Phys 342 Lecture 8

Today's lecture will be a brief, but absolutely central and vital, aside in our formalism development. We'll take this lecture to discuss the action of the time derivative operator; $\partial/\partial t$. Much of the discussion will parallel that of the position derivative $\partial/\partial x$, but its interpretation will be profoundly different.

As a reminder of what the derivative does consider a function of time, $f(t)$, that we displace its argument by a time Δt. Then, using the Taylor expansion, we have:

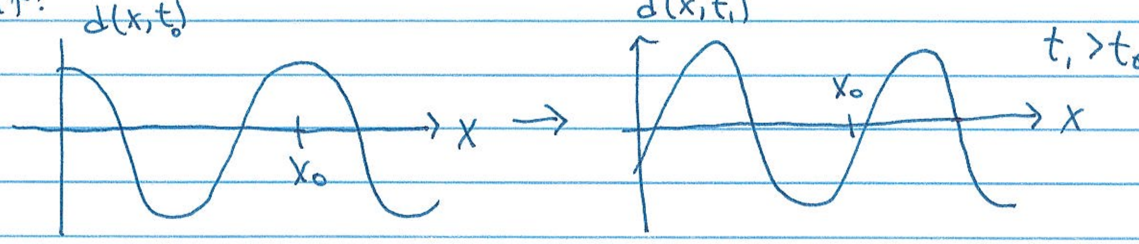
$$f(t+\Delta t) = f(t) + \Delta t \frac{\partial}{\partial t} f(t) + \frac{\Delta t^2}{2} \frac{\partial^2}{\partial t^2} f(t) + \dots$$
$$= \sum_{n=0}^{\infty} \frac{\Delta t^n}{n!} \frac{\partial^n}{\partial t^n} f(t) = e^{\Delta t \, \partial/\partial t} f(t),$$

where we should be familiar with exponentiation of a derivative from our analysis of $\partial/\partial x$. However, there's an important distinction between the spatial translation induced by $\partial/\partial x$ and the time translation induced by $\partial/\partial t$. Given a spatial origin, we can move to the left <u>or</u> right of that particular point. That is, our spatial step Δx can be positive <u>or</u> negative. However, time is not like that: regardless how we beat on, we are borne ceaselessly into the future. Given a temporal origin, we can only go but one way: forward. This correspondingly enforces a fixed sign of the time step Δt.

To determine what this sign is, we'll consider analogy with something you have already encountered in studying waves. For a wave which has displacement $d(x,t)$ at position x and time t, recall that we can express it as:

$$d(x,t) = A\cos(\omega t - kx + \phi_0),$$

where A is the amplitude of the wave, ω is the angular frequency, k is the wave number, and ϕ_0 is the initial phase. Note the relative signs between the temporal and spatial arguments in the sinusoidal function. This corresponds to a right-moving wave: as time t increases, a crest moves right!



Another way to say this is that if you were sitting at a fixed point x_0, it would feel as if you were moving left. Correspondingly, the wave moves right.

Now, we have already fixed the momentum operator to be:

$$\hat{p} = -ic_1\frac{\partial}{\partial x}, \text{ with this choice of sign.}$$

For this choice of sign of the momentum operator, we must have the opposite sign for the time translation operator, to ensure that waves moving right (positive momentum) forward in time ($\Delta t > 0$) do indeed do that.

That is, we can express the sinusoidal function $\cos(\omega t - kx)$ as:

$$\cos(\omega t - kx) = \mathrm{Re}\left[e^{-i\omega t + ikx}\right] = \mathrm{Re}\left[e^{-i(\omega t - kx)}\right]. \quad (\square)$$

We have already identified the eigenfunctions of momentum to be:

$$\hat{p}\, h_k(x) = k\, h_k(x), \text{ where}$$

$h_k(x) = e^{ikx}$, and $k = \frac{p}{c_1}$. With the "+"-sign in the exponent of the eigenfunction, $\hat{p}$ has a "−" sign. Thus, if $e^{-i\omega t}$ represents the eigenfunction of the time derivative operator, for all signs to work out, the Hermitian operator responsible for time translation must have a "+" sign.

So, if the time translation operator in exponential form is

$$\mathcal{O}(\Delta t) = e^{\Delta t \frac{\partial}{\partial t}} \quad \text{for } \Delta t > 0,$$

we can write this in the unitary operator form

$\mathcal{O} = e^{-iT}$, if we identify the Hermitian operator

T as: $T = i c_2 \frac{\partial}{\partial t}$. ~~Note~~ for some constant $c_2 \in \mathbb{R}$.

Note the "$-i$" in the exponent of the unitary operator: this follows from the − in ($\square$) and our sign convention for momentum.

~~$\hat{T} = ic_2 \frac{\partial}{\partial t}$, for some constant $c_2 \in \mathbb{R}$.~~ This is

Hermitian for all the same reasons that the momentum operator:

$$\hat{p} = -ic_1 \frac{\partial}{\partial x} \quad \text{is Hermitian.}$$

What is the physical interpretation of this time derivative operator? We can think of it, again, in much the same way as we identified $\partial/\partial x$ as the momentum operator. $\partial/\partial t$ generates a time translation of some object; that is, it forces the object to change in time. Conversely, if an object exhibits no change in time, then it just sits there, completely disengaged from every other object. When analyzing the dynamics of some system, if there is an object of that system that exhibits no change in time, then we can consistently ignore it for our analysis. Or, ~~we can set~~ it does not contribute to the system's energy: we can consistently set the energy of the unchanging object to 0. For an object or system to change in time, whatever it is, there must be a gradient of energy, so that the system changes to minimize its potential energy. For example, a ball dropped from a bridge falls to the water below because the potential energy of the ball decreases as it falls. The same thing is true for, say, deformation of squishy ball: the ball deforms to minimize its internal potential energy.

