Non-Abelian Gauge Theories Lecture 14

Before spring break, we came to a few amazing realizations about the force that binds quarks together, called the strong force. First, in our study of the process $e^+e^- \to$ hadrons, we found evidence for three types of each flavour of quark, called colour. Next, in the parton model and through deeply inelastic scattering, we showed that quarks are point-particles, and have no spatial extent. Finally, in studying $e^+e^- \to$ hadrons events in more detail, we found evidence of the particle nature of the force carrier of the strong force, called the gluon. Apparently, the gluon is similar to the photon in that it is massless and has spin-1.

So far, our understanding of the strong force has been piecemeal, with sub-theories or models describing corners of a larger structure (like the quark model or the parton model). This week, we will use this knowledge about the strong force and follow our noses to find the fundamental theory that can universally describe all of these phenomena.

To keep our discussion simple, let's consider just one massless quark, described by a spinor solution to the Dirac equation, ψ. This quark is allowed to have any of three colours, red, green, or blue, which we might denote by an index i: ψ_i. $i=1$ is red, $i=2$ is green, and $i=3$ is blue. Like in our discussion of Isospin, in this coloured world, we say that the three colours represent a symmetry. We can consider any linear combination of the colours and that produces an equally valid description of the physics.

That is, we can transform ψ_i as:

$$\psi_i \to U_{ij}\psi_j \equiv U_{i1}\psi_1 + U_{i2}\psi_2 + U_{i3}\psi_3,$$

for some numbers U_{ij}. Now, we're dealing with quantum mechanics, and so there are constraints on the U_{ij}. As a wavefunction, ψ_i represents the probability amplitude for measuring colour i. Also, we can take ψ_1, ψ_2, and ψ_3 to be an orthogonal colour basis. Therefore, to preserve probability, we must enforce:

$$|U_{i1}|^2 + |U_{i2}|^2 + |U_{i3}|^2 = 1$$

Perhaps more enlightening, we can rotate the colours into one another with a matrix $\mathbb{U}$:

$\psi \to \mathbb{U}\psi$, and conservation of probability is $\mathbb{U}^\dagger \mathbb{U} = \mathbb{1}$.

Here, I have been describing colour in ψ implicitly, and $\mathbb{U}$ is a 3×3 matrix, that is unitary. Further, as is usual, we will restrict to the case where $\det \mathbb{U} = 1$. Then, rotations of colours into one another are implemented by matrices in the group $SU(3)$. Quarks transform in the fundamental (3 dimensional) representation of $SU(3)$.

A general 3×3 complex matrix has 18 possible parameters (9 real and 9 imaginary). The unitary constraint fixes 9 of the parameters (one relation for each entry of the matrix). The $\det \mathbb{U} = 1$ constraint fixes one more entry. Therefore, an arbitrary 3×3 matrix that is in $SU(3)$ has 8 independent parameters. It is convenient to write a generic $SU(3)$

matrix with respect to 8 basis matrices, T^a, where $a = 1, \ldots, 8$. These T^a matrices are Hermitian, $(T^a)^\dagger = T^a$ and we enforce unitarity by exponentiation:

$$\mathbb{U} = e^{i\alpha^a T^a}$$

, where α^a are 8 constants, indexed by a. The matrices T^a define the Lie algebra of SU(3), named after the Norwegian mathematician, Sophus Lie. For SU(3) to form a group, it must be closed. That is, the product of two matrices must also be in the group:

$$\mathbb{U}_1 \mathbb{U}_2 = e^{i\alpha_1^a T^a} e^{i\alpha_2^b T^b} = \mathbb{U}_3 = e^{i\alpha_3^c T^c},$$

for some constants α_3^c, with the same basis matrices, T^c. As you showed in Homework 2, this matrix multiplication can be carefully accomplished by utilization of the BCH formula. The group only closes if the ~~commut~~ commutator of two matrices T^a and T^b is a linear combination of the basis matrices:

$$[T^a, T^b] = i f^{abc} T^c.$$

Here, f^{abc} are called "structure" constants and are just numbers ($\sqrt{2}$, 1, etc.), while the T^a, T^b, and T^c are matrices in the Lie Algebra of SU(3). SU(3) is called a non-Abelian group (named after another Norwegian Mathematician, Niels Abel) because the structure constants are not all 0. That is, the commutation relations are non-trivial.

