Quantum Chromodynamics Lecture 15

At the very end of last lecture, we wrote down the Lagrangian of QCD, the strong force:

$$\mathcal{L}^{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi} i \gamma \cdot D \psi$$

Here, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is called the field strength tensor and describes the kinetic energy of the gluon field A^a_μ, ψ is a quark field that carries one of three possible colours, and D_μ is the covariant derivative:

$$D_\mu = \partial_\mu - i g A^a_\mu T^a$$

There are a couple of points to discuss about this Lagrangian before continuing discussing consequences. First, the gluon is massless. If the gluon were massive there would be a term in the Lagrangian of the form:

$$\mathcal{L} \supset m^2 A^a_\mu A^{\mu a}$$, as required by the Klein-Gordon equation. However, such a term is forbidden by colour conservation as it is not invariant under a colour transformation. So, the gluon communicates colour at the speed of light, just like the photon communicates E+M at the speed of light.

Next, there are only two degrees of freedom of the gluon field A^a_μ. To see this, Let's focus on $F^a_{\mu\nu}$, and ignore interactions (set $g=0$). Then, we have:

$$F^a_{\mu\nu}\big|_{g=0} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu .$$

Those gluons that are allowed to propagate; i.e., move through space and time, must have a time derivative in the Lagrangian. This ensures that their equation of motion has a time derivative. So, we must have, say, $\mu=0$ corresponding to the time entry as:

$$F^a_{0\nu}\big|_{g=0} = \partial_0 A^a_\nu - \partial_\nu A^a_0$$

Naively, A^a_μ has four spacetime components, for each μ index. However, $\mu=0$ is not allowed to propagate! If $\mu=\nu=0$, then $F^a_{\mu\nu}$ is just zero:

$$F^a_{00} = 0, \text{ and so } A^a_0 \text{ has no time derivative}$$

in the Lagrangian! So, A^a_μ has (at most) only three degrees of freedom.

However, it's worse than that. A^a_μ transforms under colour transformations as:

$$A^a_\mu T^a \to \mathbb{U} A^a_\mu T^a \mathbb{U}^\dagger + \frac{i}{g}\mathbb{U}\left(\partial_\mu \alpha^a(x)\right) T^a \mathbb{U}^\dagger$$

We have the freedom/ability to choose $\alpha^a(x)$ as we please. Doing this fixes a gauge, and can be used to remove another degree of freedom of A^a_μ! For instance, we might impose that:

$$\partial \cdot A^a = 0.$$

This eliminates a degree of freedom along the direction which A^a_μ is propagating. To enfore this we just require that $\alpha^a(x)$ satisfies:

$$\partial^2 \alpha^a(x) = 0.$$ This is called Lorentz or Landau gauge.

So, there are only two propagating degrees of freedom of A_μ^a! These correspond to the two helicity configurations of the gluon: left- or right-handed.

In the rest of this lecture, I will just discuss consequences of the theory of QCD, beyond those that lead to its discovery.

First, a clear prediction just from the Lagrangian of QCD is that gluons interact with themselves. This is very different than electromagnetism! Recall that the photon part of the electromagnetic Lagrangian is

$$\mathcal{L}^{E\&M} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

where A_μ is the photon field. The equation of motion found from varying the Lagrangian with respect to A_μ is:

$$\partial_\mu F^{\mu\nu} = 0. = \partial^2 A^\nu - \partial^\nu \partial \cdot A$$

In Lorentz/Landau gauge, $\partial \cdot A = 0$ and so the equation of motion reduces to the Klein-Gordon equation:

$$\partial^2 A^\nu = 0.$$

That this is a linear differential equation (as are all of Maxwell's equations) has huge consequences for electromagnetic phenomena. One of the most central features of E&M is the principle of superposition. That is, the electric (or magnetic) field at any point can be found by just summing individual components of the electric field that come from different sources.

In the context of the Klein-Gordon equation, if A_1^μ and A_2^μ are two electromagnetic potentials that each satisfy the Klein-Gordon equation, then so does their sum:

$$\partial^2\left(A_1^\mu + A_2^\mu\right) = 0.$$

Another way to state the property of linearity/superposition of electromagnetism is in relation to the interactions of the photon. The photon carries no electric charge and so does not interact with itself. A photon traveling through space will pass right by another photon, without so much as a hello. Because photons do not interact with themselves, Maxwell's equations are linear; in a pithy way that David Griffiths might express it: photons cannot beget more photons.

