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Units of Particle Physics & Dimensional Analysis Lecture 2

In this lecture, we will start setting up our discussion of particle physics. And, perhaps like the early lectures of physics 101, we need to discuss how we express measurements in particle physics; that is, what our units are. Good units should be simple and respect the realm in which they are being used. Because particle physics is the realm of short distances and high energies, we need to use units that naturally and usefully express quantities.

Particle physics is the realm of both relativity and quantum mechanics. The particles we will consider will be traveling at near the speed of light c, and so c will appear in equations everywhere. For example, the particles we consider satisfy the energy-momentum relation:

$$E^2 = m^2c^4 + p^2c^2,$$

where E is the energy, m is the mass, and p is the momentum of the particle of interest. In SI units, c is

$c = 3\times10^8$ m/s, so everytime we have to use

the energy-momentum relationship, we have to lug around this huge number in SI units.

Particle physics is also the realm of quantum mechanics, the description at the shortest distances. The fundamental unit in quantum mechanics is $\hbar$, Plank's reduced constant, which quantifies units of angular momentum. It also appears in the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

We'll discuss a bit next week about generalizing the Schrödinger equation to account for relativity, but any time we want to discuss quantum phenomena, we need an $\hbar$. In SI units, $\hbar$ is

$$\hbar = 1.05 \times 10^{-34}\ \text{J}\cdot\text{s}$$

So, everytime we measure a quantum mechanical value, we have to lug around this teensy-tiny number in SI units.

So, we want to simplify our use of c and $\hbar$. Additionally, the masses or other properties of individual particles are exceptionally tiny. In SI units, the mass of the electron is

$$m_e = 9.11 \times 10^{-31}\ \text{Kg}$$

The mass of the proton, while much larger than the electron, is still tiny in SI units:

$$m_p = 1.67 \times 10^{-27}\ \text{Kg}$$

Even the most massive elementary particle, the top quark has a super tiny mass in SI units:

$$m_t = 3.1 \times 10^{-25}\ \text{Kg}$$

So, whenever we talk about the electron traveling at relativistic speeds, we need to keep track of numbers that are spread over about 40 decades! This is inconvenient.

There's another, philosophical reason to abandon SI units in particle physics. We believe, perhaps with a bit of ego, that particle physics is truly fundamental. That is, the speed of light as measured by a distant civilization would be the same as what we have measured on Earth. However, why would they use SI units? The second is defined as a part of the day, a very Earthicentric notion, and the meter is from some stick in France.

For these reasons and to express the fundamental-ness of particle physics, we introduce natural units, or "God's units" in which we set

$$\hbar = c = 1 \quad \text{(unitless)}$$

In natural units, we are able to express everything in terms of energies. In particle physics, we typically use electron-volts (eV) as the energy unit of choice as this is naturally (closer to) the scale that we work at. For example, let's see how this works for the electron mass. To express the electron mass as an energy, we multiply by c^2:

$$m_e c^2 = 2.73 \times 10^{-22}\ \text{J} = 511\ \text{KeV} \quad \text{(Kilo-eV)}$$

The mass of the proton is

$$m_p c^2 = 5 \times 10^{-19}\ \text{J} = 938\ \text{MeV} \quad \text{(mega-eV)}$$

Using natural units, we can turn distances into energies, as well. Note the Heisenberg uncertainty principle:

$$\Delta p \, \Delta x \sim \frac{\hbar}{2}$$

This tells us how to convert to natural units for distances. A distance x has the same units as $\frac{\hbar}{p}$, which you might recall as the de Broglie wavelength. Momentum p can be related to energy via the relativistic energy-momentum relation:

$$E = pc \text{, for massless particles.}$$

Then, ~~the~~ for a distance x, the quantity

$$\frac{x}{\hbar c} = \frac{1}{E} \text{, is a distance expressed in natural units.}$$

Let's see how this works in an example.

Ex What is the Bohr radius expressed in natural units? The Bohr radius in SI units is

$$a_0 = 5.3 \times 10^{-11} \text{ m}$$

Converting to natural units this is:

$$\frac{a_0}{\hbar c} = 1.7 \times 10^{15} \text{ J}^{-1} = \frac{1}{3.7 \text{ keV}}$$

Note that this corresponding energy is much larger than the ground state energy of hydrogen, which is (-)13.6 eV.

