Jets Lecture 17

Today is our final lecture on QCD, and starting next week, we'll turn our attention to the weak interactions. Our topic today will be on what is perhaps the most shocking observation and prediction of QCD: collimated streams of high-energy particles, called "jets". I spend much of my time thinking about jets in my research, so we'll get into important issues regarding jets in this lecture.

To start, I want to identify what may seem like an unrelated observation regarding QCD, from its defining Lagrangian. Let's start with a much simpler system: the action that gives rise to the Klein-Gordon equation for a massless scalar:

$$S[\phi] = \int d^4x \ \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$$

This action has many features: It is Lorentz-invariant, but it is also invariant under scale transformations. A scale transformation, or dilation, is an operation that re-scales positions by an overall factor λ:

$$x^\mu \to \lambda x^\mu, \text{ for any } \lambda > 0.$$

Let's see what this transformation does to the action above. We will need to determine how the integration measure d^4x, the derivative ∂_μ, and the field ϕ scales with λ. First, d^4x. Note that:

$d^4x = dt\, dx\, dy\, dz$, and each t, x, y, z is scaled by λ. That is

$$d^4x = dt\,dx\,dy\,dz \to d(\lambda t)\, d(\lambda x)\, d(\lambda y)\, d(\lambda z) = \lambda^4 d^4x$$

Okay, that's the integration measure. What about the derivative? The derivative four-vector ∂_μ is

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$, and each t, x, y, z is scaled by λ.

For example, $\frac{\partial}{\partial x} \to \frac{\partial}{\partial(\lambda x)} = \frac{1}{\lambda}\frac{\partial}{\partial x}$, and a similar scaling exists for all derivatives. Therefore: $\partial_\mu \to \frac{1}{\lambda}\partial_\mu$.

Now, what about the field $\phi(x)$? This scaling can be derived more rigourously within quantum field theory, but here I will just work to justify it. An action is just the time integral of a Lagrangian, ~~and~~ and therefore has dimensions of:

$$[S] = [\text{Energy}][\text{time}] = [\hbar]$$

Because we set $\hbar = 1$ in natural units, the action is dimensionless. So, what are the dimensions of the parts of the action? Well, dx in natural units has dimensions of inverse energy, while ∂_μ has units of energy (because in quantum mechanics $\partial_\mu \to p_\mu$, the momentum). Therefore, for the action to be dimensionless in natural units, the field $\phi(x)$ must have dimensions of energy. That is, $[\phi] = [\text{Energy}]$.

Because of this, we will define the scaling of ϕ with λ to be the same as that of the derivative ∂_μ, as they have the same units. Then,

$$\phi \to \frac{1}{\lambda}\phi.$$

With this scaling we can then determine how the action scales with λ. Plugging all the scales in, we have:

$$S[\phi] = \int d^4x \, \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) \rightarrow \int d^4x \, \lambda^4 \, \tfrac{1}{2}\left(\lambda^{-1}\partial_\mu \lambda^{-1}\phi\right)\left(\lambda^{-1}\partial^\mu \lambda^{-1}\phi\right)$$

$$= S[\phi]$$

That is, the action is invariant to this rescaling! Because the action is invariant, there is therefore a conservation law, by Noether's theorem. The action is sufficient to describe everything about a classical or quantum system, and such a system with an action that is invariant under these scalings is called "scale invariant".

Note that scale transformations are <u>not</u> Lorentz transformations. These dilations can be implemented on the spacetime four-vector x by a matrix Λ:

$$\Lambda = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix},$$ where λ is on the diagonal, but all other entries are 0. That is,

$\Lambda = \lambda \mathbb{1}$. Recall that the definition of a Lorentz transformation Λ is that it satisfies:

$\Lambda^T \eta \Lambda = \eta$, where η is the spacetime metric.

Testing it with the dilations, we find:

$\Lambda^T \eta \Lambda = \lambda \mathbb{1}^T \eta \lambda \mathbb{1} = \lambda^2 \eta \neq \eta$, for arbitrary λ.

Therefore dilations are <u>not</u> Lorentz transformations.

This is interesting, but is it trivial? Are all relativistic actions scale invariant? The answer is indeed no, because a dilation is not a Lorentz transformation.

