The V-A Theory   Lecture 19

Last lecture, we discussed the weird, or rather, super-weird phenomena of neutron decays. Apparently, unlike electromagnetism, the strong force, and gravity, whatever the force is that mediates neutron decay violates parity. Parity is the transformation that flips all spatial dimensions. Vectors, like position, velocity, acceleration, etc., flip under the action of parity. That is, the parity transformation P turns a velocity into minus itself:

$$P\vec{v} = -\vec{v}.$$

By contrast, there are vectors that are formed from the cross-product of two separate vectors. These are like angular momentum or magnetic field. For angular momentum for example, it is formed from the cross product of velocity and position:

$\vec{L} = \vec{r} \times \vec{p}$. Then, a parity transform does

not change angular momentum: $P\vec{L} = (P\vec{r}) \times (P\vec{p})$
$= (-\vec{r}) \times (-\vec{p}) = \vec{L}$

These objects that do not change under the action of parity are called pseudo vectors.

In 1956, C.S. Wu led an experiment to directly test the parity properties of the weak force, the force that governs the decay of neutrons, for example. The beauty of Wu's experiment was that it could directly test the parity of the weak force.

It did this because the initial state of the experiment (the spin of ^{60}Co in a magnetic field) is purely described in terms of pseudovectors. By contrast, the final state was described in terms of vectors (velocity of the electron). By observing more electrons in one direction than the other directly observed the parity violation of the weak interaction.

Very shortly after this experiment several people introduced what is called the "vector-minus-axial" or V-A theory to describe the observed phenomena of parity violation. The decay of the neutron could be described by the interaction:

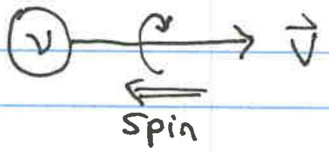
$$\mathcal{L}_{int} = \frac{4G_F}{\sqrt{2}}\left(\nu_L^{\dagger}\bar{\sigma}^{\mu}e_L\right)\left(d_L^{\dagger}\bar{\sigma}_{\mu}u_L\right)$$

Here, the spinors of the particles are denoted by the particle names ('e for the electron, for example), and all spinors are purely left-handed. That is, their spin points opposite to the direction of motion. Also in this expression The neutron decay $n \to p^+ + e^- + \nu_e$, is expressed in terms of its fundamental quark decay:

$$d \to u + e^- + \nu_e$$

Recall that if a down quark in a neutron turns into an up quark, the neutron turns into a proton. The fact that this Lagrangian involves only left-handed spinors shows how parity is violated, and actually it is violated maximally.

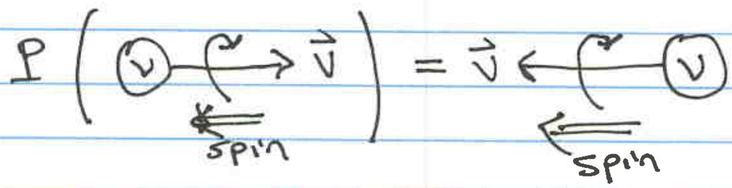
To see what this means, let's look at a left-handed neutrino just traveling through space:



Note that the spin is anti-parallel to the momentum/velocity of the neutrino so its helicity is:

$$h = \hat{p} \cdot \vec{S} = -\frac{1}{2}$$

Under a parity transformation, the spin (= angular momentum) of the neutrino is unchanged, as it is a pseudovector. However, the direction of the neutrino (its momentum) is flipped:



After this parity transformation, the neutrino's helicity is right-handed: $h = +1/2$! If parity was conserved, then the interaction must include both left and right handed particles. If parity was partially violated, then the interaction would include both left and right handed particles, but with different coefficients. The fact that there are no right-handed particle contributions in this interactions means that parity is maximally violated.

