Physics 342 Lecture 9

Now we are really cooking. Through last lecture, we had identified a wavefunction $\Psi(x,t)$ that lives in a Hilbert space:

$$\mathcal{H} = \{ \Psi(x,t) \in \mathbb{C} \mid \langle \Psi | \Psi \rangle = 1 \}$$

that describes the physical state of some system. Given a time-independent, orthonormal, complete basis for the Hilbert space, $\{f_i(x)\}_i$, the wave function can be expanded as a linear combination of the basis functions as:

$$\Psi(x,t) = \sum_i \alpha_i(t) f_i(x),$$

where the $\alpha_i(t)$ are some complex-valued functions of time t, but independent of x. L^2-normalization of the wave function means that:

$$\int dx\, \Psi(x,t)^* \Psi(x,t) = 1 = \int dx \sum_{i,j} \alpha_i^*(t) \alpha_j(t) f_i(x)^* f_j(x)$$

$$= \sum_i |\alpha_i(t)|^2,$$ because the $f_i(x)$ functions are orthonormal:

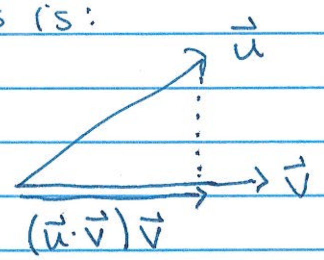
$$\int dx\, f_i(x)^* f_j(x) = \delta_{ij}.$$

We are now in a position to provide a physical interpretation of the wavefunction. To motivate it, recall the interpretation of the dot product. Consider two vectors (real-valued for now) $\vec{u}, \vec{v}$ that have both unit length:

$$\vec{u}\cdot\vec{u} = \vec{v}\cdot\vec{v} = 1.$$

Now, their dot product is: $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta = \cos\theta,$

because of their unit length. The dot product is a measure of how much of one vector lies along the direction of the other vector. A picture of this is:



If the dot product is large, then we can say that the vectors have a large overlap: Knowing one vector provides significant information about the other vector. By contrast, if the dot product is zero, the vectors have no overlap and you know very little about $\vec{u}$, say, if you know $\vec{v}$: you only know that they are orthogonal.

Now, back to the wave function. Of course, the wave function overlaps completely with itself; this is just the normalization condition:

$$\int dx\, \Psi(x,t)^* \Psi(x,t) = 1 = \langle \Psi | \Psi \rangle,$$

but it also overlaps completely with the entirety of the basis $\{f_i(x)\}$:

$$\langle \Psi | \Psi \rangle = \sum_i |\alpha_i(t)|^2 = 1.$$

Note here that $\alpha_i(t)$ is the "dot product" of Ψ with f_i:

$$\langle f_i | \Psi \rangle = \int dx\, f_i(x)^* \Psi(x,t) = \sum_j \int dx\, \alpha_j(t) f_i(x)^* f_j(x)$$

$$= \alpha_i(t).$$

So, $\alpha_i(t) = \langle f_i | \Psi \rangle$ is the "overlap" of the wavefunction with the basis element f_i. However, $\alpha_i(t)$ alone is complex-valued and unconstrained, until demanding L^2-normalization of the wavefunction. With the normalization:

$$1 = \sum_i |\alpha_i(t)|^2 = \sum_i |\langle f_i | \Psi \rangle|^2 = \sum_i \langle \Psi | f_i \rangle \langle f_i | \Psi \rangle$$

we can interpret $|\alpha_i(t)|^2$ as the actual, true fraction of the wave function Ψ that lies along the direction of $f_i(x)$. As an absolute value, it is always non-negative and is restricted to lie between 0 and 1: $0 \leq |\alpha_i(t)|^2 \leq 1$.

These considerations lead us to hypothesize that these magnitude-squared coefficients represent the probability of Ψ representing the basis element f_i. This extremely profound consequence is called the "Born Rule", after Max Born. Let's give this interpretation some more meat before we understand probability a bit more.

