QCD at Hadron Colliders Lecture 16

In our motivation for the theory of the strong interactions, we encountered two scattering processes that provided evidence for pieces of QCD. The process $e^+e^- \to q\bar{q}$, we were able to relate to the process $e^+e^- \to$ hadrons and then could calculate the R observable where,

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$ This gave us evidence

for the colour of quarks: we need to include a factor of 3 to describe the data. The other prosess we studied is $e^-q \to e^-q$, which we identified as the underlying, fundamental process in deeply-inelastic scattering of an electron off of a proton:

$$e^-p \to e^- + X, \text{ where } X \text{ is anything.}$$

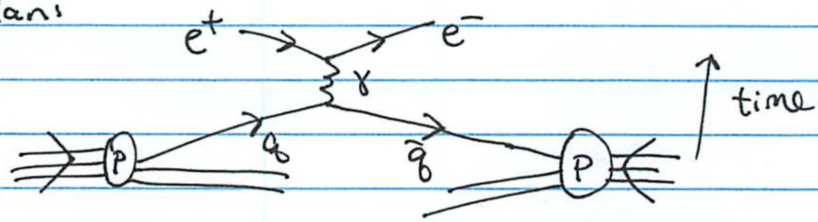
This enabled us to test the parton model, in which quarks are the point-particle constituents of the proton.

What made these two analyses so powerful was that the Feynman diagrams of the processes $e^+e^- \to q\bar{q}$ and $e^-q \to e^-q$ are related to one another by crossing symmetry. This is (essentially) an axiom of quantum field theory, and provides a deep connection between the two processes. Note that there is another crossing that is unique: we can also consider the process

$$q\bar{q} \to e^+e^-,$$ which is described by

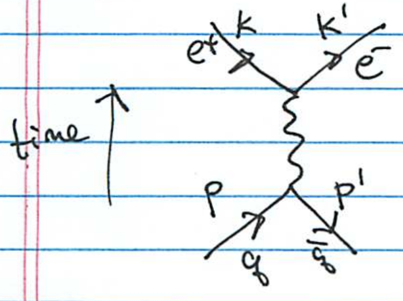
the same underlying Feynman diagram. This process is called Drell-Yan, and is a fundamental process at an experiment that collides protons, like the Large Hadron collider. In this lecture, we will discuss the manifestation of QCD at hadron colliders, and use Drell-Yan as our entry to this topic.

Let's draw a picture to see what we're dealing with in Drell-Yan:



Here, in the initial state, there are two protons that are collided at high energy. At sufficiently high energy (when the de Broglie wavelength of the proton is much smaller than its Compton wavelength), the protons explode apart and their constituents, the quarks or anti-quarks, interact directly. For Drell-Yan, a quark from one proton and an anti-quark from the other proton annihilate into a photon (or Z boson) which then splits into an electron and positron. (Interesting fact: the largest Feynman diagram in the world is on the floor of the physics building at the University of Oregon. The Feynman diagram is for the Drell-Yan process. If you're interested "Why Drell-Yan at UO?", come ask me.)

The underlying Feynman diagram for this process can be expressed in terms of the Mandelstam variables for the partons, $\hat{s}$, $\hat{t}$, and $\hat{u}$. In this process, we denote the momenta as:



and so:
$$\hat{s} = (p+p')^2 = (k+k')^2$$
$$\hat{t} = (p-k)^2 = (p'-k')^2$$
$$\hat{u} = (p-k')^2 = (p'-k)^2$$

Here, we have the initial momenta (p, p') flowing <u>inward</u> and the final momenta (k, k') flowing <u>outward</u>. This is why $\hat{t}$ and $\hat{u}$ have negative signs in their definitions.

Recall that the total cross section for $e^+e^- \to q\bar{q}$ in terms of the center-of-mass energy $\hat{s} = E_{cm}^2$ is:

$$\sigma(e^+e^- \to q\bar{q}) = 4 \frac{\pi \alpha^2}{\hat{s}} \sum_{i\,quarks} Q_i^2$$

This can be crossed into the initial state by accounting for averaging, rather than summing, over colour:

$$\sigma(q\bar{q} \to e^+e^-) = \frac{4}{9} \frac{\pi \alpha^2}{\hat{s}} \sum_{i\,quarks} Q_i^2$$

The overall factor was reduced by 1/9 because we average over the initial stat colours of the quarks (instead of sum, as we do in the final state). Okay! So we know the cross section of the fundamental process. We still need to pull the quarks out of the protons, so let's do that now.