This theory is called "vector minus axial" because its ~~the~~ is described by one linear combination of of vectors and axial vectors (also called pseudovectors). Note that under a parity transformation, this linear combination turns into the orthogonal linear combination:

$$P(V-A) = -V-A = -(V+A)$$

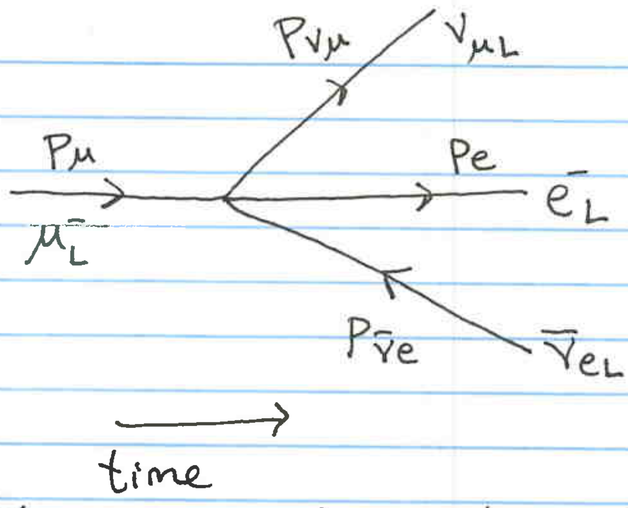
In the rest of this class, we will see what predictions this V-A theory makes. One could calculate the decay of the neutron with this theory, but this is unnecessarily challenging because you must account for all masses of final state particles. So, for simplicity, but will still demonstrate some interesting features of the V-A theory, we will study the decay of a muon. The decay of a muon is described by the interaction Lagrangian:

$$\mathcal{L}_{int} = \frac{4G_F}{\sqrt{2}}\left(\nu_{\mu L}^{\dagger}\bar{\sigma}^{\mu}\mu_L\right)\left(e_L^{\dagger}\bar{\sigma}_{\mu}\nu_{eL}\right)$$

~~This governs the decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$~~

This governs the decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$. Here, we note that there are two types of neutrinos in this decay. ν_μ is the muon-neutrino, while $\bar{\nu}_e$ is the electron anti-neutrino. If you took (or are taking J-Lab), we will predict the lifetime of the neutrino, which you (maybe?) measured in that class.

To start, as we always do, we draw the Feynman diagram that corresponds to the interaction described by the Lagrangian above. The Feynman diagram for muon decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ is:



The corresponding matrix element from this Feynman diagram with the interaction Lagrangian above is:

$$\mathcal{M}(\mu^- \to e^- + \bar\nu_e + \nu_\mu) = \frac{4G_F}{\sqrt{2}}\left(u_L^\dagger(p_{\nu_\mu})\bar\sigma^\mu u_L(p_\mu)\right)\left(u_L^\dagger(p_e)\bar\sigma_\mu v_L(p_{\bar\nu_e})\right)$$

Before we calculate this, there are a few things to note. First, note the spinor assignments: for example, the muon, as the initial state particle, has a $u_L(p_\mu)$ spinor. By contrast, the electron, as a final state particle, has a $u_L^\dagger(p_e)$ spinor. The strength of the interaction is controlled by the factor with Fermi's constant, which comes directly from the interaction Lagrangian. I have also labeled all spinors as left-handed, from the form of the V-A theory.

We will evaluate the matrix element in the frame in which the muon is at rest; that is, the muon must be massive. For simplicity, we assume that the electron and neutrinos are massless. This will then enable us to express the squared matrix element in terms of Lorentz invariant dot products, that will be independent of frame.

To evaluate the spinor products we will use a Fierz identity. We need to evaluate the spinor product:

$$\left(u_L^\dagger(p_{\nu_\mu})_a \bar{\sigma}^\mu_{ab}\, u_L(p_\mu)_b\right)\left(u_L^\dagger(p_e)_c\, \bar{\sigma}_{\mu cd}\, v_L(p_{\bar{\nu}_e})_d\right)$$

In this expression, I have written explicit spinor indices to denote the entries of the spinors, or elements of the sigma-matrices. The indices are repeated, and hence summed over, and range over 1 and 2: $a,b,c,d \in \{1,2\}$. Apparently, to evaluate this spinor product requires evaluating the matrix product:

$\bar{\sigma}^\mu_{ab}\,\bar{\sigma}_{\mu cd}$. What is this?

