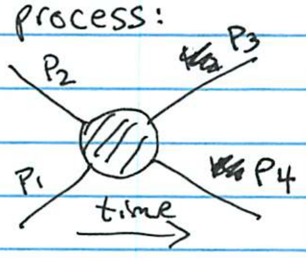
- Introduce jets from last lecture

The Parton Model Lecture 12

Last week, we discussed the process $e^+e^- \to$ hadrons and how that could be related to the calculation of the scattering process $e^+e^- \to q\bar{q}$, if you interpret the cross section appropriately. The appropriate way to interpret the cross section we argued was inclusively: if you think of the process $e^+e^- \to q\bar{q}$ inclusively (that is, anything else can happen in the final state), then that is a good approximation for the cross section of $e^+e^- \to$ hadrons. This then provided significant tests of predictions of the quark model; namely, fractional electric charges of quarks and quarks as spin-1/2 fermions. Additionally, the process $e^+e^- \to$ hadrons provided evidence for the existence of colour, of which each quark can carry one of three colours.

This week, we will discuss other experimental probes of quarks and will find that they have more strange properties. This will set up our introduction of QCD after spring break that will provide the complete fundamental description of the phenomena we are discussing. But I'm getting ahead of myself.

To frame the discussion of this lecture, I want to start by discussing the useful kinematic variables to express the experiments we will study. The fundamental scattering process is $2 \to 2$, like $e^+e^- \to q\bar{q}$, so we want useful variables to express this. Let's consider the scattering process:



with initial momenta p_1, p_2 and final momenta p_3, p_4. Note that momentum conservation imposes:

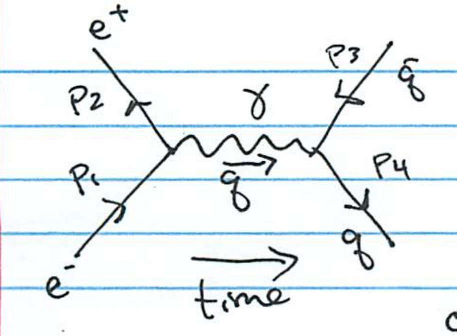
$$p_1+p_2=p_3+p_4$$

The description of the scattering process is Lorentz invariant. As such, it can only depend on particle masses (m_1, m_2, m_3, m_4) and Lorentz invariant four-vector dot products. Naïvely, there are 6 dot products that can be formed, but momentum conservation relates pairs of them:

$$s \equiv (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t \equiv (p_1+p_{\cancel{3}4})^2 = (p_2+p_3)^2$$
$$u \equiv (p_1+p_3)^2 = (p_2+p_4)^2$$

Here, I have introduced the Mandelstam variables s, t, and u named after Stanley Mandelstam (who just died very recently). Mandelstam is famous for many things in particle and theoretical physics. One thing he is famous for is proving that the exotic theory called "N=4 Supersymmetric Yang-Mills" exhibits a larger spacetime symmetry than just Lorentz transformations. This theory is actually invariant under conformal transformations; all possible transformations that maintain relative angles. As such, its interactions are extremely highly constrained, more than the interactions of the Standard Model.

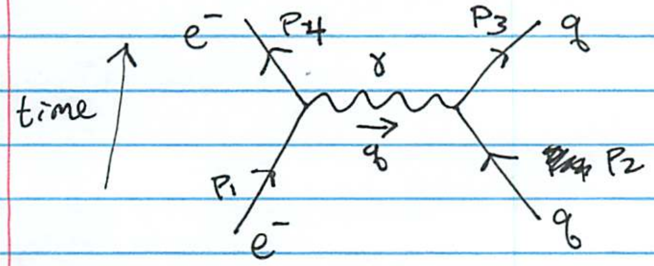
The utility of the Mandelstam variables is that they express different momentum exchanges between the initial and final state. For example, for the process $e^+e^- \to q\bar{q}$, recall that the momentum of the intermediate photon was:



$q = p_1 + p_2$, and so $q^2 = (p_1+p_2)^2 = s$.

This is therefore called an "s-channel" process. In the center-of-mass frame, $s = E_{cm}^2$.

