{a}, \vec{b} \in \mathcal{H}$. Their dot product/inner product can be expressed as:

$$\vec{a}^{\dagger}\vec{b} = \langle a|b\rangle \quad , \quad \vec{b}^{\dagger}\vec{a} = \langle b|a\rangle.$$

Note that it is not true in general that

$\langle a|b\rangle = \langle b|a\rangle$. In fact, $\boxed{\langle a|b\rangle = \langle b|a\rangle^*}$

Therefore, the normalization constraint on a vector $\vec{b} \in \mathcal{H}$ can then be expressed as:

$$\langle b|b\rangle = 1.$$

Now, let's consider some linear, unitary operator on the Hilbert space; call it $\mathbb{M}$. Then, the $(i,j)^{th}$ entry of this matrix can be written in Diracnotation as:

$$M_{ij} = \langle v_i|\mathbb{M}|v_j\rangle = \vec{v}_i^{\dagger}\mathbb{M}\vec{v}_j,$$

where $\{\vec{v}_i\}$ is some complete, orthonormal basis of the Hilbert space.

In Dirac notation, orthonormality is easy to express:

$\langle v_i|v_j\rangle = \delta_{ij}$, but what about completeness?

we had said "completeness" meant that the vector basis $\{\vec{v}_i\}_i$ can be used to find any and every element of an arbitrary linear operator $\mathbb{M}$ on the Hilbert space. To understand completeness, we need to introduce the notion of an outer product of vectors. For two vectors $\vec{a}, \vec{b} \in \mathcal{H}$, their outer product is:

$$\vec{a}\,\vec{b}^{\dagger} = |a\rangle\langle b| \quad \text{or} \quad \vec{b}\,\vec{a}^{\dagger} = |b\rangle\langle a|.$$

Recall that the inner product of the vectors is $\langle a|b\rangle = \vec{a}^{\dagger}\vec{b}$, ~~[illegible]~~ whish is just some complex number. While the outer product $|a\rangle\langle b|$ is a matrix.

If you haven't seen this before, it can be very confusing, so let's just see how this works for two-dimensional vectors. Let's take vectors

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Their inner product, $\vec{a}^{\dagger}\vec{b}$, is

$$\vec{a}^{\dagger}\vec{b} = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1^* b_1 + a_2^* b_2 .$$

On the other hand, the outer product of vectors is

$$\vec{a}\,\vec{b}^{\dagger} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} b_1^* & b_2^* \end{pmatrix} = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{pmatrix}$$

To evaluate the outer product, we still do the usual row x column matrix multiplication; its just that the rows and columns are unfamiliar. With this understanding, let's consider the orthonormal basis of vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Note that their outer product is, for example,

$$\vec{v}_1 \vec{v}_2^{\,+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

that is, $\vec{v}_i \vec{v}_j^{\,+}$ is the matrix with all 0 entries except a 1 in the i^{th} row, j^{th} column element. For the matrix $\mathbb{M}$, we know what element should be there:

$$\vec{v}_i^{\,+} \mathbb{M} \vec{v}_j = \langle v_i | \mathbb{M} | v_j \rangle = M_{ij}.$$

So, we can completely reconstruct the matrix $\mathbb{M}$ by summing over a bunch of matrices that each only have <u>one</u> non-zero entry:

$$\mathbb{M} = \sum_{i,j} M_{ij} \, \vec{v}_i \vec{v}_j^{\,+} = \sum_{i,j} \vec{v}_i \vec{v}_i^{\,+} \mathbb{M} \vec{v}_j \vec{v}_j^{\,+} = \sum_{i,j} |v_i\rangle \langle v_i | \mathbb{M} | v_j \rangle \langle v_j|.$$

For this identity to hold, we must have, for arbitrary orthonormal basis and matrix $\mathbb{M}$:

$$\sum_i |v_i\rangle \langle v_i| = \mathbb{I}, \text{ the identity matrix.}$$

This identity is called the <u>completeness relation</u>. An orthonormal basis is complete if and only if

$$\sum_i |v_i\rangle \langle v_i| = \sum_i \vec{v}_i \vec{v}_i^{\,+} = \mathbb{I}.$$

Let's see this for our 2-dimensional vectors from above.

The outer products necessary for the completeness relation are:

$$|v_1\rangle\langle v_1| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$|v_2\rangle\langle v_2| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},$$

So indeed $|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{I}.$

We could have chosen a differnt orthonormal basis; for example, one we have studied earlier is:

$$\vec{u}_1 = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

We have already shown that it is orthonormal; is it complete? The outer products are:

$$\vec{u}_1\vec{u}_1^{\,+} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} = \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}$$

$$\vec{u}_2\vec{u}_2^{\,+} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\begin{pmatrix}-\sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\sin^2\theta & -\cos\theta\sin\theta\\ -\cos\theta\sin\theta & \cos^2\theta\end{pmatrix}$$

So, indeed, this basis is complete because

$$\vec{u}_1\vec{u}_1^{\,+} + \vec{u}_2\vec{u}_2^{\,+} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{I}.$$

We'll end today connecting Dirac notation to the continuous function Hilbert space. A function $g(x) \in \mathcal{H}$ maps onto a ket state:

$$g(x) \Leftrightarrow |g\rangle$$

The inner product of two functions $h(x), g(x) \in \mathcal{H}$ is:

$\int dx\, h^*(x)\, g(x) = \langle h|g\rangle$, the continuous generalization of the dot product. The matrix element of an operator $\mathcal{O}$ on the Hilbert space is, for an orthonormal, complete basis $\{f_i(x)\}_i$,

$$\mathcal{O}_{ij} = \int dx\, f_i(x)^* \mathcal{O}\, f_j(x) = \langle f_i|\mathcal{O}|f_j\rangle$$

Its complex conjugate is:

$\mathcal{O}_{ij}^* = \langle f_i|\mathcal{O}|f_j\rangle^* = \langle f_j|\mathcal{O}^\dagger|f_i\rangle$. If $\mathcal{O}$ is Hermitian, then $\mathcal{O}_{ij}^* = \mathcal{O}_{ji} = \langle f_j|\mathcal{O}|f_i\rangle$.

Completeness in the continuous Hilbert space case just means that

$\sum_i |f_i\rangle\langle f_i| = \mathbb{I}$, where now $\mathbb{I}$ is the identity operator for which: $\langle f_i|\mathbb{I}|f_j\rangle = \int dx\, f_i(x)^* \mathbb{I}\, f_j(x) = \delta_{ij}$, for an orthonormal basis $\{f_i(x)\}_i$.

Using this notation, we will go further with our physical interpretations...