Thus, this physical picture suggest that non-trivial time dependence of some system ($\partial/\partial t \neq 0$) means that the system necessarily has non-trivial energy. So, somehow the time derivative corresponds to an energy operator. We will denote this energy operator as $\hat{H}$, as it will correspond to the <u>Hamiltonian</u>, familiar from classical mechanics. That is,

$$\hat{H} = i c_2 \frac{\partial}{\partial t}.$$

The Hamiltonian is Hermitian: $\hat{H}^\dagger = \hat{H}$, ensuring that all of its eigenvalues are real. This is indeed a good thing, as energy is always real.

If $\hat{H}$ has units of energy, we can correspondingly determine the units of c_2. We have:

$$[\hat{H}] = [E] = [c_2]\left[\frac{\partial}{\partial t}\right] = [c_2]T^{-1},$$ and so the

units of c_2 are: $[c_2] = [E]T$, or Joule-seconds in SI.

This is especially fascinating: recall that the units of the c_1 constant in the momentum operator is also Joule-seconds! Could it be that these two constants ~~are~~ just happen to have the same units, but are otherwise unrelated? Sure, I guess it is possible, but a guiding principle of science is also Occam's razor: one should make the simplest assumptions in one's hypothesis. The <u>simplest</u> assumption, motivated by the fact that these constants have the same units, is that they are just the same: $c_1 = c_2$. We will make this hypothesis, which we will be able to test, which makes it science. Actually, we'll give these constants a new name:

$$c_1 = c_2 = \hbar \quad \text{(read: "h-bar")}.$$

which is called Plank's reduced constant, or, more frequently in conversation, just h-bar. So, with this hypothesis, the momentum and energy operators are:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}, \qquad \hat{H} = i\hbar \frac{\partial}{\partial t}.$$

Both are Hermitian and so can be exponentiated to construct unitary spatial and temporal translation operators:

$$\mathcal{O}_{\text{space}}(x) = e^{\frac{i x \hat{p}}{\hbar}}, \qquad \mathcal{O}_{\text{time}}(t) = e^{-\frac{i t \hat{H}}{\hbar}}.$$

Remember the "-" sign for exponentiation of $\hat{H}$: this ensures that $t>0$ translates you forward in time.

This formulation then enables states in the Hilbert space to have time dependence, in addition to spatial dependence. So, we can consider a function $g(x,t) \in \mathcal{H}$ that encodes spatial and temporal dependence of some state of an object/system. For all of their similarities, time and space are different and this manifests itself in our definition of L^2-normalized for these time-dependent states. We really think of the function $g(x,t)$ as the complete spatial dependence of a state at time t. As such, L^2-norm is only an integration over positions:

$$1 = \int dx\, g(x,t)^* g(x,t), \quad \text{for all } t.$$

The fact that states in the Hilbert space are normalized for all time means that the volume or number of states in the Hilbert space is constant in time. This means that one can choose a fixed complete, orthonormal basis in which to define states in the Hilbert space, and that basis works for all time. For example, let's say that $\{f_i(x)\}_i$ is some time-independent orthonormal basis, such that

$$\delta_{ij} = \langle f_i | f_j \rangle = \int dx\, f_i(x)^* f_j(x).$$

Then, a general state $g(x,t) \in \mathcal{H}$ can be expressed as:

$$g(x,t) = \sum_i \alpha_i(t) |f_i\rangle = |g\rangle$$

where $\alpha_i(t)$ are some functions only of time, and all spatial dependence is carried by the f_i basis states. For g to be normalized, we must have

$$\langle g|g\rangle = 1 = \left(\sum_{i,j} \alpha_i(t)^* \alpha_j(t) \underbrace{\langle f_i | f_j \rangle}_{=\delta_{ij}} \right) = \sum_i |\alpha_i(t)|^2$$

Normalization for all time means that the time-dependent coefficients magnitude squared sum is 1, for all time.

With this time and space dependent state in the Hilbert space, we give it a special name and symbol. We denote an element of the Hilbert space as $\psi(x,t)$, where

$$\mathcal{H} = \{ \psi(x,t) \in \mathbb{C};\ x,t \in \mathbb{R} \mid \langle \psi|\psi\rangle = \int dx\, \psi(x,t)^* \psi(x,t) = 1 \}$$

and call it the "wave function". Essentially from here through the rest of the course our goal will be to study the wavefunctions of states in the Hilbert space for some system.

As a teaser for Friday's lecture, let's set the stage for an interpretation of the wave function. The wave function is L^2-normalized for all time:

$$1 = \int dx\, \Psi(x,t)^* \Psi(x,t).$$

An integral is just a continuous sum, so an interpretation of this integral is that the sum, or total, amount of some quantity represented by $\Psi^*\Psi$ is unity. Further, when expanded in a complete, orthonormal basis, this inner product is just a sum over the absolute square of some complex numbers:

$$1 = \langle \Psi | \Psi \rangle = \sum_i |\alpha_i(t)|^2, \text{ where}$$

$$\Psi(x,t) = \sum_i \alpha_i(t)\, f_i(x), \quad \text{with } \{f_i(x)\}_i \text{ the basis.}$$

The absolute square of any complex number α is non-negative:

$$|\alpha|^2 = \alpha^* \alpha \geq 0.$$

So, what do we know that its total is unity and all subparts are non-negative? What does this mean for the physical interpretation of the wavefunction Ψ? More on Friday...