To evaluate this, we need a nice way to express the sigma-matrices. For example, consider the identity matrix, $\mathbb{1}$. What are its elements? Well, it only has non-zero entries if the row and column are equal. We can then express this using the Kronecker-δ symbol:

$$\mathbb{1}_{ab} = \delta_{ab} = \delta_{a1}\delta_{b1} + \delta_{a2}\delta_{b2}$$

Recall that $\delta_{ab} = 1$ if $a=b$, and 0 otherwise. We can use the Kronecker δ to express the entries of all of the σ-matrices:

$$\mathbb{1}_{ab} = \delta_{a1}\delta_{b1} + \delta_{a2}\delta_{b2}$$
$$\sigma_{1ab} = \delta_{a1}\delta_{b2} + \delta_{a2}\delta_{b1}$$
$$\sigma_{2ab} = -i\delta_{a1}\delta_{b2} + i\delta_{a2}\delta_{b1}$$
$$\sigma_{3ab} = \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2}$$

Using these expressions, we can evaluate the matrix product $\bar{\sigma}^\mu_{ab}\bar{\sigma}_{\mu cd}$.

We have:

$$\bar{\sigma}^{\mu}_{ab}\,\bar{\sigma}_{\mu cd} = \left(\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3\right)^{\mu}_{ab}\left(\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3\right)_{\mu cd}$$

$$= \left(\delta_{a1}\delta_{b1} + \delta_{a2}\delta_{b2}\right)\left(\delta_{c1}\delta_{d1} + \delta_{c2}\delta_{d2}\right)$$

$$- \left(\delta_{a1}\delta_{b2} + \delta_{a2}\delta_{b1}\right)\left(\delta_{c1}\delta_{d2} + \delta_{c2}\delta_{d1}\right)$$

$$- \left(-i\delta_{a1}\delta_{b2} + i\delta_{a2}\delta_{b1}\right)\left(-i\delta_{c1}\delta_{d2} + i\delta_{c2}\delta_{d1}\right)$$

$$- \left(\delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2}\right)\left(\delta_{c1}\delta_{d1} - \delta_{c2}\delta_{d2}\right)$$

$$= 2\left(\delta_{a1}\delta_{c2} - \delta_{a2}\delta_{c1}\right)\left(\delta_{b1}\delta_{d2} - \delta_{b2}\delta_{d1}\right)$$

The matrix with entries $\varepsilon_{ab} \equiv \delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}$ is called the anti-symmetric symbol. In matrix form, it is:

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore, we have shown that: $\bar{\sigma}^{\mu}_{ab}\bar{\sigma}_{\mu cd} = 2\varepsilon_{ac}\varepsilon_{bd}$

We can use this to evaluate the spinor products. Writing out all indices, we have:

$$\left(u_L^\dagger(p_{\nu_\mu})_a\,\bar{\sigma}^{\mu}_{ab}\,u_L(p_\mu)_b\right)\left(u_L^\dagger(p_e)_c\,\bar{\sigma}_{\mu cd}\,v_L(p_{\bar{\nu}_e})_d\right)$$

$$= 2\left(u_L^\dagger(p_{\nu_\mu})_a\,\varepsilon_{ac}\,u_L^\dagger(p_e)_c\right)\left(u_L(p_\mu)_b\,\varepsilon_{bd}\,v_L(p_{\bar{\nu}_e})_d\right)$$

$$= -2\left(u_L^\dagger(p_{\nu_\mu})\,\varepsilon_{ac}\,u_L^\dagger(p_e)_c\right)\left(v_L(p_{\bar{\nu}_e})_d\,\varepsilon_{db}\,u_L(p_\mu)_b\right)$$

In the last line, I used that $\varepsilon_{bd} = -\varepsilon_{db}$.

