- Turn in HW

$e^+e^- \to$ hadrons Lecture 11

At the end of last lecture, we had calculated the Feynman diagrams and squared matrix elements for $e^+e^- \to \mu^+\mu^-$ scattering with definite helicities. We found

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = |\mathcal{M}(e_L^- e_R^+ \to \mu_L^- \mu_R^+)|^2 = e^4(1+\cos\theta)^2$$

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_L^- \mu_R^+)|^2 = |\mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+)|^2 = e^4(1-\cos\theta)^2$$

We're almost there to get a physical result. First, we need to combine these results in a way that is sensible for our experimental set up. Typically, we send in unpolarized electrons and positrons. That is, half of the electrons are right-handed and half are left-handed (and similar for positrons). Therefore, we should average over the initial e^+ and e^- helicities. There are four possible helicity configurations (LL, LR, RL, and RR) and so we should multiply the result by 1/4.

Also, we typically do not measure the final state muon helicities. Our experimental set up just collects muon 4-momenta. (Note that this is different than the spins being able to be measured in principle.) Therefore, we should sum the probabilities of the possible final state helicities. Therefore, our experiment ~~sums~~ averages over initial helicities and sums over final helicities. Doing this, we find:

$$\frac{1}{4}\sum_{\mu\,\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\left(2e^4(1+\cos\theta)^2 + 2e^4(1-\cos\theta)^2\right) = e^4(1+\cos^2\theta).$$

Now, we can insert this into the expression for the cross section, by Fermi's Golden Rule:

$$\sigma \equiv \frac{1}{4}\sum_{\text{spins}} \sigma = \frac{1}{2E_{e^+}}\frac{1}{2E_{e^-}}\frac{1}{|v_{e^+}-v_{e^-}|}\int \frac{d^4k_1}{(2\pi)^4} 2\pi\delta(k_1^2)\,\frac{d^4k_2}{(2\pi)^4}\,2\pi\delta(k_2^2)$$

$$\times\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right)\cdot(2\pi)^4\,\delta^{(4)}(Q-k_1-k_2)$$

where the four-vector $Q=(E_{cm},0,0,0)$. Much of this can be simplified. In the center-of-mass frame,

$2E_{e^+}=2E_{e^-}=E_{cm}$ and because the

e^+ and e^- are massless and colliding head-on,

$$|v_{e^+}-v_{e^-}|=2.$$

We can also enforce the on-shell δ-functions and plug in the expression for the ~~ex~~ squared matrix element. We then have

$$\sigma = \frac{1}{2E_{cm}^2}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{2E_{k_1}}\frac{1}{2E_{k_2}}(2\pi)^4\,\delta^{(4)}(Q-k_1-k_2)$$

$$\times\, e^4(1+\cos^2\theta).$$

The integral over $\vec{k}_2$ can be done with the δ-functions and $E_{k_1}=E_{k_2}=E_{cm}/2$ and so

$$\sigma=\frac{1}{2E_{cm}^2}\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{E_{cm}^2}(2\pi)\,\delta(E_{cm}-E_1-E_2)\,e^4(1+\cos^2\theta).$$

Continuing, we can write the remaining integrals in spherical coordinates as:

$$\sigma = \frac{1}{2E_{cm}^2}\int \frac{|\vec{k}_1|^2\, d|\vec{k}_1|\, d\cos\theta\, d\phi}{(2\pi)^3}\frac{1}{2E_{cm}^2}\, 2\pi\, \delta(E_{cm}-E_1-E_2)\, e^4(1+\cos^2\theta)$$

The integral over ϕ just gives 2π and the δ-functions fixed $|\vec{k}_1| = E_{cm}/2$. Then, we get

$$\sigma = \frac{1}{2E_{cm}^2}\frac{1}{2\pi}\cdot\frac{1}{4}\, e^4 \int \frac{d\cos\theta}{2}(1+\cos^2\theta)$$

In natural units, the fine structure constant α is

$$\alpha = \frac{e^2}{4\pi}$$ and so we find

$$\sigma = \frac{\pi\alpha^2}{2E_{cm}^2}\int_{-1}^{1} d\cos\theta\,(1+\cos^2\theta).$$

If we remove the integral, the result is called the cross section differential in the scattering angle θ or just differential cross section:

$$\frac{d\sigma}{d\cos\theta}(e^+e^-\to\mu^+\mu^-) = \frac{\pi\alpha^2}{2E_{cm}^2}(1+\cos^2\theta)$$

This angular distribution can be plotted and compared to the outcome of experiment.