This is to be contrasted with the case in QCD. The field strength tensor for the gluon A_μ^a is:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c .$$

The final term, that is quadratic in the gluon fields, is not present in electromagnetism, and is responsible for gluons interacting with themselves!

From the pure gluon component of the QCD Lagrangian, we can vary with respect to A_μ^a to determine the Euler-Lagrange equations of motion. One finds:

$$\frac{\delta \mathcal{L}^{gluon}}{\delta A_\mu^a} = 0 = \frac{\delta}{\delta A_\mu^a}\left(-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}\right)$$

$$\Rightarrow \partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu b} F^c_{\mu\nu} = 0 \quad \text{or that}$$

$$\partial^2 A^a_\nu - \partial_\nu \partial \cdot A^a + g f^{abc} \partial^\mu (A^b_\mu A^c_\nu) + g f^{abc} A^{\mu b} \partial_\mu A^c_\nu$$
$$- g f^{abc} A^{\mu b} \partial_\nu A^c_\mu + g^2 f^{abc} f^{cde} A^{\mu b} A^d_\mu A^e_\nu = 0$$

This is a highly non-linear differential equation for the gluon potential field A^a_μ! In this expression, the importance of non-linear terms is controlled by the coupling of QCD, g. If g goes to 0, the equations of motion become linear (and turn into the same expression as for E+M), but in general is non-zero.

Because this equation is non-linear, the field A^a_μ does *not* satisfy the principle of superposition. Two fields $A^a_{1\mu}$ and $A^a_{2\mu}$, each of which satisfy the equation of motion, *cannot* be summed into another solution. Because of this, QCD is a hard theory to understand! It is not known how (or even if) the equations of motion can be solved exactly.

Another thing to note is that the gluon, unlike the photon, itself carries a charge of the force that it communicates! The gluon field A^a_μ has that SU(3) colour index "a", which can take one of 8 possible values (because there are 8 basis matrices of SU(3)). We say that gluons carry colour, and in a Griffiths's way we might say that:

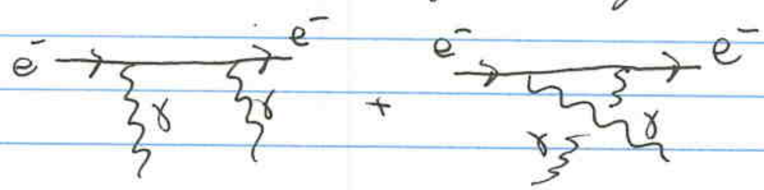
Gluons beget more gluons.

We aren't yet out of the woods in terms of weirdness of QCD.

In class, we have defined the fine-structure constant α as a measure of the strength of electromagnetism. α controls the cross-section for $e^+e^- \to$ hadrons scattering, as well as setting the size of the electric potential between charged particles. I have also said that its value is $\alpha \cong 1/137$, which allowed us to state that corrections to the cross section for $e^+e^- \to \mu^+\mu^-$ that we calculated in class were small, and so could be ignored. But, this is quantum mechanics, and so everything is only as good as our measurements allow. So, to determine the fine structure constant, we need to describe the measurement we would perform.

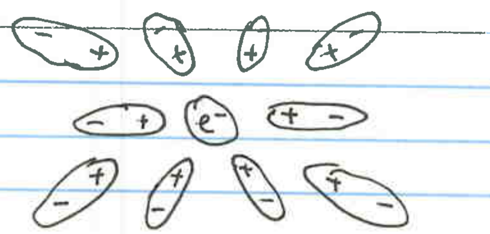
One way in principle to measure α is the following. It will happen to be impractical, but provides insight into the physics of what is happening. Let's imagine we have an electron sitting in space. To measure α, we need to know how strongly a photon couples to that electron. So, we could shine a laser onto the electron and observe what happens. So, schematically, the set up is illustrated as:

This can be represented by the Feynman diagrams:



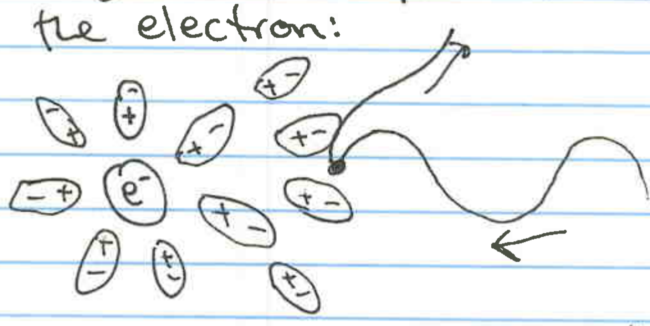
This is called "Compton Scattering" after Arthur Compton. In setting the electron out like this, however, some strange things happen.

Because electromagnetic waves propagate through vacuum, the vacuum must be a medium. As a medium, it is a dielectric; it can be polarized in the presence of a ~~magnetic~~ charged particle, like an electron. How does the electron polarize the vacuum? It creates an electric field in which virtual particles have a preferred orientation! That is, because of the negative charge of the electron, virtual positrons and electrons orient themselves in this field as so:

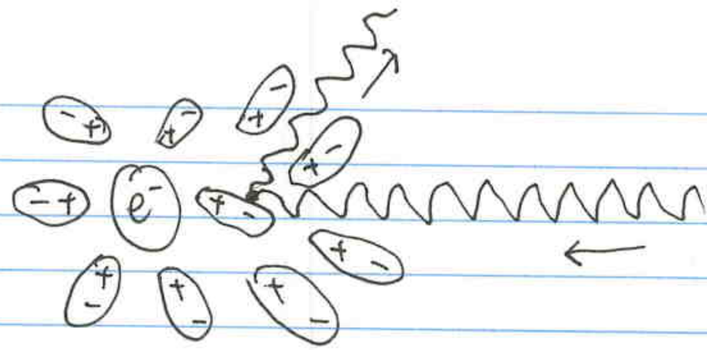


Here, I have denoted e^+e^- pairs by the blobs with + and − inside. These virtual particles effectively screen the electric charge of the electron making it appear smaller when you are a further distance away. In turn, it appears that α is smaller when you are further away!

So, with this insight, let's go back to our experiment of shining light on the electron. If the wavelength of light is long (that is, it has low energy), then the light is scattered by the virtual particles before it gets close to the electron:

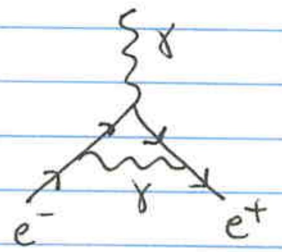


So, a low energy photon sees a small value of α. By contrast, if the photon is high energy or short wavelength, it penetrates further into the cloud:



So, this photon would see a larger value of α! That is, the fine structure constant depends on the energy/wavelength at which it is measured. To denote this, we refer to it as a "running coupling" (its value changes or "runs" with energy) and write $\alpha(Q)$, for energy Q.

This energy dependence can be calculated in quantum field theory. The diagram which describes the first approximation to the energy dependence of α is:



This diagram represents corrections to the electron-positron-photon Feynman diagram vertex. Note that this diagram has a "loop" in it. That is, by conservation of energy and momentum, <u>any</u> four-momenta can flow in the loop. Consideration of how this diagram changes with the momentum that is flowing in the loop enables a determination of the running coupling $\alpha(Q)$.

By the way, Naomi Gelder, who graduated last year, worked to calculate this diagram, but for muons for her thesis.

The way that this energy dependence is typically expressed is via a object that is called the β-function ("beta-function") for α. The β-function encodes the derivative of α with energy:

$$\beta(\alpha) \equiv Q \frac{d\alpha}{dQ}$$

The leading-in-α term of the β-function for α calculated from the diagram above is:

$$\beta(\alpha) = \frac{2}{3}\frac{\alpha^2}{\pi} + \cdots$$

We can solve the differential equation for α as a function of energy Q:

$$Q\frac{d\alpha}{dQ} = \frac{2}{3}\frac{\alpha^2}{\pi} \Rightarrow \frac{d\alpha}{\alpha^2} = \frac{2}{3\pi}\frac{dQ}{Q}$$

