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The Gluon   Lecture 13

In our study thus far of attempting to understand the quarks and their interactions we have learned a huge amount:

- Particles composed of quarks (hadrons) arrange themselves into irreducible representations of flavour symmetry groups (i.e., isospin)
- There seem to be three "colours" of ~~quar~~ each flavour of quark, which is necessary to account for the measured value of R (the ratio of $e^+e^- \to$ hadrons cross section to $e^+e^- \to \mu^+\mu^-$)
- The parton model predicts that quarks are point particles; their description is independent of the energy/wavelength that probes them.

While these observations are extremely informative, they also seem to raise more questions:

- What is the force that binds the quarks together in hadrons?
- What exactly is colour? What are the transformations that relate the three colours?
- Related to this, if colour is conserved, what is the symmetry?

In this lecture, we will posit part of a solution to these problems, and after spring break, we will introduce <u>the</u> theory that answers <u>all</u> of them: Quantum Chromodynamics.

Our goal in this lecture will be pretty constrained: we will just attempt at making predictions for what the force carrier that binds quarks together in hadrons. This also happens to be my favorite particle: the gluon.

What might this "gluon" be? Well, if it is responsible for binding quarks, then we might postulate that it is something like a photon. Electromagnetism, through its force carrier the photon, is responsible for binding the proton and the electron into hydrogen. So, it is feasible that the gluon is a spin-1 particle and massless, just like the photon. Okay, so let's assume that the gluon is spin-1 and massless and see what the predictions are.

Additionally, the force that the gluon carries <u>cannot</u> be electromagnetism; that is, the gluon is <u>not</u> the photon. Perhaps this is obvious, but it is a very important point. Within the quark model, the argument for this is simple. There exist bound states of quarks for which all quarks have the <u>same</u> electric charge; for example, the Ω^- baryon is the bound state of three strange quarks. It is not possible for three particles of the same electric charge to form a bound state in electromagnetism. Additionally, this means that particles like the electron or muon (collectively called "leptons") do not feel the force carried by the gluon.

So, we now have a model for the <u>strongforce</u>, the force that binds hadrons together. It is carried by the gluon, which only talks directly to the quarks, and <u>not</u> the leptons. What predictions does this simple model make?

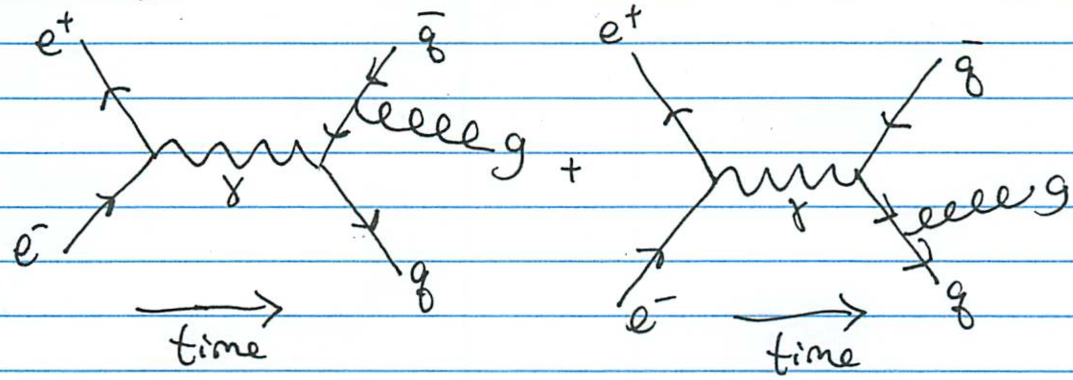
Let's go back to our good friend e^+e^- collisions, in which we first found evidence for quarks. Can we "see" a gluon in e^+e^- collisions?

Because gluons do not talk directly to electrons or positrons, we cannot just produce gluons from e^+e^- collisions. Gluons can, however, be radiated from quarks, just like photons can be radiated from accelerating charges (like electrons, for example). So, the simplest process in which gluons can be produced in e^+e^- collisions is:

$$e^+e^- \to q\bar{q}g$$

We will calculate the Feynman diagram and the cross section for this process.

By the rules of Feynman diagrams, we need to sum over all possible diagrams we can draw, consistent with the interactions that we define. For the process $e^+e^- \to q\bar{q}g$, there are two diagrams that we must sum together:



That is, the final-state gluon could have been emitted off of either the final state quark or anti-quark, and there is no measurement we can perform to distinguish them. Here, we distinguish the photon and the gluon (both spin-1 particles) by their symbol: the photon is a wavy line, while the gluon is a curly line (like a spring!).