We can also calculate the total cross section by integrating over $\cos\theta$:

$$\sigma = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

This cross section can then be used to calculate the total number of $e^+e^-\to\mu^+\mu^-$ events in a collision experiment.

To put this in perspective, the fine structure constant

$$\alpha \cong 0.0073 \cong 1/137 \quad \text{and so}$$

$$\sigma(e^+e^- \to \mu^+\mu^-) \cong 80\,\text{nb} \cdot \left(\frac{1\,\text{GeV}}{E_{cm}}\right)^2$$

So, $e^+e^- \to \mu^+\mu^-$ occurs about a millionth as likely as proton-proton collisions at high energies.

This is called an inclusive cross section ~~for~~ because it is a good approximation to the cross section for including anything else. This process is denoted by

$$e^+e^- \to \mu^+\mu^- + X, \text{ where } X \text{ can be anything.}$$

That is, X can be nothing (like we've studied here), or it can be a photon γ, or 5 photons, or an e^+e^- pair, etc. That is,

$$e^+e^- \to \mu^+\mu^- + X = (e^+e^- \to \mu^+\mu^-) + (e^+e^- \to \mu^+\mu^-\gamma) + \ldots$$

The process is a good approximation to the inclusive cross section because α is small. Note that the cross section for $e^+e^- \to \mu^+\mu^-\gamma$ scales like:

$$\sigma(e^+e^- \to \mu^+\mu^-\gamma) \propto \alpha^3, \text{ which is about 100 times}$$

smaller than $\sigma(e^+e^- \to \mu^+\mu^-)$. In general, the cross section scales like α to the power equal to the number of final state particles. That is, we can write

$$\cancel{\sigma(e^+e^- \to \mu^+\mu^- + X) \cong (80\,\text{nb} +}$$

$$\sigma(e^+e^- \to \mu^+\mu^- + X) \cong 80(1+\overset{\mathcal{O}(\alpha)}{\downarrow}) \text{ nb} \left(\frac{1\text{ GeV}}{E_{cm}}\right)^2.$$

To excellent approximation $\sigma(e^+e^- \to \mu^+\mu^- + X) \cong \sigma(e^+e^- \to \mu^+\mu^-)$; nevertheless, one can calculate the corrections suppressed by α and find excellent agreement with data.

Very importantly, $\sigma(e^+e^- \to \mu^+\mu^-)$ is <u>not</u> a good approximation for the <u>exclusive</u> process

$$e^+e^- \to \mu^+\mu^- + \text{nothing}.$$

We won't discuss why this is the case in this class, but the basic idea is that "nothing" is quite a subtle subject in particle physics.

This discussion of inclusive cross sections leads into the observation and prediction of $e^+e^- \to$ hadrons events. At a lepton collider, e^+ and e^- are accelerated and made to collide head-on. In addition to processes like $e^+e^- \to \mu^+\mu^-$, also observed is the production of hadrons in the process $e^+e^- \to$ hadrons. Hadrons, like pions, protons, neutrons and the like are complex, composite particles. We would like to predict the cross section for $e^+e^- \to$ hadrons.

How do we do this? Well, one way is to calculate the individual rates for $e^+e^- \to \pi^0\pi^0$, $e^+e^- \to \pi^+\pi^-$, $e^+e^- \to p^+p^-, \ldots$, and then sum them all together. At the very least, this is enormously computationally challenging as we have to consider a huge number of processes. As it turns out, despite significant effort by thousands of theorists, no one has made significant progress on the "direct" calculation of $e^+e^- \to$ hadrons.

So, what do we do? Do we give up?