Okay, let's see how this works. Early in the class, we introduced the Dirac Lagrangian, from which the Dirac equation can be derived. It is:

$$\mathcal{L} = \bar{\psi} i \gamma \cdot \partial \psi$$

, where we assume ψ is massless.

Here, we have implicitly taken the inner product over colours. That is $\bar{\psi}$ and ψ should be thought of as colour vectors and we dot them appropriately:

$$\bar{\psi}\psi \equiv \bar{\psi}^i \psi_i \text{ , where } i=1,2,3 \text{ are the colours.}$$

What happens to the Dirac Lagrangian under a colour rotation? We know how ψ transforms:

$\psi \to \mathbb{U}\psi$, and as its conjugate, $\bar{\psi}$ transforms as:

$\bar{\psi} \to \bar{\psi}\mathbb{U}^\dagger$, that is, we need to transpose and complex conjugate. Then, the Dirac Lagrangian becomes:

$$\mathcal{L} = \bar{\psi} i\gamma\cdot\partial\psi \to \bar{\psi}\mathbb{U}^\dagger i\gamma\cdot\partial\,\mathbb{U}\psi = \bar{\psi} i\gamma\cdot\partial\psi,$$

where we have used that $\mathbb{U}$ is just a constant, unitary matrix. So, the Dirac Lagrangian is invariant under SU(3) transformations!

But does this actually make sense? We have argued for properties of colour transformations from quantum mechanics. Are there requirements from special relativity? The answer is yes, and this will have important consequences for the theory we develop.

So far, we have assumed that the SU(3) colour transformation is identical everywhere in space. We wrote the colour matrix as:

$\mathbb{U} = e^{i\alpha^a T^a}$, for constants α^a, independent of spacetime position.

Do we have to make this assumption? No, and actually special relativity would prefer that we do not.

Consider two people, call them Emmy and Albert, located at opposite sides of the universe from one another. They are studying colour transformations of quarks and each define a basis for colour. Emmy's basis is, say, ψ_i, while Albert's is $\psi_{i'}$. Because there are three colours and probability is conserved, there exists some unitary matrix that relates the two bases:

$$\psi_i = U_{ii'} \psi_{i'}.$$

So, in principle, Emmy and Albert could figure out this unitary transformation and align their colour bases.

However, they are very far apart, and can only exchange information at the speed of light! Therefore, they cannot instantaneously align their colour bases. Well, they didn't have to be on opposite sides of the universe; they could have been in the same room, or even only separated by the radius of a proton. Regardless, they could not instantaneously align their colour bases. So, we should, more naturally, choose a different colour basis at every spacetime point, and then the different bases will be reconciled and related to one another by an object that implements a colour rotation that travels at the speed of light. Spoiling the punchline slightly, this will turn out to be the gluon.

So, instead of considering unitary transformations that are independent of spacetime position, we will consider transformations that are general functions of position:

$$\mathbb{U}(x) = e^{i\alpha^a(x)T^a}$$, where now, the coefficients

$\alpha^a(x)$ depend on position four-vector x. We are still allowed to use the same basis matrices T^a, because we are always describing $SU(3)$ colour.

While this change might seem small, it has profound consequences. Let's look at the Dirac Lagrangian again with this unitary transformation:

$$\bar{\psi} i\gamma\cdot\partial\psi \to \bar{\psi}\mathbb{U}^\dagger i\gamma\cdot\partial\,\mathbb{U}\psi.$$

Now, the derivative doesn't commute with the matrix $\mathbb{U}$. Instead, we find:

$$i\gamma\cdot\partial\,\mathbb{U} = i\Big(i\big(\gamma\cdot\partial\,\alpha^a(x)\big)T^a\,\mathbb{U} + \mathbb{U}\,\gamma\cdot\partial\Big)$$

$$= \mathbb{U}\,i\gamma\cdot\partial - T^a\,\mathbb{U}\,\gamma\cdot\partial\,\alpha^a(x).$$