Throughout this class, we will employ natural units as they will make expressions and algebra <u>much</u> easier. From natural units, one can always uniquely go to any other unit system by restoring the factors of c and $\hbar$. You just have to remember what the quantity is (a length, time, mass, etc.).

For the rest of this lecture and into next week, we will review special relativity in some detail. In this week's homework, you will play around with natural units and gain an intuition for its utility.

The review of special relativity for this class will be very practical: we won't think about trains or falling in an elevator. I want to, however, as the book does, emphasize the importance of symmetries in particle physics. Symmetries are a recurring theme and fundamental guiding principle in particle physics. In any field of physics, how something works is straightforward and requires just measuring appropriate quantities. For example, how two blocks collided just requires measuring their initial and final velocities. Why the blocks collided how they did is much more interesting and actually explains the mechanism of their collision; i.e., what laws of nature govern their collision. We know what laws govern their collision: conservation of energy and conservation of momentum. With these two simple principles, we can uniquely predict what will happen when any two blocks collide!

The goal of particle physics is to explain what happens at the shortest distance scales. Explanation is provided by conservation laws that highly restrict the possible outcomes of an experiment. So, to proceed we need two things: we need to identify the conservation laws of particle physics and we need to determine how those conservation laws restrict the possible physics.

Determining the conservation laws can be done experimentally. We can measure quanties before and after particles interact to see what remains the same. Determining how those conservation laws restrict physics is much more subtle and exciting. So, how do we do it?

Determinging how conservation laws restrict physics is so important it is elevated to a theorem in physics, and is possibly the most important and general result in all of theoretical physics. It is called Noether's Theorem, after Emmy Noether, a German mathematician. Noether proved using the Lagrangian formulation of classical mechanics, that a symmetry of the Lagrangian has a corresponding conservation law. For example, time translation invariance corresponds to energy conservation, gauge invariance in E+M corresponds to charge conservation, etc. So, given a conservation law that we determine empirically, we can write down the Lagrangian of the system, demanding that it has the appropriate symmetry! We'll review Lagrangians for some systems next week. Here, I want to develop an understanding for how these symmetries are constraining, and review relativistic kinematics.

Before we go on, I think it is of utmost importance to learn history along with physics, and importantly, about who the physicists in history were. Apologies to those in Phys 367 last semester as you've already heard this, but I want to discuss Emmy Noether a bit more. Noether, in addition to her theorem, made fundamental contributions to abstract algebra and the calculus of variations. In a time when it was exceptionally challenging for women to get professorships, in the early 20th century, Noether found an ally in David Hilbert. Hilbert was possibly the most famous mathematician in the world at the time and lobbied hard at his institution at Göttingen for Noether to teach. Getting substantial protests due to her gender, Hilbert famously replied to his colleagues:

"I do not see that the sex of the candidate is an argument against her admission as lecturer. After all, we are a university, not a bath house."

Noether was allowed to stay in Göttingen but was not paid for teaching or research for many years.

It was in Göettingen that she proved her famous theorem. However, her time there was short-lived due to the rise of the National Socialist party in the 1930's. Noether, a Jew, was expelled from Göttingen and relocated to Bryn Mawr College in Pennsylvania. She died in 1935.

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Let's see how we can use the consequences of Noether's theorem in our formulation of special relativity and its utility for particle physics. Before we work with special relativity, let's work with the more familiar example of vectors in three-dimensions. Let's say we want to describe the physics of a system in which angular momentum is conserved. Noether tells us that angular momentum conservation corresponds to invariance of the system under rotations. For example, we might be considering a particle in a central potential, like gravity or E&M. Positions and velocities of the particles are represented by vectors; $\vec{x}$, $\vec{v}$, etc. If angular momentum is conserved, then how are these vectors allowed to appear in the calculation of properties of the system? That is, what function of the vectors $\vec{x}$, $\vec{p}$, etc., is consistent with rotational invariance?