What do we need to know to evaluate this diagram? Well, we know how to evaluate just the stuff involving the photon from our analysis last week. There are two truly new things from just $e^+e^- \to q\bar{q}$ scattering: what the wave function of an external gluon is and what an intermediate fermion (quark) propagator is. First, the external gluon wave function.

Because we are assuming that the gluon is a spin-1 massless particle, its wave function should satisfy the <u>same</u> equation as the photon. Additionally, this wave equation must be the Klein-Gordon equation for the gluon to be an on-shell, external, relativistic particle. I will just provide motivation for the external wave function here, not a complete, detailed derivation.

In electromagnetism, the Lagrangian can be written as a function of the field strength tensor $F_{\mu\nu}$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \text{ where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and A_μ is called the four-vector potential. The equation of motion that follows from this Lagrangian (varying with respect to A_μ) is:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu \equiv \partial^2 A^\nu - \partial^\nu \partial \cdot A = 0$$

This corresponds to two of Maxwell's equations: Gauss's Law and Ampère's Law. (The two other equations follow from something called the Bianchi identity.) This isn't quite enough, though. Note that the equation of motion is <u>not</u> the Klein-Gordon equation. As written in terms of a vector potential A_μ, electromagnetism exhibits a gauge invariance: I am allowed to change the vector potential A_μ by a derivative term and no observable (that is, no $\vec{E}$ or $\vec{B}$ fields) change. Under the transformation

$A_\mu \to A_\mu + \partial_\mu \Lambda$, for some function Λ, note that the field strength is unchanged:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) = F_{\mu\nu}.$$

We can exploit this gauge symmetry to enforce that

$\partial \cdot A = 0$ and so $\partial^2 A_\mu = 0$, which is

the Klein-Gordon equation. The constraint that

$\partial \cdot A = 0$ is called the Lorentz gauge.

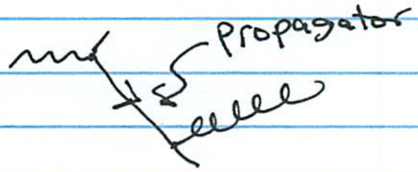
We can then solve the Klein Gordon equation in Lorentz gauge to find:

$$A_\mu = \varepsilon_\mu e^{-ip\cdot x}, \quad \text{with } p^2 = 0 \text{ and } p\cdot\varepsilon = 0 = \varepsilon\cdot\varepsilon$$

ε_μ is called the polarization vector and note that the conditions $\varepsilon\cdot\varepsilon = p\cdot\varepsilon = 0$ imply that there are two independent components. These can be identified with the two helicities of the photon, $\varepsilon_{L\mu}$ and $\varepsilon_{R\mu}$. The polarization vector ε_μ specifies the external wavefunction of the photon's spin.

Okay, because we postulated that the gluon was spin-1 and massless, just like the photon, we have the same polarization vector and constraints as for a photon.

Okay, now onto the intermediate quark propagator. The way that we can think about the quark propagator (or any propagator) is as the Green's function for the appropriate equation of motion. Let's see what I mean in more detail. The quark propagator occurs in the part of the diagram like:



Vertices, where particles interact, correspond to particles at the same spatial position (just like the nodes in a circuit diagram). So, we think of the propagator as representing a quark produced, say, at the spatial point that corresponds to the vertex with the photon. It is then annihilated at the vertex with the gluon. That is, the propagator for the quark is the solution to the Dirac equation with a source at a single point. Let's call this solution $G(x)$. Then

$$i\gamma\cdot\partial\, G(x) = i\delta(x),$$

for a massless quark propagator. $G(x)$ is indeed a Green's function. To solve this, we can Fourier transform, which then turns into:

$$\gamma\cdot p\,\tilde{G}(p) = i \quad \text{or that} \quad \tilde{G}(p) = \frac{i}{\gamma\cdot p}.$$

Here $\tilde{G}(p)$ is the Fourier transform of $G(x)$. $\tilde{G}(p)$ is the propagator for a massless quark.