That is, the Dirac Lagrangian transforms as:

$$\mathcal{L} = \bar{\psi} i\gamma\cdot\partial\psi \to \bar{\psi}\mathbb{U}^\dagger i\gamma\cdot\partial\,\mathbb{U}\psi = \mathcal{L} - \bar{\psi}T^a\left[\gamma\cdot\partial\,\alpha^a(x)\right]\psi,$$

which is no longer invariant! This is just another way of saying that colour bases at different spacetime points are in general different. To restore invariance of the Lagrangian, we need to introduce an object that can rotate colour bases at the speed of light. Recall that we want the Lagrangian to be invariant under colour rotations because we believe that this is a symmetry of nature. As a symmetry, by Noether's theorem, this means that colour is conserved in interactions. This will highly restrict the possible interactions of the quarks (which is a good thing for predictivity!).

So, let's introduce a field A^a_μ that has a transformation that exactly cancels this. That is, we will consider the augmented Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial + g\gamma\cdot A^a T^a)\psi, \text{ where } g \text{ is a number}$$

that will be defined soon. For this to be invariant, the field A^a_μ must transform inhomogeneously:

$$A^a_\mu \to A^a_\mu + \Delta A^a_\mu, \text{ where } \Delta A^a_\mu \text{ is the transformation.}$$

Let's see what this is.

Performing a colour rotation, we have:

$$\bar{\psi}(i\gamma\cdot\partial + g\gamma\cdot A^a T^a)\psi \to \bar{\psi}\mathcal{U}^\dagger(i\gamma\cdot\partial + g\gamma\cdot A^a T^a + g\gamma\cdot\Delta A^a T^a)\mathcal{U}\psi$$

$$= \bar{\psi}i\gamma\cdot\partial\psi - \bar{\psi}\gamma\cdot\partial\alpha^a(x)T^a\psi + \bar{\psi}\mathcal{U}^\dagger(g\gamma\cdot A^a T^a + g\gamma\cdot\Delta A^a T^a)\mathcal{U}\psi$$

For this to be invariant, we then need everything to the right of $\bar{\psi}i\gamma\cdot\partial\psi$ to equal $\bar{\psi}g\gamma\cdot A^a T^a\psi$:

$$-\bar{\psi}\gamma\cdot\partial\alpha^a(x)T^a\psi + \bar{\psi}\mathcal{U}^\dagger(g\gamma\cdot A^a T^a + g\gamma\cdot\Delta A^a T^a)\mathcal{U}\psi = \bar{\psi}g\gamma\cdot A^a T^a\psi$$

or that:

$$-\partial_\mu \alpha^a(x) T^a + g\,\mathbb{U}^\dagger A_\mu^a T^a \mathbb{U} + g\,\mathbb{U}^\dagger \Delta A_\mu^a T^a \mathbb{U} = g A_\mu^a T^a.$$

We can solve for ΔA_μ^a by multiplying by $\mathbb{U}$ on the left and $\mathbb{U}^\dagger$ on the right, using $\mathbb{U}^\dagger \mathbb{U} = \mathbb{1}$:

$$-\mathbb{U}\left(\partial_\mu \alpha^a(x)\right) T^a \mathbb{U}^\dagger + g A_\mu^a T^a + g\,\Delta A_\mu^a T^a = g\,\mathbb{U} A_\mu^a T^a \mathbb{U}^\dagger$$

or that

$$\Delta A_\mu^a T^a = \frac{1}{g}\mathbb{U}\left(\partial_\mu \alpha^a(x)\right) T^a \mathbb{U}^\dagger + \mathbb{U}\left(A_\mu^a T^a - \mathbb{U}^\dagger A_\mu^a T^a \mathbb{U}\right)\mathbb{U}^\dagger$$

An exceptionally nice way to package the field A_μ^a is in a covariant derivative D_μ which is defined as:

$$D_\mu = \partial_\mu - ig A_\mu^a T^a.$$

The transformation law above implies the exceptionally simple transformation of D_μ under a colour rotation:

$D_\mu \to \mathbb{U} D_\mu \mathbb{U}^\dagger$. This is why it is called "covariant".

