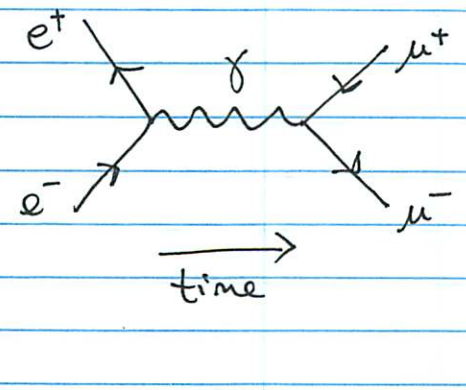
- Residue theorem
- HW due today & Thursday

Electron-Positron Annihilation Lecture 10

Over the first five weeks of this class, we introduced tools of particle physics: from natural units, to group theory, and Fermi's Golden Rule. This week and for the rest of this class, we are using these tools to understand experimental results and to construct physical theories for explanation. Today, we will go in some detail in the calculation of a Feynman diagram, which will form the basis of our understanding of Quantum Chromodynamics, or QCD.

We will first calculate the Feynman diagram for the process $e^+e^- \to \mu^+\mu^-$, which we studied a bit two weeks ago. To calculate the Feynman diagram, we will make the simplifying assumption that both the electron and muon are massless. That is, we will consider the electron and muons in this collision to have energy much greater than their masses. This is actually the realm in which we will mostly work in this class, so this is an excellent approximation. In this massless approximation, we are also able to express the spin of the electon and muon in terms of its helicity, which will greatly simplify the analysis.

So, our goal this lecture is to calculate:



with $m_e = m_\mu = 0$.

To accomplish this, there are several things we need to do:

- Determine the massless solutions to the Dirac equation to describe the external electrons and muons
- Identify the allowed helicity configurations of the process $e^+e^- \to \mu^+\mu^-$
- Actually calculate the Feynman diagrams for the helicity configurations that are allowed.

So, let's begin. The massless Dirac equation is

$$i\gamma \cdot \partial \psi = 0$$

To solve this, we will work with γ matrices in the Weyl or Chiral representation, where

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \text{ with } \begin{array}{l} \sigma_\mu = (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3) \\ \bar{\sigma}_\mu = (\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3) \end{array}$$

where the σ_i are the Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Plugging this into the Dirac equation, we find that it separates into two equations:

$$i\sigma \cdot \partial \psi_R = 0$$
$$i\bar{\sigma} \cdot \partial \psi_L = 0$$

These are called the Weyl equations. The solutions to $i\sigma\cdot\partial\psi_R = 0$ are right-handed helicity fermions, while the solutions to $i\bar{\sigma}\cdot\partial\psi_L = 0$ are ~~left~~-handed helicity fermions. These names will be clear in a minute.

Let's solve $i\sigma\cdot\partial\psi_R = 0$ in the standard way. We will write

$\psi_R = u_R e^{-ip\cdot x}$, for some spinor u_R (like a two-component vector) and four-momentum p. We then have

$$i\sigma\cdot\partial\psi_R \rightarrow (\sigma\cdot p)u_R = 0$$

First, for illustration, we will consider $\vec{p}$ aligned along the $+\hat{z}$ direction and $E_p > 0$. Then, as you showed in a homework,

$$\sigma\cdot p = \sigma_0 p_0 - \sigma_3 p_z = \begin{pmatrix} p_0 & 0 \\ 0 & p_0 \end{pmatrix} - \begin{pmatrix} p_z & 0 \\ 0 & -p_z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}$$

Note that $p_0 = E$ and by the masslessness condition $E = +|\vec{p}|$. Therefore the solution to $(\sigma\cdot p)u_R = 0$ is

$u_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. That is, this fermion is "spin up",

or its spin is aligned with the direction of motion, +-helicity. Out of convention, this spinor is normalized by the energy of the fermion as:

$$u_R = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \psi_R = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iEt+iEz}.$$

(This normalization isn't arbitrary; it is required for probabilities to sum to 1.)

In the more general case where the momentum is at an angle θ with respect to the $+\hat{z}$ axis and at an angle ϕ about the $+\hat{z}$ axis, we can still determine the solution to the Dirac equation. In this case:

$$(\sigma \cdot p)\, u_R = 0 = \left(\sigma_0 E - \sigma_1 E \sin\theta\cos\phi - \sigma_2 E \sin\theta \sin\phi - \sigma_3 E\cos\theta\right) u_R = 0$$

That is,

$$E \begin{pmatrix} 1-\cos\theta & -e^{-i\phi}\sin\theta \\ -e^{i\phi}\sin\theta & 1+\cos\theta \end{pmatrix} u_R = 0$$

The properly normalized solution to this eigenvalue equation is

$$u_R = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

Note that for $\theta \to 0$, $u_R \to \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is the solution for spin-up along the $\hat{z}$-axis. Note that

$$u_R^{\dagger} u_R = 2E, \text{ as required.}$$

The negative energy solutions (anti-particles) and the left-handed solutions can be found similarly, so I won't discuss it more here. I will just write down their results, and make an observation.