Now, we want to figure out what these ε

anti-symmetric symbols are doing. Let's focus on the matrix product $\varepsilon_{ac} u_L^\dagger(p_e)_c$ first. Recall for a momentum vector p at an angle θ with respect to the $\hat{z}$ axis an angle ϕ about the $\hat{z}$ axis, the spinor is:

$$u_L^\dagger(p)_c = \sqrt{2E}\begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix}.$$

Now, let's act on this with the ε symbol:

$$\varepsilon_{ac} u_L^\dagger(p)_c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ac} \sqrt{2E}\begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix}_c$$

$$= \sqrt{2E}\begin{pmatrix} -\cos\theta/2 \\ -e^{i\phi}\sin\theta/2 \end{pmatrix}_a = -u_R(p)_a$$

The ε-symbol turns $u_L^\dagger(p)$ into $u_R(p)$! We can do the same thing with $v_L(p_{\bar{\nu}_e})_d \varepsilon_{db}$. We have:

$$v_L(p)_d \varepsilon_{db} = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}_d \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{db}$$

$$= \sqrt{2E}\begin{pmatrix} -e^{i\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix}_b = -v_R^\dagger(p)_b$$

Cool! Putting this into the spinor products, we have:

$$\left(u_L^\dagger(p_{\nu_\mu})\bar{\sigma}^\mu u_L(p_\mu)\right)\left(u_L^\dagger(p_e)\bar{\sigma}_\mu v_L(p_{\bar{\nu}_e})\right)$$

$$= -2\left(u_L^\dagger(p_{\nu_\mu})\, u_R(p_e)\right)\left(v_R^\dagger(p_{\bar{\nu}_e})\, u_L(p_\mu)\right)$$

Then, the matrix element is:

$$\mathcal{M}(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu) = -2\frac{4G_F}{\sqrt{2}}\left(u_L^\dagger(p_{\nu_\mu})\, u_R(p_e)\right)\left(v_R^\dagger(p_{\bar{\nu}_e})\, u_L(p_\mu)\right)$$

The absolute squared matrix element for muon decay is:

$$\left|\mathcal{M}(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu)\right|^2 = 32 G_F^2 \left|u_L^\dagger(p_{\nu_\mu}) u_R(p_e)\right|^2 \left|v_R^\dagger(p_{\bar{\nu}_e}) u_L(p_\mu)\right|^2$$

We have evaluated some of these before in homework, or in lecture. The first squared spinor product is:

$$\left|u_L^\dagger(p_{\nu_\mu}) u_R(p_e)\right|^2 = 2 p_{\nu_\mu} \cdot p_e .$$

The second squared spinor product we might be tempted to write the same (or the related) expression for. However, we have to be a bit careful, because we require that the muon is massive. So, we have to treat the spinor $u_L(p_\mu)$ carefully.

The simplest way to do this is to evaluate it in one frame, and then Lorentz boost to an arbitrary frame. Let's work in the frame where the muon is at rest and the electron anti-neutrino is traveling along the $+\hat{z}$ direction. Then:

$$v_R^\dagger(p_{\bar{\nu}_e}) = \sqrt{2E_{\bar{\nu}_e}} \begin{pmatrix} 0 & -1 \end{pmatrix} \quad \text{and} \quad u_L(p_\mu) = \sqrt{m_\mu}\, \xi .$$

The expression for $v_R^\dagger$ is familiar, but for $u_L(p_\mu)$ it is probably not. The overall factor of $\sqrt{m_\mu}$ is normalization (just like the $\sqrt{2E}$ for massless spinors. ξ is an arbitrary two-component spinor normalized such that $\xi^\dagger \xi = 1$. This represents the spin of the muon in its rest frame. (Recall that helicity for massive particles is not Lorentz invariant and so a "left-handed" spinor has both components of spin.) To evaluate the spinor product, we need to correspondingly average over the spin of the muon (as its direction of spin is not well-defined.). This can be accomplished by setting ξ to represent ½ probability to be either spin up or spin down:

$$\xi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$ Now, evaluating the spinor product, we have:

$$\left|v_R^\dagger(p_{\bar{\nu}_e})u_L(p_\mu)\right|^2 = \left|\sqrt{2E_{\bar{\nu}_e}}\begin{pmatrix}0 & -1\end{pmatrix}\sqrt{\frac{m_\mu}{2}}\begin{pmatrix}1\\1\end{pmatrix}\right|^2$$

$$= m_\mu E_{\bar{\nu}_e}.$$

This is the rest-frame evaluation of the Lorentz-invariant dot product $p_\mu \cdot p_{\bar{\nu}_e}$.