Integrating both sides, we have:

$$\frac{1}{\alpha(Q_0)} - \frac{1}{\alpha(Q)} = \frac{2}{3\pi}\ln\frac{Q}{Q_0} \quad \text{or that}$$

$$\alpha(Q) = \frac{\alpha(Q_0)}{1 - \frac{2}{3\pi}\alpha(Q_0)\ln\frac{Q}{Q_0}}$$

Here, Q_0 is some reference energy scale (like the Z boson mass, for example) and $\alpha(Q_0)$ is the value of the fine structure constant at that energy. Note that this indeed expresses the expectation for the energy dependence of α: as Q increases, the denominator decreases, and so $\alpha(Q)$ increases. Again, we have a nice picture of this as resulting from polarization of the vacuum.

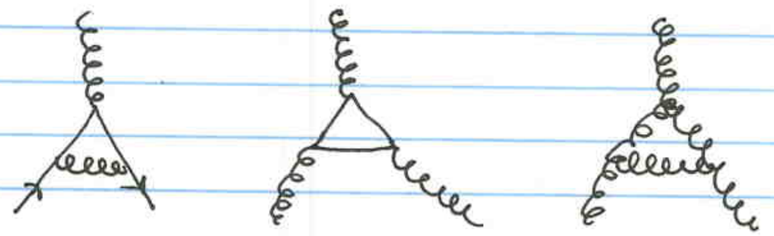
Our definition of the fine structure constant is in terms of the fundamental electric charge (in natural units):

$$\alpha = e^2/4\pi$$

In the QCD Lagrangian, we introduced the coupling g that controls the strength of the non-linear terms. When used in Feynman diagrams and to calculate cross sections, g will be squared, and so we introduce the strong coupling α_s:

$$\alpha_s = \frac{g^2}{4\pi}$$

Not surprisingly, the value of α_s, like α, also depends on energy. In QCD, we can calculate appropriate diagrams to find the β-function of α_s. The diagrams we need to compute are:



That is, we need to determine the corrections to the coupling of quarks to gluons (and gluons to gluons) g with these one-loop diagrams. This calculation is more involved than determining the β-function for α, but the same procedure applies. The β-function for α_s was first calculated by Gerardus 't Hooft, David Politzer, David Gross, and Frank Wilczek (who was a 20 year old graduate student at the time). They found:

$$\beta(\alpha_s) = \frac{\alpha_s^2}{2\pi}\left(-11 + \frac{2}{3} n_f\right)$$

This result is arguably one of the most important in particle physics. Unlike α, there are two contributions to the β-function for α_s. First, the $\frac{2}{3} n_f$ contribution (which is positive) is the contribution from quarks. Here, n_f is the number of quarks that can contribute to the β-function. Its interpretation is exactly like the

β-function for α. Pairs of quarks and antiquarks pop in and out of the vacuum and effectively screen the colour charge of a particle. Therefore this contribution works to increase α_s as energy increases.

However, because the gluon itself carries colour, gluons can pop in and out of the vacuum to affect the colour charge of a particle! This is totally different than for α, and has no analog in classical mechanics. Apparently gluons are responsible for strong anti-screening: gluons work to decrease the size of α_s as it is probed at higher and higher energies! The standard Model has 6 quarks (up, down, strange, charm, top, bottom), and so n_f is at most 6. Even with all of the Standard Model quarks around,

$$-11 + \frac{2}{3}\cdot 6 = \frac{-33+12}{3} = -\frac{21}{3} = -7$$

and so the β-function of α_s is negative! Apparently, the strong force QCD gets "weaker" at higher energies.

Solving the β-function equation for $\alpha_s(Q)$ we find:

$$\alpha_s(Q) = \frac{\alpha_s(Q_0)}{1 + \frac{\alpha_s(Q_0)}{2\pi}\left(11 - \frac{2}{3}n_f\right)\ln\frac{Q}{Q_0}}$$

As $Q \to \infty$, $\alpha_s(Q) \to 0$. This feature is called "asymptotic freedom": at asymptotically high energies, quarks interact more and more weakly, becoming like free (non-interacting) particles.

This feature of asymptotic freedom will have huge consequences for the phenomena of QCD at high energies.