This is still slightly weird because $\gamma\cdot p$ is a matrix, and we have to invert it. To do this, we can use the anti-commutation relation of the γ matrices:

$$\frac{1}{\gamma\cdot p} = \frac{\gamma\cdot p}{(\gamma\cdot p)(\gamma\cdot p)} = \frac{\gamma\cdot p}{p_\mu p_\nu \gamma^\mu \gamma^\nu} = \frac{\gamma\cdot p}{\frac{1}{2}p_\mu p_\nu \{\gamma^\mu,\gamma^\nu\}} = \frac{\gamma\cdot p}{p^2}$$

Thus, we can write the propagator for a massless quark as:

$$\tilde{G}(p) = \frac{i\gamma\cdot p}{p^2}$$

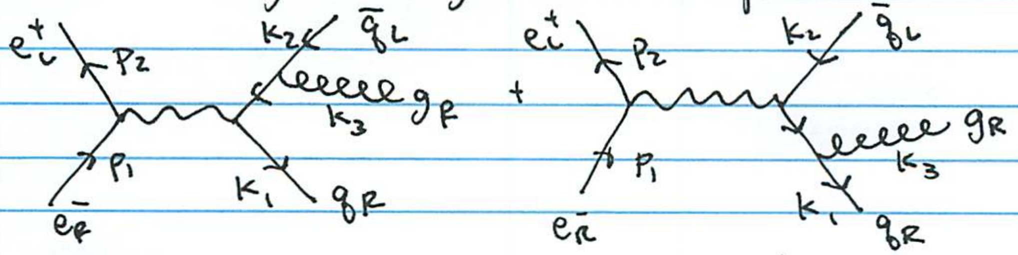
For two-component spinors of definite helicity, the corresponding propagators are:

$$\tilde{G}_R(p) = \frac{i\bar{\sigma}\cdot p}{p^2}, \quad \tilde{G}_L(p) = \frac{i\sigma\cdot p}{p^2}$$

Okay! We're now ready to evaluate the Feynman diagrams. As we did last week, we will evaluate the Feynman diagrams for explicit helicity configurations. We'll do this explicitly for one helicity configuration, and then I will quote the result for the others. Let's consider the process:

$$e^-_R e^+_L \to q_R \bar{q}_L g_R$$

where I will orient all momenta outward. (This is different from what we have done before, but will make the notation simpler.) That is, the sum of momentum of all particles is 0. The Feynman diagrams for this process are:



$$= v_R^\dagger(p_2)\bar{\sigma}^\mu e\, u_L(p_1)\frac{1}{(p_1+p_2)^2} u_R^\dagger(k_1)\, e\,\sigma_\mu \frac{\bar{\sigma}\cdot(-k_2-k_3)}{2k_2\cdot k_3}\, g\,\sigma\cdot\varepsilon_R(k_3)\, v_L(k_2)$$

$$+ v_R^\dagger(p_2)\bar{\sigma}^\mu e\, u_L(p_1)\frac{1}{(p_1+p_2)^2} u_R^\dagger(k_1)\, g\,\sigma\cdot\varepsilon_R(k_3)\frac{\bar{\sigma}\cdot(k_1+k_3)}{2k_1\cdot k_3}\, e\,\sigma_\mu v_L(k_2)$$

I'll need to describe a few things about these expressions.

First, the left side of the diagrams is just the electromagnetic interactions of electrons and positrons, which is the same for both diagrams. Then, on the right of these diagrams, we have denoted the strength of the gluon coupling by g and included the appropriate quark propagators. Note that the first diagram has a relative sign in the quark propagator with respect to the second. This is because the momentum flows against the quark arrow in the first diagram, and with it in the second. Conservation of energy and momentum for this process is

$$p_1+p_2+k_1+k_2+k_3=0, \text{ with all momentum flowing } \underline{\text{out}}.$$

Using the Fierz identity that you studied in the homework, we can simplify this expression further:

$$\mathcal{M}(e_R^- e_L^+ \to q_R \bar{q}_L g_R) = -\frac{2e^2 g}{(p_1+p_2)^2} u_R^\dagger(k_1)\, u_L(p_1)\, v_R^\dagger(p_2) \frac{\bar{\sigma}\cdot(k_2+k_3)}{2k_2\cdot k_3}\, \sigma\cdot\varepsilon_R(k_3)\, v_L(k_2)$$

$$+ \frac{2e^2 g}{(p_1+p_2)^2} u_R^\dagger(k_1)\, \sigma\cdot\varepsilon_R(k_3) \frac{\bar{\sigma}\cdot(k_1+k_3)}{2k_1\cdot k_3}\, u_L(p_1)\, v_R^\dagger(p_2)\, v_L(k_2).$$

Further simplifications can be made by exploiting spinor identities. For example, we can express

$$\bar{\sigma}\cdot(k_1+k_3) = u_R(k_1)\,u_R^\dagger(k_1) + u_R(k_3)\,u_R^\dagger(k_3).$$

Similarly, we can express $2k_1\cdot k_3$ as spinor products using the Fierz identity:

$$2k_1\cdot k_3 = \frac{1}{2}\left(u_R^\dagger(k_1)\,\sigma^\mu v_L(k_1)\right)\left(v_R^\dagger(k_3)\,\bar{\sigma}_\mu u_L(k_3)\right) = u_R^\dagger(k_1)\,u_L(k_3)\,v_R^\dagger(k_3)\,v_L(k_1).$$

You will verify this in the homework. We almost have everything written as spinor products; the only thing that remains is the polarization of the gluon, $\varepsilon_R(k_3)$.