Therefore, once the dust has settled, the squared matrix element is:

$$\left|\mathcal{M}(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu)\right|^2 = 64G_F^2(p_e \cdot p_{\nu_\mu})(p_\mu \cdot p_{\bar{\nu}_e}).$$

The only decay product we can observe is the electron, so let's express the matrix element squared in terms of the electron momentum. The final state is described by three-body phase space, and so we use the x_i variables, where

$$x_i = \frac{2Q \cdot p_i}{Q^2}, \text{ for } i = e, \nu_\mu, \bar{\nu}_e.$$

Note that for muon decay $Q^2 = m_\mu^2$, and $Q = p_\mu$. Then,

$$p_\mu \cdot p_{\bar{\nu}_e} = Q \cdot p_{\bar{\nu}_e} = m_\mu^2 \frac{x_{\bar{\nu}_e}}{2} \quad \text{and}$$

$$2p_e \cdot p_{\nu_\mu} = Q^2 - 2Q \cdot p_{\bar{\nu}_e} = m_\mu^2(1 - x_{\bar{\nu}_e})$$

The squared matrix element is then written in terms of $x_{\bar{\nu}_e}$ as:

$$|\mathcal{M}(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu)|^2 = 16\, G_F^2\, m_\mu^4\, x_{\bar{\nu}_e}(1 - x_{\bar{\nu}_e}).$$

This can then be plugged into Fermi's Golden Rule for decays. The decay rate Γ is:

$$\Gamma = \frac{1}{2E_\mu} \int d\Pi_3\, |\mathcal{M}(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu)|^2$$

$$= \frac{1}{2m_\mu} \frac{m_\mu^2}{128\pi^3} \int_0^1 dx_e \int_0^1 dx_{\bar{\nu}_e}\, \Theta(x_e + x_{\bar{\nu}_e} - 1)\, 16\, G_F^2\, m_\mu^4\, x_{\bar{\nu}_e}(1 - x_{\bar{\nu}_e}).$$

$$= \frac{G_F^2 m_\mu^5}{16\pi^3} \int_0^1 dx_e \int_{1-x_e}^1 dx_{\bar{\nu}_e}\, x_{\bar{\nu}_e}(1 - x_{\bar{\nu}_e}).$$

In the second line, I used the expression for three-body phase space that you derived in Homework 4. We can't measure the electron anti-neutrino momentum fraction $x_{\bar{\nu}_e}$, so we integrate over it:

$$\Gamma = \frac{G_F^2 m_\mu^5}{96\pi^3} \int_0^1 dx_e\, (3x_e^2 - 2x_e^3)$$

From here, we can define the decay rate differential in the electron energy fraction, x_e:

$$\frac{d\Gamma}{dx_e} = \frac{G_F^2 m_\mu^5}{96\pi^3}(3x_e^2 - 2x_e^3),$$ which the book shows agrees wonderfully with data.

You can also calculate the total decay rate by integrating over x_e. Doing this, we find:

$$\Gamma(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu) = \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

The Fermi constant is $G_F \cong 1.2 \times 10^{-5}\ \text{GeV}^{-2}$ and the muon mass is $m_\mu = 106\ \text{MeV}$. Plugging in the numbers, the decay rate is then:

$$\Gamma(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu) \cong 3.2 \times 10^{-10}\ \text{eV}.$$

Turning this into a lifetime (in seconds) by adding factors of $\hbar$, we have:

$$\tau = \frac{\hbar}{\Gamma} \cong 2.1 \times 10^{-6}\ \text{s}.$$

The PDG says that the lifetime of the muon is 2.2×10^{-6} s. So, we're very close! Awesome!

While the V-A theory makes nice predictions like this, theoretically, it leaves a lot to be desired. Most importantly, it provides no explanation for the force; that is, there is no force carrier analagous to the photon or gluon. How do we resolve this? More next time...