The t-channel process is just a different time-ordering of the same scattering. The invariant mass of the intermediate photon is t if:

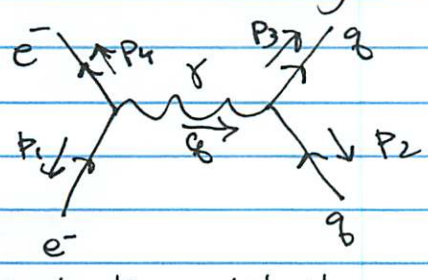


Note that I have relabeled the particle momenta so that p_1 and p_2 are always the initial momentum.

Then, the momentum flowing through the photon is

$$(p_1 - p_4)^2 = q^2,$$ which isn't quite t. We aren't

treating the initial and final momentum on the same footing: initial momentum flows into the interaction while final momentaum flows out. If instead we consider all momentum flowing out of the process as



then $q^2 = (p_1 + p_4)^2 = t$.

Note that this didn't matter for s-channel processes, because we always summed together two initial or two final state momenta. Note that, nicely, with this convention the sum of all momenta is 0

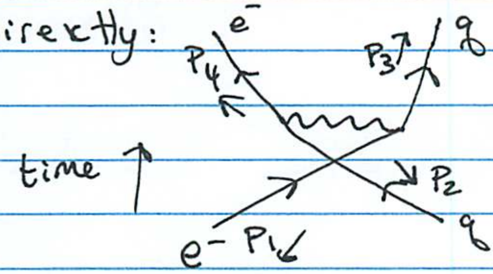
$$p_1 + p_2 + p_3 + p_4 = 0.$$

Specializing to massless particles and going to the center-of-mass frame, we can write:

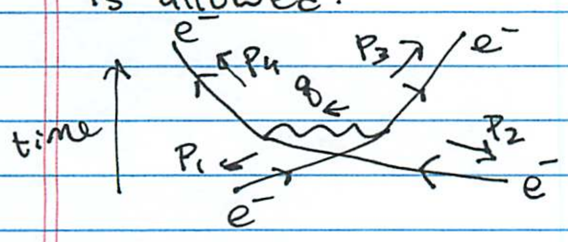
$$p_1 = -\frac{E_{cm}}{2}(1,0,0,1) \quad , \quad p_3 = \frac{E_{cm}}{2}(1, -\cos\phi\sin\theta, -\sin\phi\sin\theta, -\cos\theta)$$
$$p_2 = -\frac{E_{cm}}{2}(1,0,0,-1) \quad , \quad p_4 = \frac{E_{cm}}{2}(1, \cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

and so $(p_1+p_4)^2 = 2p_1\cdot p_4 = -\frac{E_{cm}^2}{2}(1-\cos\theta)$.

The u-channel would be yet a further different time ordering, but happens to not exist for electron-quark scattering. The reason for this is because electromagnetism does not allow for electrons to turn into quarks directly:



is not allowed. It is allowed when scattering identical particles, like $e^-e^- \to e^-e^-$ scattering. Then, the following diagram is allowed:



Then, the momentum q flowing through the photon is:

$$u = q^2 = (p_1+p_3)^2 = -\frac{E_{cm}^2}{2}(1+\cos\theta) \text{, in the center-of-mass frame.}$$

For all massless particle scattering, note that we have the relationship:

$$s+t+u=0$$

which is a Lorentz-invariant constraint.

That is, for $2 \to 2$ scattering there are only two independent phase space variables: the center-of-mass collision energy and the scattering angle.

In the Mandelstam variables, we can rewrite the expression for the cross section of $e^+e^- \to q\bar{q}$ that we derived last week. For one flavour i of quarks with electric charge Q_i, we found

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to q_i\bar{q}_i) = \frac{3}{2}\frac{\pi\alpha^2}{E_{cm}^2}(1+\cos^2\theta)\, Q_i^2$$

Note that $E_{cm}^2 = s$ and $t-u = E_{cm}^2\cos\theta = s\cos\theta$.
Therefore,

$$\cos\theta = \frac{t-u}{s} = \frac{t-(-s-t)}{s} = \frac{2t}{s}+1$$

Then $1+\cos^2\theta = \frac{4}{2E_{cm}^4}(u^2+t^2) = \frac{2}{E_{cm}^4}(u^2+t^2) = \frac{2(u^2+t^2)}{s^2}$.