Right now, I will just postulate the form for the polarization $\varepsilon_R^\mu(k_3)$ in terms of spinors. You'll verify in homework that it satisfies the properties we expect (right-handed helicity, etc.). I claim that:

$$\varepsilon_R^\mu(k_3) = \frac{1}{\sqrt{2}} \frac{u_L^\dagger(r)\,\bar{\sigma}^\mu u_L(k_3)}{u_L^\dagger(r)\,u_R(k_3)}$$

Here, r is an arbitrary momentum four-vector that represents the gauge freedom of the vector potential. Note that:

$$k_3 \cdot \varepsilon_R(k_3) = \frac{1}{\sqrt{2}} \frac{u_L^\dagger(r)\,\bar{\sigma}\cdot k_3\, u_L(k_3)}{u_L^\dagger(r)\,u_R(k_3)} = 0,$$

by the Dirac equation, $\bar{\sigma}\cdot k_3\, u_R(k_3) = 0$. Because of the freedom to choose any r, let's take $r = k_2$. The matrix element is independent of the choice of r.

With this choice, let's evaluate the σ-product that appears in the first diagram:

$$\bar{\sigma}\cdot(k_2+k_3)\,\sigma\cdot\varepsilon_R(k_3)\,v_R(k_2) = \left(u_R(k_2)u_R^\dagger(k_2) + u_R(k_3)u_R^\dagger(k_3)\right)\frac{1}{\sqrt{2}}\frac{u_L(k_3)\,u_L^\dagger(k_2)}{u_L^\dagger(k_2)\,u_R(k_3)}\,v_L(k_2)$$

$$= 0!$$

This is awesome! Let's see how this works a bit more slowly. Note that:

$$u_R(k_3)u_R^\dagger(k_3)\,u_L(k_3) = \bar{\sigma}\cdot k_3\, u_L(k_3) = 0,$$

by the Dirac equation. Also, note that

$$u_L^\dagger(k_2) = v_R^\dagger(k_2) \text{ and } v_R^\dagger(k_2)\,v_L(k_2) = 0.$$

To see this, note that: $\bar{\sigma}\cdot k_2\, v_L(k_2) = 0 = v_R(k_2)\,v_R^\dagger(k_2)\,v_L(k_2)$, and so $v_R^\dagger(k_2)\,v_L(k_2) = 0$. So, one diagram is just 0 with an appropriate choice of gauge!

The matrix element is then:

$$\mathcal{M}(e_R^- e_L^+ \to q_R \bar{q}_L g_R)$$

$$= \frac{\sqrt{2}e^2 g}{(p_1+p_2)^2} \frac{u_R^\dagger(k_1) u_L(k_3) u_L^\dagger(k_2) \bar{\sigma}\cdot(k_1+k_3) u_L(p_1) v_R^\dagger(p_2) v_L(k_2)}{u_L^\dagger(k_2) u_R(k_3)\, u_R^\dagger(k_1) u_L(k_3) v_R^\dagger(k_3) v_L(k_1)}$$

$$= -\frac{\sqrt{2}e^2 g}{(p_1+p_2)^2} \frac{u_L^\dagger(k_2)\bar{\sigma}\cdot p_2\, u_L(p_1)\, v_R^\dagger(p_2) v_L(k_2)}{u_L^\dagger(k_2) u_R(k_3) v_R^\dagger(k_3) v_L(k_1)}$$

$$= -\frac{\sqrt{2}e^2 g}{v_R^\dagger(p_1) v_L(p_2)} \frac{u_L^\dagger(k_2) u_R(p_2) v_R^\dagger(p_2) v_L(k_2)}{u_L^\dagger(k_2) u_R(k_3) v_R^\dagger(k_3) v_L(k_1)}$$