Then, we can write our new Lagrangian as:

$\mathcal{L} = \bar{\Psi} i\gamma\cdot D \Psi$. This transforms as:

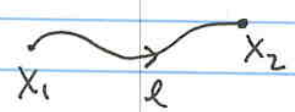
$$\mathcal{L} = \bar{\Psi} i\gamma\cdot D\Psi \to \bar{\Psi}\mathbb{U}^\dagger i\gamma^\mu\left(\mathbb{U} D_\mu \mathbb{U}^\dagger\right)\mathbb{U}\Psi = \bar{\Psi} i\gamma\cdot D\Psi,$$

which is invariant! Awesome!

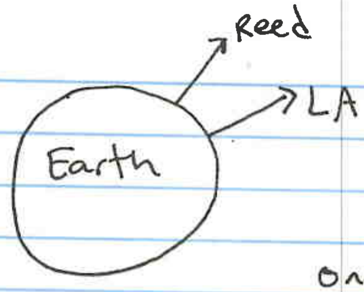
We aren't out of the woods yet. Currently, the field A_μ^a is not allowed to propagate; that is, in the Lagrangian, it has no kinetic energy and therefore no velocity. This is still not complete then, because we don't yet have a way to ~~[illegible]~~ communicate between different spacetime points about what colour bases are being used. Let's work to include this now.

I don't have much time in the remainder of this lecture to justify everything, so I present some physical arguments, and you'll explore more in Homework. The covariant derivative D_μ enables robust and well-defined differentiation when the theory has a symmetry under which fields transform. In fact, the covariant derivative satisfies the Leibniz rule for differentiation, while the regular derivative ∂_μ does not. Therefore, in this theory with SU(3) colour whenever we think "derivative" or "differentiation" we should think covariant derivative, D_μ.

A fundamental quantity that can be formed out of the covariant derivative is the curvature. Effectively, the curvature is a measure of how "badly" two neighboring points' colour bases disagree. How can we define this? Let's consider two neighboring points, x_1 and x_2. We imagine traveling from x_1 to x_2 like so:



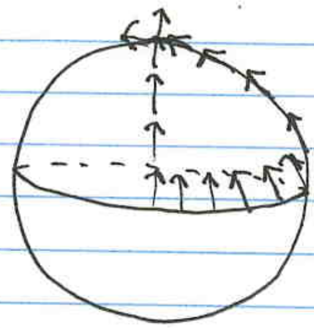
As we travel from x_1 to x_2, we want to measure how much our colour basis is rotated. Then, once at point x_2, we can compare before and after. Actually, this comparison isn't actually well defined. We can't in a well-defined way compare quantities at two separated points. As an illustration of this, consider two people, one here at Reed and the other in Los Angeles. If you ask both of those people to point "up" they will point something like:



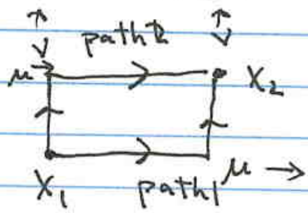
These "up" directions are not the same! It's unclear how one can compare them, and actually one can't. What you can do, though, is compare at the same point!

To measure the curvature, we will travel from x_1 to x_2 on two different paths and compare how much the colour basis has rotated. If there is no rotation, then it shouldn't matter how we got from x_1 to x_2; it is "flat". If there is high curvature, then two paths should be very different!

For those of you who have taken general relativity, the notion of curvature and measuring it by comparing two curves along which a vector is parallel transported should be familiar. The idea is the same here, but with this abstract colour symmetry group. For those of you who haven't taken general relativity, I'll provide a simple picture that illustrates this phenomena. Let's say you are on the equator of the Earth holding an arrow (that represents a vector.). You're supposed to always keep the arrow pointed in the same direction, north, as you travel. Let's take two different paths to the North Pole: just head due north, and first travel along the equator for a bit, then travel due north. Because the Earth is curved, when you reach the north pole your vectors will be at a relative angle to one another! This relative angle is a measure of the curvature of the Earth. If the Earth was flat, the arrows would be pointed in the same direction! An illustration of this is:



Cool! How do we do this for colour SU(3)? Consider traveling to x_2 from x_1 along two paths: in the μ direction and then in the ν direction, and vice-versa. The picture is:



To get to point x_2, we need to translate; that is, we need momentum. We get momentum from D_μ!