First, the observation. The eigenvalue equation for the negative energy solutions to the right-handed Weyl equation is:

$$(\sigma \cdot p)\, v_R = \left(-\sigma_0 E - \sigma_1 E\sin\theta\cos\phi - \sigma_2 E\sin\theta\sin\phi - \sigma_3 E\cos\theta\right) v_R = 0$$

or that $$-E\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} v_R = 0.$$

On the other hand, the left-handed, positive energy solution satisfies the eigenvalue equation:

$$(\bar{\sigma}\cdot p)\, u_L = 0 = \left(\sigma_0 E + \sigma_1 E\sin\theta\cos\phi + \sigma_2 E\sin\theta\sin\phi + \sigma_3 E\cos\theta\right) u_L = 0$$

or that $$E\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} u_L = 0$$

That is, as two component spinors, v_R and u_L satisfy the same eigenvalue equation! This is also true for u_R and v_L. Therefore, the spinor solutions to the Dirac equation are:

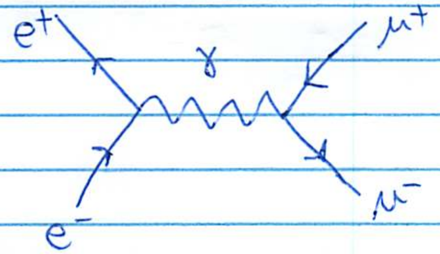
$$u_R = v_L = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}, \quad u_L = v_R = \sqrt{2E}\begin{pmatrix} e^{-i\phi}\sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

Note that for each of them, $u^\dagger u = 2E$ and

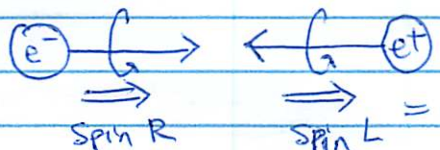
$$u_R^\dagger u_L = v_L^\dagger v_R = 0.$$

Whew, we found the solutions to the massless Dirac equation! Now, we need to work on the second aspect of this lecture: determining the allowed helicity configurations of the electrons and muons. We discussed this a couple weeks ago when we introduced Feynman diagrams, so I'll just remind of the results from that discussion.

For the process $e^+e^- \to \mu^+\mu^-$, the Feynman diagram is:

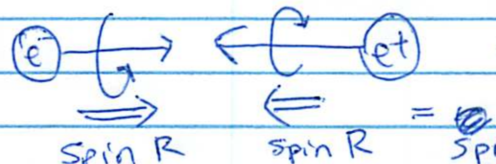


The intermediate photon is spin-1, so the helicities of the e^+e^- pair <u>and</u> the $\mu^+\mu^-$ pair must combine into spin-1. This means that, for electrons colliding in the center of mass frame:



is allowed

while



is <u>not</u> allowed.

So, the allowed helicity configurations are all of those for which the electron and positron have opposite helicities, and the μ^+ and μ^- have opposite helicities. These possibilities are:

$$e^+_L e^-_R \to \mu^+_L \mu^-_R, \qquad e^+_R e^-_L \to \mu^+_L \mu^-_R,$$

$$e^+_L e^-_R \to \mu^+_R \mu^-_L, \qquad e^+_R e^-_L \to \mu^+_R \mu^-_L.$$

The helicities of the particles are denoted by the subscripts. A helicity configuration like

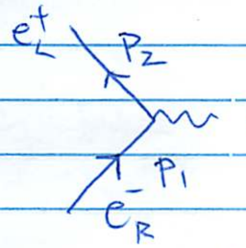
$$e^+_L e^-_L \to \mu^+_R \mu^-_L$$

has 0 probability to occur. That is, its corresponding Feynman diagram is 0. Only the four helicity configurations listed above have non-zero Feynman diagrams.

There's a bit more simplification we can do. Recall from our discussion of Feynman diagrams that we assign appropriate spinors or their conjugates to initial and final state particles and anti-particles. We have the rules:

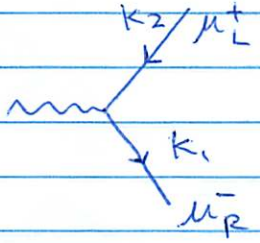
Final particle $\Rightarrow u^{+}(p)$ Final anti-particle $\Rightarrow v(p)$
Initial particle $\Rightarrow u(p)$ Initial anti-particle $\Rightarrow v^{+}(p)$

So, in the Feynman diagram calculation we will have to calculate factors such as:



$= V^{+}_{L}(p_2)\,\sigma^{\mu} u_R(p_1)$

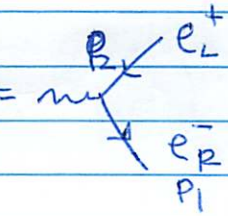
and



$= u^{+}_{R}(k_1)\,\bar{\sigma}_{\mu} V_L(k_2)$

Recall that the σ-matrices are there to ensure that the helicities of the e^+e^- or $\mu^+\mu^-$ pair align.
Now, for some magic simplification/identification.
We noted that $V_L = u_R$, for example, so we can make the identifications:

$$= V_L^\dagger(p_2)\sigma^\mu u_R(p_1) = u_R^\dagger(p_2)\sigma^\mu V_L(p_1) =$$



That is, we have turned initial state interactions into final state interactions by just identifying spinors. The magic comes when we relate this to complex conjugation. Note that

$$\left(V_L^\dagger(p_2)\sigma^\mu u_R(p_1)\right)^* = u_R^\dagger(p_1)\sigma^\mu V_L(p_2)$$

So, apparently, complex conjugation relates initial- and final states!

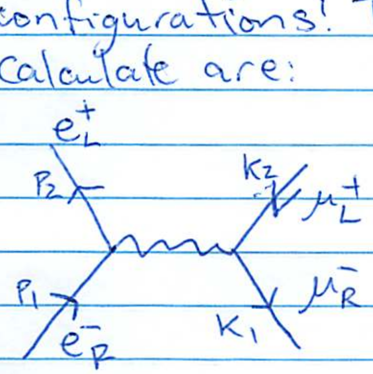
This observation immediately relates some of the matrix elements we need to calculate. This implies that:

$$\mathcal{M}(e_L^+ e_R^- \to \mu_L^+ \mu_R^-) = \mathcal{M}(e_R^+ e_L^- \to \mu_R^- \mu_L^+)^*$$

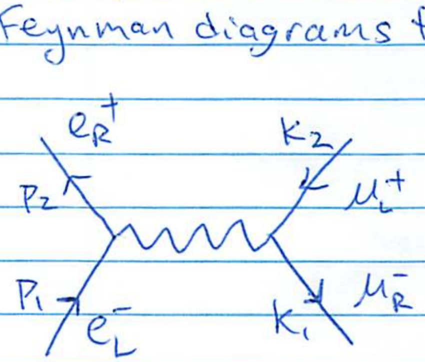
and

$$\mathcal{M}(e_R^+ e_L^- \to \mu_L^+ \mu_R^-) = \mathcal{M}(e_L^+ e_R^- \to \mu_R^+ \mu_L^-)^*$$

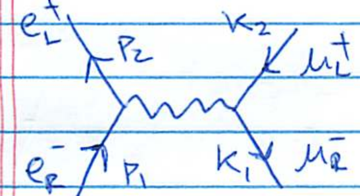
So, we only need to calculate two non-zero helicity configurations! The two Feynman diagrams to calculate are:



and



A couple of weeks ago, we found that the first diagram can be expressed as:



$$= v_L^\dagger(p_2)\sigma^\mu u_R(p_1)\frac{e^2}{(p_1+p_2)^2}u_R^\dagger(k_1)\bar{\sigma}_\mu v_L(k_2)$$

This is Lorentz invariant, and so can be evaluated in any frame. Let's choose the center-of-mass frame, when the e^+ and e^- have net 0 momentum and collide head on. Then, we can write

$$p_1 = \frac{E_{cm}}{2}(1,0,0,1), \quad p_2 = \frac{E_{cm}}{2}(1,0,0,-1)$$

where E_{cm} is the total energy of the e^+e^- system. The spinors are then

$$v_L^\dagger(p_2) = \sqrt{\cancel{2}\frac{E_{cm}}{\cancel{2}}}\begin{pmatrix}0 & 1\end{pmatrix}, \quad u_R(p_1) = \sqrt{E_{cm}}\begin{pmatrix}1\\0\end{pmatrix}$$

because $2E = E_{cm}$. Then, the product:

$$v_L^\dagger\sigma^\mu u_R = E_{cm}\begin{pmatrix}0 & 1\end{pmatrix}(\mathbb{1},\sigma_1,\sigma_2,\sigma_3)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= E_{cm}\begin{pmatrix}0 & 1\end{pmatrix}\left(\hat{x}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}+\hat{y}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= E_{cm}(\hat{x}+i\hat{y})$$

We can also evaluate the product:

$u_R^\dagger(k_1)\bar{\sigma}_\mu v_L(k_2) = \cancel{E_{cm}}$, which requires knowing the momenta k_1 and k_2. By energy-momentum conservation $E_{\mu^+} = E_{\mu^-} = E_{cm}/2$ and we can choose the frame where

$$k_1 = \frac{E_{cm}}{2}\left(1, \sin\theta, 0, \cos\theta\right), \quad k_2 = \frac{E_{cm}}{2}\left(1, -\sin\theta, 0, -\cos\theta\right)$$