Let's say we pull the quark out of proton 1 with momentum fraction x_1 of proton 1's momentum, while the antiquark comes from proton 2 with momentum fraction x_2. If we call the momentum of proton 1 $\underline{P}_1$ (and $\underline{P}_2$ for proton 2) then we define the center-of-mass proton collision energy as:

$$s \equiv (\underline{P}_1 + \underline{P}_2)^2 = 2\underline{P}_1 \cdot \underline{P}_2 \text{ so that } \hat{s} = x_1 x_2 s = 2(x_1 \underline{P}_1) \cdot (x_2 \underline{P}_2).$$

Here, we assume that we are working at sufficiently high energy so as to ignore the proton mass. The probability distributions of the momentum fractions x_1 and x_2 are defined by the parton distribution functions $f_q(x)$. The cross-section differential in both momentum fractions can therefore be written as:

$$\frac{d\sigma}{dx_1 dx_2}(q\bar{q} \to e^+e^-) = \frac{4}{9}\frac{\pi\alpha^2}{\hat{s}} \sum_q Q_q^2 f_q(x_1) f_{\bar{q}}(x_2)$$

or, written as the cross section for $pp \to e^+e^-$:

$$\frac{d\sigma}{dx_1 dx_2}(pp \to e^+e^-) = \frac{4}{9}\frac{\pi\alpha^2}{s} \sum_q Q_q^2 \frac{f_q(x_1)}{x_1} \frac{f_{\bar{q}}(x_2)}{x_2}$$

Here, q represents all possible quarks (or anti-quarks) that can be pulled out of the proton.

This is interesting, but we can't directly measure the momentum fractions x_1 and x_2. So, we want to re-express them in terms of things we can measure. Two useful quantities with which to express the momentum fractions are through the invariant mass and rapidity of the electron-positron pair. The invariant mass, Q^2, is just $\hat{s}$:

$$Q^2 = \hat{s} = x_1 x_2 s$$

The rapidity y is defined as: $y = \frac{1}{2} \ln \frac{E+p_z}{E-p_z}$, and can be directly measured by measuring the sum of the energies and z-component of momenta of the e^+ and e^-. The total energy of the e^+e^- pair is just the sum of the energies of the quark and anti-quark:

$$E = x_1 \frac{E_{cm}}{2} + x_2 \frac{E_{cm}}{2} = \frac{x_1 + x_2}{2} E_{cm}$$

Similarly, the p_z of the e^+e^- pair is equal to the p_z of the quark and anti-quark by momentum conservation:

$$p_z = x_1 \frac{E_{cm}}{2} - x_2 \frac{E_{cm}}{2} = \frac{x_1 - x_2}{2} E_{cm}$$

Here, we have aligned the momenta of the protons along the z-axis. Now, from these expressions, we can calculate the rapidity:

$$y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \frac{1}{2}\ln\frac{\frac{x_1+x_2}{2}E_{cm} + \frac{x_1-x_2}{2}E_{cm}}{\frac{x_1+x_2}{2}E_{cm} - \frac{x_1-x_2}{2}E_{cm}} = \frac{1}{2}\ln\frac{x_1}{x_2}.$$

To write the cross section interms of Q^2 and y we can solve for x_1 and x_2 as:

$$x_1 = \sqrt{\frac{Q^2}{s}}\,e^{y}, \quad x_2 = \sqrt{\frac{Q^2}{s}}\,e^{-y}$$

We also need the Jacobian factor from the change of variables. To do this, we calculate all derivatives and then take the determinant of the derivative matrix:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial Q^2} & \frac{\partial x_1}{\partial y} \\ \frac{\partial x_2}{\partial Q^2} & \frac{\partial x_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{e^{y}}{2\sqrt{sQ^2}} & \sqrt{\frac{Q^2}{s}}\,e^{y} \\ \frac{e^{-y}}{2\sqrt{sQ^2}} & -\sqrt{\frac{Q^2}{s}}\,e^{-y} \end{vmatrix}$$

$$= \frac{1}{s}.$$

Putting it all together, we predict that the differential cross section is:

$$\frac{d\sigma}{dQ^2 dy}(pp\to e^+e^-) = \frac{d\sigma}{dx_1 dx_2}(pp\to e^+e^-)\cdot J$$

$$= \frac{4}{9}\frac{\pi\alpha^2}{Q^2 s}\sum_q Q_q^2 f_q\left(\sqrt{\frac{Q^2}{s}}\,e^{y}\right) f_{\bar{q}}\left(\sqrt{\frac{Q^2}{s}}\,e^{-y}\right)$$

This is an incredible result! Rapidity dependence only comes in the parton distributions, and therefore a measurement of the rapidity is very sensitive to the pdfs.