The differential cross section can be written in the invariant way by noting that

$$dt = \frac{s}{2}\, d\cos\theta \Rightarrow d\cos\theta = \frac{2}{s}\, dt.$$

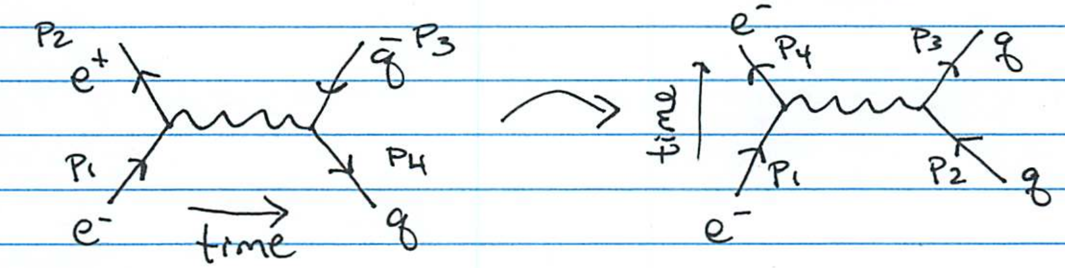
Then, we have,

$$\frac{d\sigma(e^+e^- \to q_i\bar{q}_i)}{d\cos\theta} = \frac{d\sigma(e^+e^- \to q_i\bar{q}_i)}{dt}\cdot\frac{dt}{d\cos\theta}$$ or that

$$\frac{d\sigma(e^+e^- \to q_i\bar{q}_i)}{dt} = \frac{2}{s}\cdot\frac{3}{2}\frac{\pi\alpha^2}{s}Q_i^2\,\frac{2}{s^2}(u^2+t^2)$$

$$= \frac{6\pi\alpha^2 Q_i^2}{s^2}\,\frac{u^2+t^2}{s^2}$$

These relationships and this formalism is exceptionally powerful. We can simply write down the cross section for the process $e^- q \to e^- q$. In terms of diagrams, let's see what we are doing:



In going from the Feynman diagram for $e^+e^- \to q\bar{q}$ to $e^- q \to e^- q$, we made the exchanges of momenta:

$$p_1 \to p_1, \; p_2 \to p_4, \; p_3 \to p_2, \; p_4 \to p_3.$$

In terms of the Mandelstam variables, this is:

$$s = (p_1 + p_2)^2 \longrightarrow (p_1 + p_4)^2 = t$$
$$t = (p_1 + p_4)^2 \longrightarrow (p_1 + p_3)^2 = u$$
$$u = (p_1 + p_3)^2 \longrightarrow (p_1 + p_2)^2 = s$$

Extremely importantly, the value of the Feynman diagram cannot depend on the direction of time, essentially because its value is Lorentz invariant. Therefore, to determine the cross-section for the process $e^- q \to e^- q$, all we need to do is make the above replacements for s, t, and u!

Making these replacements, we find:

$$\frac{d\sigma(e^- q \to e^- q)}{dt} = \frac{2\pi\alpha^2 Q_i^2}{s^2} \, \frac{s^2 + u^2}{t^2}.$$

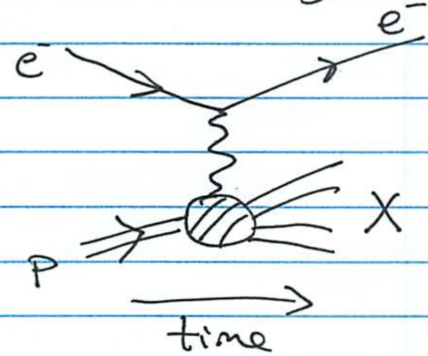
Note that some things changed, somethings did not. First, we dropped the factor of 3 from colour because a quark is in the initial state, so we average over initial state colours (because we can't prepare a "pure" colour quark). Thus, we divide by 3. Next, the overall factor of $1/s^2$ does not change. This comes from the factor of $1/E_A \cdot 1/E_B$ in the cross section that is independent of the matrix element. In the rest of the expression, we replaced s, t, and u appropriately.