Then, $u_R^\dagger(k_1) = \sqrt{E_{cm}}\left(\cos\theta/2 \;\; \sin\theta/2\right)$

$$V_L(k_2) = \sqrt{E_{cm}}\begin{pmatrix} +\sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

The product is then:

$$u_R^\dagger(k_1)\,\bar{\sigma}_\mu\, V_L(k_2) = E_{cm}\left(\cos\theta/2 \;\; \sin\theta/2\right)\left(\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3\right)\begin{pmatrix} +\sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

$$= E_{cm}\left(\hat{x}\cos\theta - i\hat{y} - \hat{z}\sin\theta\right)$$

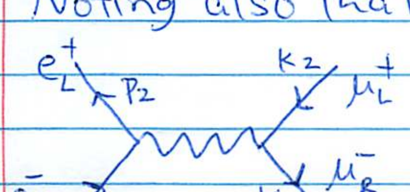
That is,

$$V_L^\dagger \sigma^\mu u_R = E_{cm}\left(0, 1, i, 0\right)$$
$$u_R^\dagger \bar{\sigma}^\mu V_L = E_{cm}\left(0, \cos\theta, -i, -\sin\theta\right)$$

The dot product is therefore:

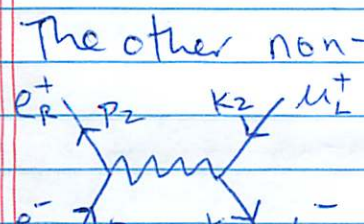
$$V_L^\dagger \sigma^\mu u_R\, u_R^\dagger \bar{\sigma}_\mu V_L = E_{cm}^2\left(-\cos\theta - 1\right) = -E_{cm}^2\left(1 + \cos\theta\right)$$

Noting also that $(p_1 + p_2)^2 = E_{cm}^2$, the Feynman diagram is:



$$= -e^2\left(1 + \cos\theta\right) = \mathcal{M}\left(e_R^- e_L^+ \to \mu_R^- \mu_L^+\right)$$

The other non-trivial Feynman diagram is



$$= V_R^\dagger(p_2)\sigma^\mu u_L(p_1)\,\frac{e^2}{(p_1+p_2)^2}\, u_R^\dagger(k_1)\bar{\sigma}_\mu V_L(k_2)$$

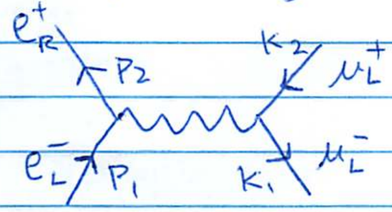
We can calculate the first spinor product to be:

$$V_R^\dagger(p_2)\sigma^\mu u_L(p_1) = E_{cm}\begin{pmatrix}1 & 0\end{pmatrix}(\mathbb{1},\sigma_1,\sigma_2,\sigma_3)\begin{pmatrix}0\\1\end{pmatrix}$$
$$= E_{cm}(\hat{x}-i\hat{y})$$

The dot product for this helicity configuration is:

$$V_R^\dagger(p_2)\sigma^\mu u_L(p_1)\, u_R^\dagger(k_1)\bar{\sigma}_\mu V_L(k_2) = E_{cm}^2(-\cos\theta+1) = +E_{cm}^2(1-\cos\theta)$$

Then, the feynman diagram is:



$$= e^2(1-\cos\theta) = \mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+)$$

Woohoo! We've calculated our first Feynman diagrams. One more word, and then we'll pick up this next lecture. Now, that we calculated these amplitudes, how do we find probabilities? Well, in principle, we can measure if a particle is an e^+, e^-, μ^+ or μ^- by a charge and mass measurement. Additionally, we can, in principle, measure the spin of the electrons and muons (via Stern-Gerlach, or such). This means that the collisions

$$e_R^- e_L^+ \to \mu_R^- \mu_L^+ \quad \text{and} \quad e_L^- e_R^+ \to \mu_R^- \mu_L^+$$

correspond to distinct measurable, physical processes. Therefore, they cannot interfere quantum mechanically.

Then, the probability that an electron-muon collision occurred with given spin configurations is found by squaring individual amplitudes. For collisions that produce muons at an angle θ with respect to the electron-positron beams in the center-of-mass frame, we then have:

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = |\mathcal{M}(e_L^- e_R^+ \to \mu_L^- \mu_R^+)|^2 = e^4(1+\cos\theta)^2,$$

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_L^- \mu_R^+)|^2 = |\mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+)|^2 = e^4(1-\cos\theta)^2.$$

We'll discuss how to put these into a cross section and its implications for QCD next lecture.