Let's consider adding a mass to this system. Then, the action is:

$$S[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2\right]$$

Now, let's apply this dilation to this action. The kinetic term (with derivatives) is just the same as what we studied earlier. The mass term is different. Its scaling is:

$$S_{m^2}[\phi] = -\frac{m^2}{2}\int d^4x\, \phi^2 \rightarrow -\frac{m^2}{2}\int d^4x\, \lambda^4 (\lambda^{-1}\phi)^2 = \lambda^2 S_{m^2}[\phi]$$

Importantly, the mass m is just a number and so does not scale with λ. This demonstrates that the theory with a mass is <u>not</u> scale invariant. Again, this isn't a problem, perse, because dilations are not Lorentz transformations.

So, what about the scaling of QCD? The QCD Lagrangian with massless quarks is

$$S[A_\mu^a, \psi] = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} i\gamma\cdot D\psi\right]$$

which we discussed last week. Recall that the field strength tensor $F_{\mu\nu}^a$ is:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c .$$

Using the same principles as for the action of ϕ, this implies that the field A_μ^a has dimensions of energy, and so will scale like λ^{-1} under a dilation. Using the definition of the covariant derivative:

$$D_\mu = \partial_\mu - ig A_\mu^a T^a$$

we also find the units of the quark field ψ to be:

$$[\psi] = [\text{energy}]^{3/2}.$$

Therefore, under a dilation the quark field ψ scales like $\lambda^{-3/2}$. These scalings imply that the QCD Lagrangian (or action) is scale invariant:

$$S[A_\mu^a, \psi] \to \int d^4x\, \lambda^4 \left[-\frac{1}{4} \lambda^{-2} F_{\mu\nu}^a \lambda^{-2} F^{\mu\nu a} + \lambda^{-3/2} \bar{\psi} i\gamma\cdot D \lambda^{-1} \psi \lambda^{-3/2} \right]$$

$$= S[A_\mu^a, \psi].$$

Therefore, we expect that QCD is not only Lorentz invariant and gauge invariant, but also scale invariant. We have seen how Lorentz and gauge invariante highly restrict interactions of particles. We'll discuss what this scale invariance does in a second.

First, our argument for scale invariance in QCD was a litte fast. In this argument, we assumed that the coupling g that appears in the field strength or the covariant derivative is scale invariant. That is, we assumed that the coupling g is the same regardless of with what wavelength we probe it or at what energy we probe. However, we know this isn't true! The coupling g, or the strong coupling α_s, changes with energy according to the β-function of QCD:

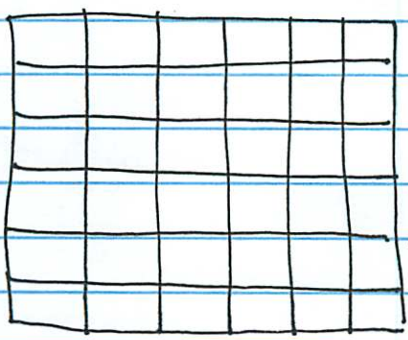
$$Q \frac{dg}{dQ} = \beta(g) \neq 0.$$

Therefore, the β-function breaks the scale-invariance of QCD! This is entirely a quantum phenomena; i.e., would not exist if $\hbar = 0$.

By the way, if you are interested, the equation that governs the violation of scale invariance in quantum field theory is called the Callan-Symanzik equation.

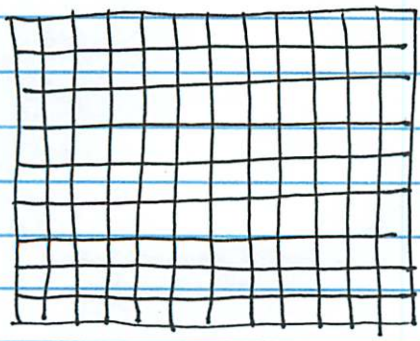
Nevertheless, there is a sense in which the violation of scale invariance is weak in QCD: the value of the β-function is relatively small, so we can consider QCD as a scale-invariant theory, with corrections that violate scale invariance (i.e., the running coupling).

So, how do we think about scale invariance, with these caveats? Imagine we probe the system with some wavelength λ_1, which can be visualized as viewing the system with some pixel size set by λ_1:



$\} \sim \lambda_1$

Scale invariance says that if we probe the system at another wavelength λ_2, then the physics is identical:



$\} \lambda_2 < \lambda_1$

We can freely "zoom in" or "zoom out" of our system and we see the same physical phenomena. Mathematical structures that have this property were studied and defined in detail by Benoit Mandelbrot in the 1970's. He called them "fractals", and there are a huge number of fractal systems in nature (for example, romanesco broccoli). His seminal paper on fractals is called "How long is the coast of Britain?"