Let's assume we have a Hilbert space $\mathcal{H}$ such that

$$\mathcal{H} = \{ \Psi(x,t) \in \mathbb{C} \mid \int dx\, \Psi(x,t)^* \Psi(x,t) \}.$$

On this Hilbert space, there are unitary, linear transformations such that:

$$\mathcal{O}^+ \mathcal{O} = \mathbb{I},$$ and such a unitary operator can

be expressed as the exponentiation of some Hermitian, linear operator T, such that $T = T^\dagger$, and

$$\mathcal{O} = e^{iT}.$$

The only basis-independent data of the operator T are its eigenvalues and, as a Hermitian operator, all of the eigenvalues of T are real-valued. These two facts motivate the eigenvalues of T as possible results of a physical experiment. Further, we can provide an orthonormal basis on $\mathcal{H}$ with the eigenvectors of T; call them $\{|v_i\rangle\}_i$, such that

$$T|v_i\rangle = \lambda_i |v_i\rangle .$$

For distinct eigenvalues $\lambda_i \neq \lambda_j$, then different basis elements ~~...~~ are orthogonal and can be fixed to be L^2-normalized:

$$\langle v_i | v_j \rangle = \delta_{ij},$$

and we will assume completeness: $\mathbb{I} = \sum_i |v_i\rangle\langle v_i|$.

Thus, the <u>physical</u> interpretation of the eigenvalue equation

$$T|v_i\rangle = \lambda_i |v_i\rangle$$

is that if your system is in the state $|v_i\rangle$, then the outcome of measuring the quantity that corresponds to the Hermitian operator T is <u>always</u> λ_i.

Now let's consider a general wavefunction on this Hilbert space with this basis, $|\psi\rangle$. We can write

$|\psi\rangle$ expanded in the basis as:

$|\psi\rangle = \sum_i \alpha_i |v_i\rangle$, where we suppress/ignore any temporal dependence in the coefficients α_i. Given the state of the system $|\psi\rangle$, when we measure the quantity that corresponds to $\hat{T}$, we must get <u>some</u> value, but that value can be <u>any</u> eigenvalue of $\hat{T}$. This is just the statement of normalization of $|\psi\rangle$:

$$\langle\psi|\psi\rangle = \sum_i |\alpha_i|^2 = 1.$$

The interpretation of the magnitude-squared term $|\alpha_i|^2$ <u>individually</u> is that it is the <u>probability</u> that doing this measurement, the outcome will be the eigenvalue λ_i for state/eigenfunction $|v_i\rangle$.

That is, given the state ψ, if we perform the measurement corresponding to $\hat{T}$ over and over and over on this state ψ, we expect that the fraction of the measurements that we perform for which the outcome is λ_i is $|\alpha_i|^2$.

We'll see <u>many</u> examples of this in action in the coming weeks. For the rest of this lecture, we'll study some properties of probability that we'll need for going forward.

You are likely familiar with probability, if only colloquially. There are three axioms of probability which we have said already, but in a different language:

1) The probability of an event (= physical measurement) E is a non-negative real number:

$$P(E) \geq 0, \quad P(E) \in \mathbb{R}.$$

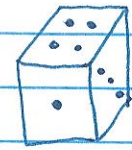
2) The total probability of any possible event occuring is unity:

$$\sum_E P(E) = 1$$

3) The probability of ~~mutual~~ a collection of mutually exclusive events $\{E_i\}$ occurring is just the sum of their individual probabilities:

$$\sum_i P(E_i) = P_{tot}.$$

We had seen axiom 1 from the normalization of the wavefunction: the magnitude square coefficients of $|\psi\rangle$ in the eigenstate basis $\{|v_i\rangle\}$ are non-negative: $|\alpha_i|^2 \geq 0$. The second axiom also follows from the normalization of the wavefunction: $\langle\psi|\psi\rangle = 1$; i.e., when we measure something on the state ψ, there is always <u>some</u> result. Finally, axiom 3 follows from the orthogonality of the basis $\{|v_i\rangle\}_i$. Orthogonality of two states $|v_i\rangle, |v_j\rangle$, $i \neq j$, means that their corresponding eigenvalues are distinct: $\lambda_i \neq \lambda_j$. Thus, outcomes of measuring the quantity that corresponds to T of λ_i and λ_j are mutually exclusive; i.e., they are <u>distinct</u> experimental outcomes.