In going from the first to second line, I used momentum conservation to re-write $k_1+k_2 = -p_1-p_2-k_3$, and enforced the Dirac equation. The square of this matrix element is:

$$|\mathcal{M}(e_R^- e_L^+ \to q_R \bar{q}_L g_R)|^2 = e^4 g^2 \frac{(p_2\cdot k_2)(p_2\cdot k_2)}{(p_1\cdot p_2)(k_2\cdot k_3)(k_1\cdot k_3)}$$

The other matrix elements squared are found by simply permuting indices. For example,

$$|\mathcal{M}(e_R^- e_L^+ \to q_L \bar{q}_R g_R)|^2 = e^4 g^2 \frac{(p_2\cdot k_1)(p_2\cdot k_1)}{(p_1\cdot p_2)(k_2\cdot k_3)(k_1\cdot k_3)}$$

In homework 4, you played with the x_i phase space variables. Recall that

$$x_i = \frac{2Q\cdot k_i}{Q^2}, \text{ for } i=1,2,3. \text{ Then } x_1+x_2+x_3=2,$$

where $Q=(E_{cm},0,0,0)$, in the center-of-mass frame. Using these x_i coordinates, the matrix element squared becomes:

$$|\mathcal{M}(e_R^- e_L^+ \to q_R \bar{q}_L g_R)|^2 = \frac{e^4 g^2}{2E_{cm}^2} \frac{x_2^2 (1+\cos\theta)^2}{(1-x_1)(1-x_2)}$$

Here, $\cos\theta$ is the scattering angle; the angle between the electron-positron beam and the final state anti-quark. For just studying the dynamics of the gluon, we need to integrate over $\cos\theta$.

Note the physics contained in this expression. The (squared) matrix element diverges when either $x_1 \to 1$ or $x_2 \to 1$. Physically, this corresponds to either $k_2 \cdot k_3 \to 0$ or $k_1 \cdot k_3 \to 0$. For two massless four-vectors, this can occur either when $k_3 \to 0$ (the energy of the gluon is small) or when $\vec{k}_3 \parallel \vec{k}_1$, or $\vec{k}_2$, called the "collinear limit". The existence of divergences in the matrix element in the soft and/or collinear limits will have profound physical consequences for observing phenomena of the gluon.

Summing over spins and integrating over $\cos\theta$, we find the cross section differential in x_1, x_2:

$$\frac{d\sigma(e^+e^- \to q\bar{q}g)}{dx_1 dx_2} = \sigma(e^+e^- \to q\bar{q}) \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}.$$

Here, $\alpha_s \equiv \frac{g^2}{4\pi}$ is the strong coupling constant and $C_F = \frac{4}{3}$ is the factor that accounts for the possible different colours of the gluon emitted off of the final state quarks.

One can compare this differential cross section to events in e^+e^- collisions in which three collimated streams of particles (called "jets") are observed, and construct x_1, x_2, x_3 for the jets. This then directly tests the spin-1 nature of the gluon. As shown in the textbook, the agreement of this cross-section with the data is phenomenal. Apparently, the gluon is spin-1!

By the way, the formalism applied in this lecture for calculating matrix elements with two-component spinors is called "spinor helicity" and is wideley used for modern calculations involving complicated Feynman diagrams. In spinor helicity, the two-component spinors are denoted as:

$$u_R(p) = v_L(p) \equiv p\rangle\,, \qquad u_R^\dagger(p) = v_L^\dagger(p) \equiv [p$$

$$u_L^\dagger(p) = v_R^\dagger(p) \equiv \langle p\,, \qquad u_L(p) = v_R(p) \equiv p]$$

With this notation, note that

$$u_L^\dagger(p)\, u_R(k) = \langle pk\rangle \quad \text{and} \quad \left(u_L^\dagger(p)\, u_R(k)\right)^* = [kp].$$

Also, $[pk]\langle kp\rangle = [p\,\bar{\sigma}\cdot k\,p\rangle = \langle k\,\bar{\sigma}\cdot p\,k] = 2k\cdot p.$

In this notation, the matrix element that we calculated is written as:

$$\mathcal{M}(e_R^- e_L^+ \to q_R \bar{q}_L g_R) = -\sqrt{2}\,e^2 g \frac{\langle k_2 p_2\rangle\langle p_2 k_2\rangle}{\langle p_1 p_2\rangle\langle k_2 k_3\rangle\langle k_3 k_1\rangle} = \sqrt{2}\,e^2 g \frac{\langle k_2 p_2\rangle^2}{\langle p_1 p_2\rangle\langle k_1 k_3\rangle\langle k_3 k_2\rangle}$$

which is exceptionally compact! Actually, the spinor helicity formalism enables simple determination of the matrix element, just from some simple, physical requirements! More in the homework...