Okay, this is nice, fancy words, what does this tell us about the physics of QCD? Last lecture, we introduced the DGLAP parton evolution equation. It governed the energy evolution of the parton distribution function for a parton (a quark or gluon) in the proton. For quarks, we argued that the DGLAP equation is:

$$Q^2 \frac{d f_q(x,Q^2)}{dQ^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{qg \leftarrow q}\left(\frac{x}{z}\right) f_q(z,Q^2)$$

The DGLAP equation describes the emission of an arbitrary number of gluons off of a quark as the energy/wavelength which you probe the quark changes. Up to the running of α_s, it does so in a scale invariant manner. The left side of the equation scales with λ however the pdf $f_q(x,Q^2)$ scales with λ:

$$Q^2 \frac{d f_q}{dQ^2} \rightarrow (\lambda^{-2}Q^2) \frac{d(\lambda^{\mathfrak{f}} f_q)}{d(\lambda^{-2}Q^2)} = \lambda^{\mathfrak{f}} Q^2 \frac{d f_q}{dQ^2},$$

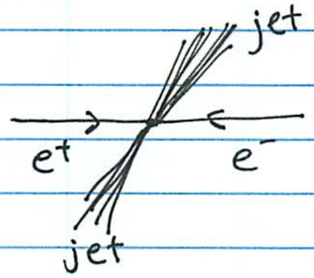
where $\lambda^{\mathfrak{f}}$ is the pdf scaling. On the right side, there are no explicit scales whatsoever: z and x are energy fractions, and so do not scale with λ! The only thing that might scale (ignoring the running of α_s) is $f_q(z,Q^2)$. This scales the same as $f_q(x,Q^2)$ on the left, and so DGLAP describes scale-invariant gluon emission!

Let's think about this a bit more physically. DGLAP says there is no such thing as a "bare" quark. A quark is always associated with an arbitrary number of gluons that are approximately collinear to the direction of the quark. Thus, in an experiment ~~wee~~ where a quark is produced, like in $e^+e^- \to q\bar{q}$ events, that quark (or anti-quark) will always be associated with an arbitrary number of collinear particles.

This collimated stream of particles is called a jet: it is the manifestation of the approximate scale invariance of QCD, in addition to the relative smallness of α_s (~ 0.1).

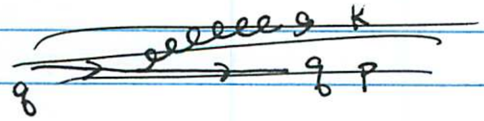
This prediction of jets produced in $e^+e^- \to$ hadrons events was somewhat scandalous in the late 1960's and early 1970's. There were very few people in the world who thought that jets may exist. It wasn't until 1974, when the property of asymptotic freedom was verified in QCD, that people took the predictions of jets seriously.

What we see in high energy e^+e^- scattering is something like:

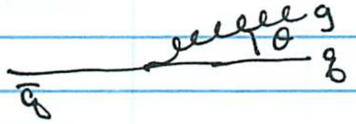


As collimated streams of particles, jets invoke water jets, and one source of the etymology of "jet" in particle physics is from the Jet d'Eau, a spectacular water fountain that sits in Lake Geneva. To my knowledge, the first use of the term "jet" was by James Bjorken and Stanley Brodsky in 1970.

In the remainder of this class, I want to make one more prediction of jets, which will illustrate more properties of (approximate) scale invariance in QCD. ~~Consider the emission of a gluon from a quark:~~



Let's make a prediction of the thrust observable in the limit when $\tau \to 1$. We found earlier that $\tau \to 1$ corresponds to the limit in which the gluon in $e^+e^- \to q\bar{q}g$ events becomes collinear. This is a singular limit, in the sense of the matrix element describing this configuration diverges like $1/\theta$, where θ is the angle of the split gluon:



Last lecture, we argued (in a different way) that this singular limit means that an arbitrary number of collinear gluons can (and will!) be emitted from the quark and antiquark. How do these numerous gluons affect the differential cross section of thrust?