To gain some familiarity with probability, let's consider a fair, six-sided die, with each side individually numbered with pips: 

A "fair die" means that the outcome of a roll of the die is equally likely to be any of the sides. Well, when we roll the die, <u>some</u> side must be up, so the total probability of any side being up must be 1. Different sides being up correspond to mutually exclusive outcomes, so, by the axioms of probability imply that the probability for any one side of a fair die to be up is $1/6$. That is, if you roll a fair die N times, as $N \to \infty$, you <u>expect</u> that $N/6$ rolls are 1, $N/6$ rolls are 2, etc.

Okay, but what if you just roll the die once: what value do you <u>expect</u> the die to return? One answer is simply that you expect each side evenly, so there is no well-defined answer. Indeed, but you could also interpret this question as what the <u>average</u> outcome of one roll would be. That is, roll the die $N \to \infty$ times, and ask what is the mean value of the outcome of these rolls? To calculate this average, we simply sum up the ~~total~~ values of ~~the~~ each ~~rolls of~~ die for every one of the N rolls. If the outcome of i has probability p_i, then you expect that $p_i N$ of the rolls return i, and their total sum of pips is $i p_i N$. To find the sum of all possible rolls, we then sum over i:

$$\text{Total} = \sum_{i=1}^{6} i p_i N$$

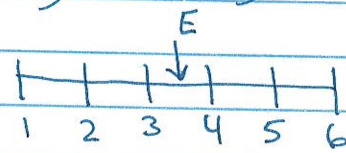
The mean is then this total divided by the number of rolls N. We call this mean the <u>expectation value</u> E:

$$E = \sum_{i=1}^{6} i p_i .$$

With $p_i = 1/6$ for a fair die, this expectation is:

$$E = \frac{1}{6}\sum_{i=1}^{6} = \frac{1}{6}\,\frac{6(6+1)}{2} = \frac{7}{2} = 3.5$$

This value is exactly halfway between the possible rolls of the die:



(Note that the center-of-mass is like an expectation value for the "mass" of an object. If we interpret $m_i / \sum_i m_i = p_i$, then the center-of-mass can be expressed as:

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i} = \sum_i x_i p_i .)$$

In the language of our Hilbert space, we've already seen this expectation value, though in a different language. Given a state $|\psi\rangle$, expanded in the eigenbasis of a Hermitian operator T as

$|\psi\rangle = \sum_i \alpha_i |v_i\rangle$, what is the expectation

value of the outcome of a measument of the quantity of which T corresponds? Well, the outcomes are just the eigenvalues λ_i with probabilities $|\alpha_i|^2$, so the expectation value is:

$$E = \sum_i \lambda_i |\alpha_i|^2 = \sum_{i,j} \alpha_i^* \alpha_j \langle f_i | T | f_j \rangle = \langle \psi | T | \psi \rangle .$$

That is, the expectation value of T on the state Ψ is:

$$E_T = \langle \Psi | T | \Psi \rangle = \int dx\, \Psi(x,t)^* \, T \, \Psi(x,t).$$

This is also, equivalently, the matrix element of T in a basis that includes the state $|\Psi\rangle$ that lies in the Ψ^{th} row and column on the diagonal of T. That is, expectation values of a Hermitian operator just correspond to its matrix elements, in some basis.

That's it for today; next week we will (finally!) get to the fundamental equation of quantum mechanics.