In our discussion of deeply-inelastic scattering, a similar observation was used to argue that quarks were point particles. If the parton distributions are independent of Q^2, then the structure of quarks is independent of the wavelength at which they are probed. That is, quarks are point particles. But, is this actually true?

Last week, we discussed the property of coupling dependence on energy scale. That is, as one probes a system with different photon wavelengths, one sees that the strength of the force changes. In QCD, this means that the strength with which gluons couple to quarks depends on the invariant mass of the quark-gluon system. So, because of this, we don't necessarily expect the pdf to actually be independent of Q^2. So, how does it depend on Q^2 and can we calculate the dependence?

To motivate this discussion, let's remind ourselves about the cross section we calculated for $e^+e^- \to q\bar{q}g$. We had found:

$$\frac{d\sigma(e^+e^- \to q\bar{q}g)}{dx_1\, dx_2} = \sigma(e^+e^- \to q\bar{q}) \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Here, recall that the x_i variables are defined from the final state momenta p_i as:

$$x_i = \frac{2Q\cdot p_i}{Q^2}, \quad \text{where in the center-of-mass, } Q = (E_{cm}, 0, 0, 0).$$

Now, in the differential cross-section above, I want to take the limit in which the quark and gluon become collinear. That is, we will expand in the limit where the angle between the quark and gluon becomes small and we keep the leading terms. The physical picture we have is:

, where θ is the splitting angle.

A nice way to express this in terms of the quark and gluon energy fractions:

$$z \equiv \frac{E_q}{E_q + E_g} = \frac{x_1}{x_1 + x_3} = \frac{x_1}{2 - x_2}, \quad 1 - z \equiv \frac{E_g}{E_q + E_g} = \frac{2 - x_1 - x_2}{2 - x_2}.$$

In the collinear limit, we can set $x_2 = 1$; that is, when the quark and gluon become collinear, then the anti-quark has energy $E_{cm}/2$. In this limit,

$$z = x_1, \quad 1 - z = 1 - x_1.$$

We also want to express the angle between the quark and gluon. Note that

$$\frac{2 p_1 \cdot p_3}{Q^2} = \frac{2 E_1 E_3}{E_{cm}^2}(1 - \cos\theta) = \frac{x_1 x_3}{2}(1 - \cos\theta)$$

$$= 1 - x_2 = \frac{Q^2 - 2 Q \cdot p_2}{Q^2}$$

In the collinear limit, $1 - \cos\theta = \frac{\theta^2}{2} + \dots$, and so

$$\theta^2 = \frac{4(1 - x_2)}{x_1(2 - x_1 - x_2)} \rightarrow \frac{4(1 - x_2)}{x_1(1 - x_1)}$$

With these expressions for z and θ^2, we will change variables from x_1 and x_2 to z and θ. We have:

$$z = x_1, \quad 4(1 - x_2) = z(1 - z)\theta^2$$

or that: $x_2 = 1 - \frac{z(1-z)}{4}\theta^2$

The Jacobian of this change of variables is:

$$J = \left|\frac{\partial x_2}{\partial \theta^2}\right| = \frac{z(1-z)}{4} = \frac{1-x_2}{\theta^2}$$

Then, in the collinear limit of quark and gluon in $e^+e^- \to q\bar{q}g$ becomes:

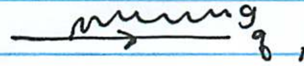
$$\frac{d\sigma(e^+e^- \to \bar{q}(qg))}{dz\, d\theta^2} = \sigma(e^+e^- \to q\bar{q}) \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z} \frac{1}{\theta^2}$$

This form of the cross section tells us many things. First, the expression diverges in the collinear limit, when $\theta^2 \to 0$ and in the soft gluon limit, when $1-z \to 0$. This might be very disconcerting if we attempt to interpret the cross section as a probability. Probabilities can't be larger than 1, so what is going on?

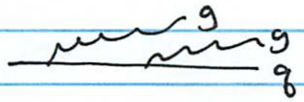
The answer is simply that we can't interpret this cross section as a probability! If you are taking quantum II now, you might be familiar with degenerate perturbation theory. If you have a quantum system with states that are degenerate, that is, have equal energy, then if you perform a perturbative analysis you can find divergences. However, those divergences exactly cancel if you sum over all degenerate state contributions! Only when all degenerate states are included can you get a finite, normalizable probability. So, what's the story here for collinear quark-gluon splitting?