Recall that we showed that the probability distribution for emission of a gluon at angle θ and energy fraction $1-z$ off of a quark (which has energy fraction z) is:

$$P_q(z,\theta^2) = \frac{\alpha_s}{2\pi} C_F \frac{1+z^2}{1-z} \frac{1}{\theta^2} dz\, d\theta^2,$$

which holds when $\theta \to 0$. The thrust observable τ was defined as:

$$\tau = \max\{x_i\}$$

If we let the quark and gluon have energy fractions of x_1 and x_3, respectively, then the antiquark has the largest energy fraction x_2. In the limit when the gluon becomes collinear, last lecture we showed that:

$$\tau = x_2 = 1 - \frac{z(1-z)}{4}\theta^2 \quad \text{or that} \quad 1-\tau = \frac{z(1-z)}{4}\theta^2.$$

As a further simplification, we will work in the soft and collinear limit in which the gluon has both lowenergy and is collinear. This corresponds to $z \to 1$, and so the probability distribution becomes:

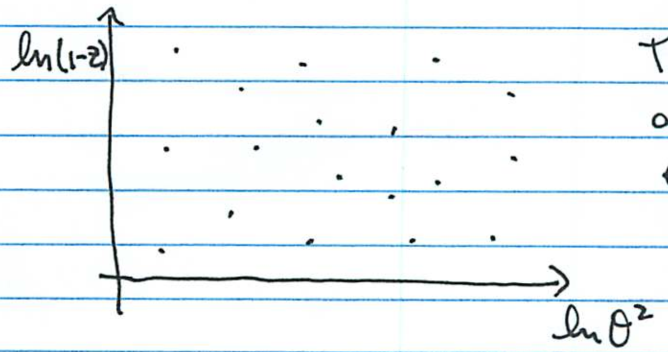
$$P_q(z,\theta^2) \to \frac{\alpha_s}{\pi} C_F \frac{dz}{1-z} \frac{d\theta^2}{\theta^2}$$ and the thrust is

$$\tau \to 1 - \frac{1-z}{4}\theta^2 \text{ or }, \quad 1-\tau \to \frac{(1-z)\theta^2}{4}.$$

There is a beautiful way to visualize this system and calculate the thrust in the soft and collinear limit. The probability $P_q(z,\theta^2)$ governs the splitting probability of an arbitrary number of soft and collinear gluons. Note that it is a flat distribution in $\ln(1-z)$ and $\ln\theta^2$:

$$P_q(z,\theta^2) = \frac{\alpha_s}{\pi} C_F \frac{dz}{1-z}\frac{d\theta^2}{\theta^2} = \frac{\alpha_s}{\pi} C_F \, d\ln(1-z)\, d\ln\theta^2$$

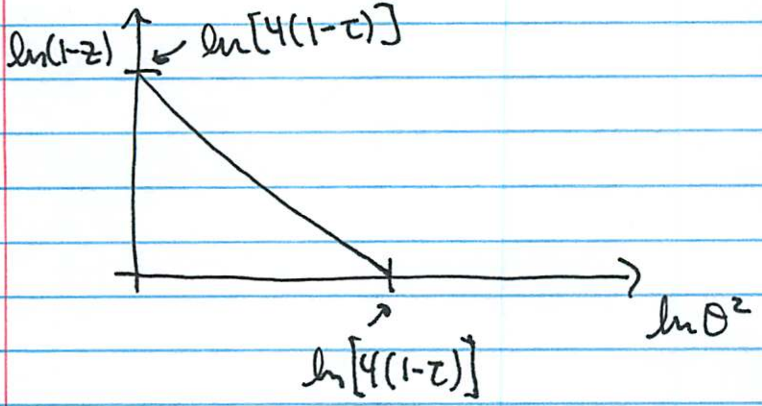
Therefore, we can imagine gluons uniformly populating the $\ln(1-z)$, $\ln\theta^2$ plane:



This plane is called a Lund diagram and each dot corresponds to an emitted gluon with some energy fraction $1-z$ emitted at an angle θ. A measured value of thrust corresponds to a straight line on this plane:

$$\ln(1-\tau) = \ln(1-z) + \ln\theta^2 - \ln 4$$

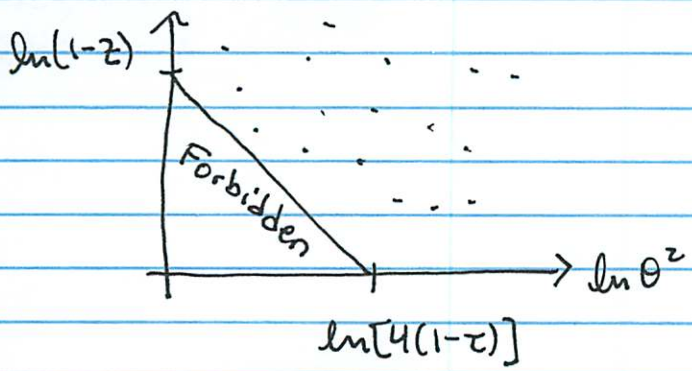
We'll draw this line on the Lund diagram:



On this diagram, I have denoted the x- and y-intercepts; that is, the value of $\ln\theta^2$ or $\ln(1-z)$ when the other is 0.

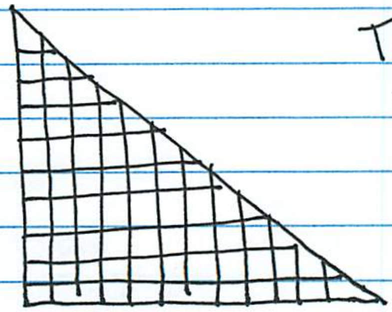
Importantly, note that emissions that land above this line contribute a tiny (negligible) value to thrust, while emissions below the line will increase the value of $1-\tau$ beyond its measured value (that is, those gluons have larger $1-z$ or θ^2). Let's calculate the cumulative distribution of thrust in this picture; i.e., we will calculate the probability that the thrust is no larger than the measured value of $4(1-\tau)$. We will call this cumulative probability distribution $\Sigma(\tau)$.

Recall that emissions in the Lund diagram are uniformly distributed, but for the measured value of thrust, we must forbid emissions that land below the line:



To determine the probability that no gluon emissions land in the triangle, we imagine breaking it up into tiny pieces and forbidding any emission in each of the pieces.

The picture is:



The probability that no gluon was emitted into one of the squares is set by the area of a square:

$$P(\text{no gluon}) = 1 - \frac{\alpha_s}{\pi} C_F \times (\text{area of square})$$

To find the total probability that no gluons land anywhere in the triangle, we need to multiply the probabilities of each square together:

$$P(\text{no gluon in triangle}) = \prod_{i=1}^{N} \left(1 - \frac{\alpha_s}{\pi} C_F \cdot (\text{area of square } i)\right)$$

As the number of squares $N \to \infty$, this probability transmogrifies into an exponential:

$$P(\text{no gluon in triangle}) = \exp\left[-\frac{\alpha_s}{\pi} C_F \cdot (\text{area of triangle})\right]$$

The area of the forbidden triangle is just

area of triangle $= \frac{1}{2} \ln^2 [4(1-\tau)]$ and so the

probability of no gluon in the triangle or the cumulative probability of τ in the soft and collinear limits is:

$$P(\text{no gluon in triangle}) = \Sigma(\tau) = \exp\left[-\frac{\alpha_s}{\pi} \frac{C_F}{2} \ln^2(4(1-\tau))\right]$$

This result is remarkable, and this exponential factor is called the Sudakov form factor. It is responsible for exponentially suppressing the soft and collinear region of phase space (when $\tau \to 1$), which is physical behavior. Divergences in Feynman diagrams are turned into suppressions!

Note in this Sudakov factor we have summed a series in α_s to all powers in α_s. This procedure is called "resummation", and is an aspect of a more general procedure in quantum field theory called renormalization. The history of renormalization ~~is~~ is long and often fraught with some fighting. The opponents of renormalization (which included Feynman) thought of it as a terribly ill-defined procedure for "sweeping infinities under the rug". Nevertheless, the property of renormalizability, that is, the ability to systematically account for and eliminate infinities in a quantum field theory is necessary for predictivity. Gerardus 't Hooft and Martinus Veltman proved that the theory of the Standard Model is renormalizable, and therefore we can predict within it.

Renormalization was put on a sound theoretically mathematical and physical footing by Ken Wilson. Using renormalization, that is, consistently dealing with infinities, he was able to solve an outstanding problem in condensed matter physics, called the Kondo problem. After his work, renormalization became a standard tool for particle physicists and condensed matter physicists alike.