That we can so easily determine the cross section for processes with the same underlying Feynman diagrams by replacing s, t, and u, is called "crossing symmetry." We say that the process $e^+e^- \to q\bar{q}$ is just found by crossing $e^-q \to e^-q$ (and vice versa).

What did all of this get us? Well, we have a new way to test the quark model. If we are able to collide electrons on quarks, then we can test the hypothesis that quarks are point particles; that is, they have no spatial extent. If quarks are point particles, then they should look the same, no matter what energy we scatter with them. That is, they should look the same regardless of the wavelength that we probe them with. So, how do we test this?

Well, we need to figure out how to get a sample of quarks with which we can scatter electrons. I don't know how to just get a blob of quarks, but the quark model helps us out: we can scatter electrons off of protons. At sufficiently high energies, the electrons should probe its constituent quarks, according to the quark model. Both electrons and quarks are electrically charged and so can exchange photons in their scattering.

So, the process we are considering is the following (in schematic Feynman-like diagrams):



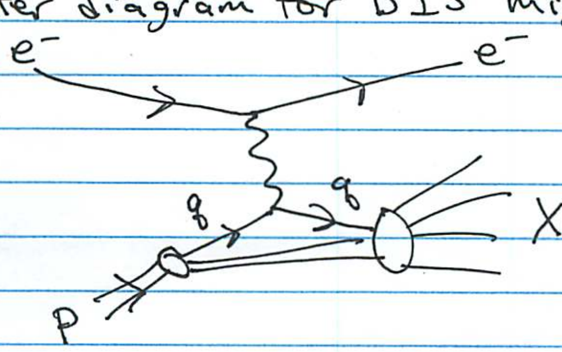
An electron is scattered off of a proton and the final configuration consists of an electron and X, which is some collection of hadrons.

We want to be completely inclusive on the final state (for reasons that will be clear shortly) and so the process we consider is:

$$e^- p \to e^- + X \text{, where } X \text{ is anything.}$$

If the energy of the electron changes in this interaction, then the collision is inelastic, and the proton explodes into other particles. This inelastic process is called deeply inelastic scattering, or DIS.

In DIS, the photon is not interacting with the proton, but rather with its constiuents, the quarks. A better diagram for DIS might be:



Let's break this diagram apart and understand its pieces to make predictions in this model.

By the way, the model in which quarks are the "parts" of a proton and therefore govern high-energy interactions is called the "parton model", introduced by Richard Feynman.

First, let's go to the center-of-mass frame for simplicity, though the analysis will be completely Lorentz invariant. ~~We can draw a scattering picture of this interaction as~~ Let the momentum of the initial and final electrons be k_e and k_e', respectively, and the momentum of the initial proton be P. Then, the momentum flowing through the photon is:

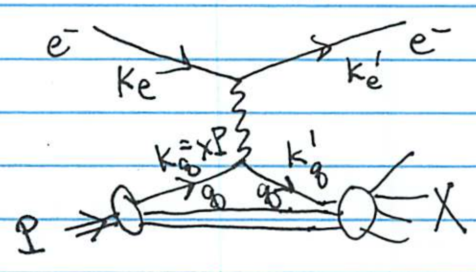
$q = k_e - k_e'$. Note that q is space-like:

$$q^2 = (k_e - k_e')^2 = -2\, k_e \cdot k_e' < 0, \quad \text{for } k_e^2 = (k_e')^2 = 0.$$

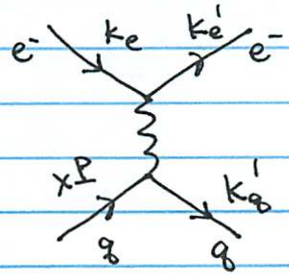
Often, we will denote $-q^2 \equiv Q^2$, which repesents the momentum transferred from the electron to the proton. In the parton model picture, the electron actually interacts with an individual quark. What is the momentum of the quark? Well, for a high-energy proton, the quark will just carry a fraction x of the total proton momentum. Let's call the momentum of the initial quark k_q and k_q'. Then,

$$k_q = x P \quad \text{and} \quad k_e + k_q = k_e' + k_q'.$$

Then, the picture we have is:



Now, with these kinematics, we need to calculate some things. First, there is the hard electron-quark scattering. We know the differential cross section for this. It is:



$$\Rightarrow \quad \frac{d\sigma}{d\hat{t}} = \frac{\pi 2 \alpha^2 Q_i^2}{\hat{s}^2} \; \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$

Here, we introduce $\hat{s}$, $\hat{t}$, and $\hat{u}$ which are the Mandelstam variables fore the interacting partons. Note thate:

$$\hat{s} = (k_e + xP)^2 = 2x\, k_e \cdot P = xs,$$ where s is the Mandelstam variable for e-p scattering.