Let's visualize what's going on. We could have just a quark traveling along:

, or a quark with one collinear gluon:



, or two collinear gluons:



, or three, or <u>any</u> number of

collinear gluons emitted off of the quark. The important point is that all of these configurations are degenerate! Each of them is individually divergent, but when you sum over the q, qg, qgg, qggg, …, states you find a finite result! We'll talk about how to do this in a second.

Another feature of this limit is the presence of a function of z that governs the distribution of the quark's energy fraction. This function is universal: it is the same for <u>every</u> quark that splits collinearly to a gluon. As such, it is called the universal collinear splitting function $P_{qg\leftarrow q}(z)$:

$$P_{qg\leftarrow q}(z) = C_F\,\frac{1+z^2}{(1-z)}$$

with this observation that there are collinear divergences when gluons and quarks get close, we can compute the energy dependence of the pdf, $f_q(x)$! In doing this, we will also demonstrate the way that an arbitrary number of emissions off of a quark can be included.

To do this, we will imagine probing the proton at a scale $Q^2+\delta Q^2$ and then a slightly smaller scale Q^2. Here, δQ^2 is a small energy scale that we will eventually take to 0.

To do this, we will need the differential cross section for the collinear splitting. In terms of the splitting angle, this is

$$\frac{d\sigma}{dz\,d\theta^2} = \frac{\alpha_s}{2\pi}\frac{1}{\theta^2}P_{qg\leftarrow q}(z).$$

In terms of the invariant mass of the quark and gluon, Q^2,

$$Q^2 = z(1-z)E^2\theta^2,$$ where E is the total energy of the quark and gluon. The cross section in z and Q^2 is then

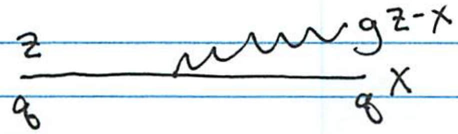
$$\frac{d\sigma}{dz\,dQ^2} = \frac{\alpha_s}{2\pi}\frac{1}{Q^2}P_{qg\leftarrow q}(z)$$

Then, if we look at the proton at scale $Q^2+\delta Q^2$ and find a quark with energy fraction x, we have

$f_q(x, Q^2+\delta Q^2)$. Then, we want to look at the proton at a slightly lower scale of Q^2. There are two possibilities for what can happen. First, nothing can happen; we just see the pdf at scale Q^2:

$$f_q(x, Q^2).$$

Next, there could have been a collinear emission of a gluon at this lower scale that <u>decreased</u> the energy fraction of the quark from some value z, down to x. The picture of this is:



The probability for this to happen is:

$$\frac{\delta Q^2}{Q^2}\int_0^1 dx' \int_0^1 dz \, \frac{\alpha_s}{2\pi} P_{qg\leftarrow q}(\cancel{z}x') \, f_q(z, Q^2)\, \delta(x - z x')$$

$$= \frac{\delta Q^2}{Q^2}\int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{qg\leftarrow q}\left(\frac{x}{z}\right) f_q(z, Q^2)$$

There are a few moving parts, so let me explain. In the first expression z is the momentum fraction of the quark with respect to the proton before splitting and x after splitting. x' is the relative momentum fraction of the splitting that ranges between 0 and 1. We can integrate over x' with the δ-function to get the second line. The factor of $1/z$ is the Jacobian from the δ-function.

Combining these three terms, we have:

$$f_q(x, Q^2+\delta Q^2) = f_q(x, Q^2) + \frac{\delta Q^2}{Q^2}\int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi} P_{qg\leftarrow q}\left(\frac{x}{z}\right) f_q(z, Q^2)$$

Then, in the limit as $\delta Q^2 \to 0$, this turns into a differential equation:

$$\boxed{Q^2 \frac{d f_q(x, Q^2)}{d Q^2} = \int_x^1 \frac{dz}{z}\frac{\alpha_s}{2\pi} P_{qg\leftarrow q}\left(\frac{x}{z}\right) f_q(z, Q^2)}$$

This differential equation determines the energy dependence of the parton distribution function. It is among the most important results in QCD and is called the DGLAP evolution equation after its discoverers: Yuri Dokshitzer, Vladimir Gribov, Lev Lipatov, Guido Altarelli, and Giorgio Parisi.