On path 1, we get to x_2 by first traveling in μ then ν: $D_\mu D_\nu$. On path 2, we do the opposite: $D_\nu D_\mu$. To compare the two paths, we just take the difference of these derivative products:

$$D_\mu D_\nu - D_\nu D_\mu = [D_\mu, D_\nu] \equiv -ig F^a_{\mu\nu} T^a$$

This commutator is called the curvature tensor in geometry or the Yang-Mills field strength tensor $F^a_{\mu\nu}$. It is named after C.N. Yang and Robert Mills who first constructed the non-Abelian case in the 1950's. $F^a_{\mu\nu}$ is a measure of the curvature of the symmetry group SU(3). Note that if the curvature is small, $F^a_{\mu\nu}$ is small, and the radius of curvature is large. This corresponds to a long wavelength or low energy excitation produced by A^a_μ. By contrast, if the curvature is high, $F^a_{\mu\nu}$ is large and wavelengths are small. Thus, this corresponds to a high energy excitation. This motivates $F^a_{\mu\nu}$ as a measure of the kinetic energy of the field A^a_μ! As a kinetic energy, we just need to square it, and we have the Lagrangian!

Just a couple more details and we're done for today. Using the commutator definition above, we find $F^a_{\mu\nu}$ to be:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Just like the covariant derivative, $F^a_{\mu\nu}$ transforms covariantly under a colour rotation:

$$F^a_{\mu\nu} T^a \to e^{i\alpha^b(x)T^b} F^a_{\mu\nu} T^a e^{-i\alpha^b(x)T^b}$$

$F^a_{\mu\nu}$ has naked Lorentz indices (and so tranforms under boosts and rotations) and is not invariant under colour rotations. However, this can be made invariant by squaring and taking the trace over colour matrices:

$$\mathrm{tr}\left[F^a_{\mu\nu} T^a F^{\mu\nu b} T^b \right]$$ is Lorentz- and colour invariant!

To see colour invariance, note that the trace is cyclic. For three matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}$, their trace is:

$$\mathrm{tr}[\mathbb{A}\mathbb{B}\mathbb{C}] = \mathrm{tr}[\mathbb{B}\mathbb{C}\mathbb{A}] = \mathrm{tr}[\mathbb{C}\mathbb{A}\mathbb{B}]$$

By convention, we can normalize the T^a matrices so that

$\mathrm{tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$, which is called the killing form.

Finally, the gauge-invariant, Lorentz-invariant description of the theory of colour symmetry, is the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi} i \gamma \cdot D \psi.$$

The $-\frac{1}{4}$ comes from matching kinetic energies. The field strength $F^a_{\mu\nu}$ creates gluons, the force carrier that talks to the quarks through the covariant derivative. This theory combines every observation we have discussed about the strong force: spin-$\frac{1}{2}$ point particle quarks, 3 colours, and a force carrier gluon. As it is the quantum theory of colour, it is called Quantum Chromodynamics, or QCD. Extremely importantly, connecting to our original motivation, note that there is no mass in the Lagrangian of QCD for the gluon field A_μ^a. The gluon is necessarily massless, which is a requirement for colour conservation.

If you're interested in the mathematical aspects of today's lecture, it goes under the name of "fibre bundles" and the field A_μ^a is called the connection. A book that discusses this and much more in great detail that I highly recommend is Nakahara's "Geometry, Topology, and Physics".

By the way, general relativity can be formulated in the same language as SU(3) colour. In this way we think of general relativity as the gauge theory of Lorentz transformations. That is, we allow every point in space to have a different Lorentz transformation. This exactly corresponds to diffeomorphisms.