Similarly,
$$\hat{t} = (k_e - k_e')^2 = q^2 = -Q^2.$$

To define $\hat{u}$, it is useful to define y:

$$y = \frac{2P\cdot(k_e - k_e')}{2P\cdot k_e} = 1 + \frac{-2xP\cdot k_e'}{2xP\cdot k_e} = 1 + \frac{\hat{u}}{\hat{s}}$$

Importantly, note that y is observable: it only involves proton and electron momenta. Then,

$$\hat{u} = \hat{s}(y-1) \quad \text{or} \quad \hat{u}^2 = \hat{s}^2(1-y)^2 = x^2(1-y)^2 s^2.$$

Substituting these expressions in, we can write the differential cross section as:

$$\frac{d\sigma}{dQ^2}(e^-q \to e^-q) = 2\pi\alpha^2 Q_i^2 \cdot \frac{1+(1-y)^2}{Q^4}.$$

Okay, getting close. This isn't the whole story; this is just the subprocess $e^-q \to e^-q$ scattering in e-p collisions.

We need to get the quark out of the proton in the first place. To model this, note that the quark from a high energy proton will have a momentum fraction x of the total proton momentum. In the parton model, this fraction x for a quark q has a probability distribution $f_q(x)$, which is called a parton distribution function, or pdf. The probability of extracting a quark parton from a proton with momentum fraction in $[x, x+dx]$ is

$$P_q([x, x+dx]) = f_q(x)\, dx.$$

Extremely importantly, this is independent of Q. That is, quarks are point particles: regardless of the energy/wavelength with which they are probed, they look the same/have the same pdf.

Then, the cross section differential in x and Q^2 for a quark q is

$$\frac{d\sigma}{dx\,dQ^2} = 2\pi\alpha^2 Q_q^2 f_q(x) \frac{1+(1-y)^2}{Q^4}$$

Note that $y = 1 + \frac{\hat{u}}{\hat{s}} = 1 + \frac{-\hat{s}-\hat{t}}{\hat{s}} = \frac{Q^2}{xs}$ or that, $Q^2 = xys$.

Changing variables to x and y, the differential cross section is: $\frac{d\sigma}{dx\,dy} = 2\pi\alpha^2 s \cdot x\, Q_q^2 f_q(x) \cdot \frac{1+(1-y)^2}{Q^4}$

The quark/parton model predicts multiple quarks in the proton, so we need to sum over all of them to get the complete cross-section:

$$\frac{d\sigma}{dx\,dy} = 2\pi\alpha^2 s \left[\sum_q x\, Q_q^2 f_q(x)\right] \frac{1+(1-y)^2}{Q^4} = 2\pi\alpha^2 s \cdot F_2 \cdot \frac{1+(1-y)^2}{Q^4}$$

Here, F_2 is called a form factor, which in the parton model, is independent of Q.

The final thing to note is that x is actually observable. The beautiful thing about the parton model is that the $e^-q \to e^-q$ subprocess is actually elastic scattering! Therefore, the final state quark is on-shell:

$$k_q'^2 = 0 = (q + xP)^2 = q^2 + 2xq\cdot P \Rightarrow x = \frac{Q^2}{2q\cdot P}$$

So, everything in this expression is an observable!

Again, the ~~quark~~ parton model predicts that quarks are pointlike, and so we can test that F is indeed independent of Q. In the late 1960's and early 1970's experiments at SLAC and elsewhere demonstrated that the form factor was independent of Q, evidence for point-like quark. This feature of the parton model is called Bjorken scaling, after James "BJ" Bjorken.

Today, we know that Bjorken scaling is only approximate, but the deviations are exactly predicted in QCD.