Of course the answer is no, we just have to think about the problem differently. This goes back to our understanding of inclusive cross sections and the constituents of hadrons. As we discussed with the quark model, hadrons organize themselves into representations of isospin or SU(3) flavour. All observed representations have a dimension larger than 2 or 3, which would be the dimensionality of the fundamental representation of isospin and SU(3) flavor, respectively. So, this suggests that all representations of hadrons are formed by taking products of smaller representations, and the existence of fundamental constituents, called quarks.

We don't, nor have we ever, observed quarks directly for reasons we will discuss in a few weeks. Nevertheless, with this observation, we can make progress on understanding $e^+e^- \to$ hadrons. First, this is an inclusive process; we aren't demanding anything specific about the observed hadrons (like that they are all pions, or something), and there could be anything else produced in the collision. So, with that understanding, considering processes like

$$e^+e^- \to \pi^+\pi^- + X$$

is wrong; while this includes some of the possible final states, it misses most of them, like

$e^+e^- \to \pi^0\pi^0$, for example.

However, while we don't understand the magic responsible for it, if we consider processes in which quarks are produced, this can include the production of ~~[illegible]~~ any collection of hadrons in the final state, consistent with energy and momentum conservation. For example, considering the inclusive process

$$e^+e^- \to u\bar{u}, \text{ where } u \text{ is an up quark}$$

describes both processes where neutral and charged hadrons are produced:

$$e^+e^- \to \pi^0\pi^0 \to (u\bar{u})(u\bar{u}), \quad e^+e^- \to \pi^+\pi^- \to (u\bar{d})(\bar{u}d),$$

Again we have to be inclusive: $e^+e^- \to u\bar{u} + X$ can produce $\pi^0\pi^0$ and $\pi^+\pi^-$ final states (and many others!).

So, to good approximation, we can predict the $e^+e^- \to$ hadrons cross section from the inclusive cross section

$$e^+e^- \to q\bar{q}, \text{ where we sum over}$$

all possible final state quarks allowed by energy and momentum conservation. Quarks are spin-1/2 particles, just like muons, so we can reuse our results from that study to determine

$$e^+e^- \to \text{hadrons} \cong e^+e^- \to q\bar{q} + X$$

So, how do we do this?

By the quark model, quarks must be electrically charged. What are their charges? The quark model predicts that the proton consists of two up quarks and a down quark (uud) and a neutron is two down quarks and an up (ddu). The electric charge of the proton and neutron is just the sum of electric charges of the quarks that compose them. Then,

$$2q_u + q_d = 1 \quad , \quad 2q_d + q_u = 0$$

Or that $q_u = 2/3$, $q_d = -1/3$. The quark model predicts that quarks have fractional electric charges! Fascinating, but let's keep going.

Quarks, like electrons and muons are electrically charged, therefore they couple to electromagnetism and can be produced in essentially the exact same way as $e^+e^- \to \mu^+\mu^-$. The only difference is that we need to include the charge of the quarks in the calculation, and sum over all quarks; That is, sum over all flavours of quarks. Importantly, however, this sum is done at the level of the squared matrix element, and not of individual Feynman diagrams. While we have never directly observed quarks, in principle we could, and could therefore distinguish them by their masses, charges, etc.

This principle then tells us that the total cross section for $e^+e^- \to$ hadrons, to good approximation, is just

$$\sigma(e^+e^- \to \text{hadrons}) \cong \sum_{\text{quarks}} \sigma(e^+e^- \to q\bar{q}) = \sum_{\substack{\text{quarks} \\ i}} \frac{4\pi\alpha^2}{3E_{cm}^2} Q_i^2 ,$$

where Q_i is the electric charge of quark i.

This can be compared to experiment. Actually, a better measurement is the ratio of the cross sections

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)},$$

which reduces to the sum over quark charges. By 1974, four quarks were known; up, down, strange and charm, with charges:

$$Q_u = 2/3, \quad Q_d = -1/3, \quad Q_c = 2/3, \quad Q_s = -1/3$$

Thus, we (naively) predict that R for energies above two times the mass of the charm quark is

$$R \cong \sum_{\substack{\text{quarks} \\ i}} Q_i^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{10}{9}$$

The observed value is actually extremely close to

$R = 10/3$, three times larger than this prediction.

Hmm, does this mean that we are wrong? No, just incomplete. Along with other evidence we will discuss soon, this cross section ratio discrepancy is evidence that there are actually 3 copies of each type of quark. These three copies of quarks each carry a different colour, typically called blue, red and green (though has nothing to do with visible light!). Therefore, there are actually three times as many quark final states available as we naively expected. Including this factor of 3, we find,

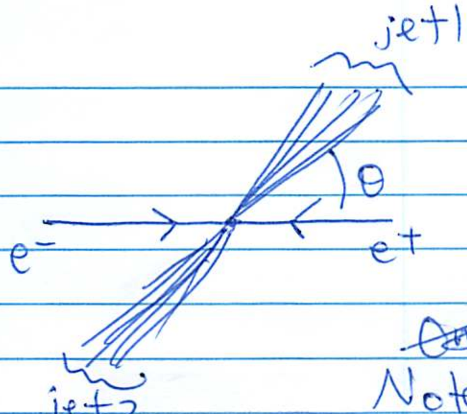
$R = 10/3$, for center-of-mass collision energies a bit higher than twice the charm quark mass ($m_c \simeq 1.3$ GeV). This simple picture and calculation agrees beautifully with experiment, as ~~a~~ a plot in the textbook shows.

Okay, so this R measurement is evidence for both fractional charges in the quark model and for the existence of color. Don't worry, we'll find a lot more evidence for both. One thing I said and you let me get away with was that quarks are spin-½ particles. How do we know? Well, let's go back to the differential cross-section for $e^+e^- \to \mu^+\mu^-$:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2E_{cm}^2}(1+\cos^2\theta)$$

If we translated this to quarks, the overall coefficient would be affected by quark charges and colour, but the shape, the $1+\cos^2\theta$, would remain. This shape, which is peaked for $\theta = 0$ or π, is indicative of the final state particles being fermions, spin-½.

So, if we can measure the angular distribution of the final state quarks, then we would have evidence of their spin. For reasons we will discuss over the next several weeks, hadrons produced in $e^+e^- \to$ hadrons are not just uniformly distributed througout your experiment. ~~At~~ At high energies, they form collimated streams of ~~p~~ hadrons, called jets. These jets are a manifestation of the underlying quarks. That is, in an $e^+e^- \to$ hadrons event, we will observe something like:



where the two jets are composed of numerous hadrons (pions, protons, etc.)

~~One can measure the θ~~

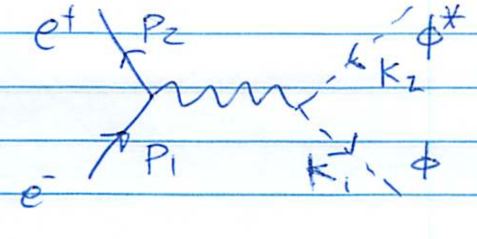
Note that dominantly, by energy momentum conservation, two back-to-back jets will be produced. Then, we have the relationship between cross sections:

$$\sigma(e^+e^- \to \text{hadrons}) \cong \sigma(e^+e^- \to q\bar{q}) \cong \sigma(e^+e^- \to 2\,\text{jets})$$

We can then measure the scattering angle θ that the jets make with the electron beam axis, and compare to the $1+\cos^2\theta$ expectation. The textbook shows a plot from the ALEPH experiment that demonstrates beautiful agreement.

Therefore, just from thinking about the consequences of the quark model for $e^+e^- \to$ hadrons, we have evidence for both colour and the spin-1/2 nature of quarks. Cool!

If you think that I was a bit fast with the spin-1/2 argument, good. It is <u>possible</u> that the $1+\cos^2\theta$ shape is always there, regardless of quark spin. Well, we can test this in the simple model where quarks are spin-0 scalars; let's call them ϕ. In this case, we must evaluate the Feynman diagram:



You'll calculate this in the homework.