To understand this, let's be concrete and consider two two-dimensional vectors $\vec{a}$, $\vec{b}$. To understand rotational invariance we need to figure out how they transform under a rotation. A rotation is a <u>linear</u> operation; that is, it can be represented as a matrix acting on the vector:

$$\vec{a} \to \mathbb{M}\vec{a}\,, \text{ where}$$

$$\mathbb{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \text{ where the angle of rotation is } \theta.$$

Note that a rotation changes the vector $\vec{a}$, and so the vector is not invariant to rotations. However, note that the rotation of the transpose of the vector is:

$$\vec{a}^T \to (\mathbb{M}\vec{a})^T = \vec{a}^T\mathbb{M}^T, \text{ where}$$

$$\mathbb{M}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Also note that the rotation matrix times its transpose is the identity matrix $\mathbb{1}$:

$$\mathbb{M}^T\mathbb{M} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\mathbb{1}}$$

This then tells us that $\vec{b}^T\vec{a}$ is invariant under rotations!

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That is, under a rotation $\mathbb{M}$, $b^T a$ is:

$$\vec{b}^T\vec{a} \to (\mathbb{M}\vec{b})^T \mathbb{M}\vec{a} = \vec{b}^T \mathbb{M}^T \mathbb{M}\vec{a} = \vec{b}^T \mathbb{1}\vec{a} = \vec{b}^T\vec{a}.$$

Therefore, for angular momentum to be conserved, the system must be described by quantities like $\vec{b}^T\vec{a}$. Of course, this is nothing more than the dot product:

$$\vec{b}^T\vec{a} = \vec{b}\cdot\vec{a} = \sum_{i=1}^{2} b_i a_i,$$

where I have introduce summation notation over the elements of the vectors. This can be equivalently written as

$$\vec{b}\cdot\vec{a} = \sum_{i=1}^{2} b_i a_i = \sum_{i,j} b_i \delta_{ij} a_j, \text{ where } \delta_{ij} \text{ is the}$$

Knonecker-δ,
$$\delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{else.} \end{cases}$$

This may seem a bit tautological, but now consider the rotation of this expression:

$$\sum_{i,j} b_i \delta_{ij} a_j \longrightarrow \sum_{i,j} b_i \mathbb{M}_{ik} \delta_{k\ell} \mathbb{M}_{\ell j} a_j$$

There are a couplethings I did to write this. I am using the Einstein summation notation: repeated indices are summed over. That is,

$$\mathbb{M}_{\ell j} a_j = \mathbb{M}_{\ell 1} a_1 + \mathbb{M}_{\ell 2} a_2, \text{ which}$$

implements the rotation on vector $\vec{a}$ (or, rather the ℓ^{th} component).

Note that this is invariant to rotations if and only if:

$$\mathbb{M}_{ik}\,\delta^{\cancel{kl}}_{kl}\,\mathbb{M}_{lj} = \delta_{ij}$$

However, this is nothing more than the $(i,j)^{th}$ entry of the matrix

$$(\mathbb{M}^T\mathbb{M})_{ij} = \delta_{ij}\,,$$ by the form of the rotation matrix.

Therefore, an equivalent way to define the rotation matrix is as all matrices that leave the Kronecker-δ invariant:

$\mathbb{M}$ is a rotation matrix iff $\mathbb{M}^T\mathbb{1}\mathbb{M} = \mathbb{1}$. $\quad(*)$

This is what we mean by invariance under rotations: vectors are combined with the Kronecker-δ/dot product, and rotation matrices do not change this.

We'll talk about this more with groups in a couple of weeks, but note a few things at this stage. The expression $(*)$ defines a rotation matrix in any dimension, D. We aren't restricted to $D=2$, as our analysis started from. The set of all matrices of Dimension D that satisfy $(*)$ is called the orthogonal group in dimension D, or $O(D)$.

The result $(*)$ and generalizations of it will be helpful in working in special relativity. If we want to describe a system in which relativistic energy, momentum, and angular momentum are conserved, then, just like for rotations, we want to identify vectors and how to take dot products appropriately. This will also define the space-time metric of flat space, often denoted as $\eta_{\mu\nu}$.

We're about out of time today, but just a teaser for next lecture. Recall the energy-momentum relationship in special relativity (in natural units!):

$$E^2 = m^2 + \vec{p}^{\,2} \quad \text{(Isn't that much prettier?)}$$

Depending on your frame, you will in general measure different energies E and momenta $\vec{p}$. However, you always measure the same mass m in any frame. Then, a better way to write this expression is:

$$m^2 = E^2 - \vec{p}^{\,2}.$$

Now, both sides are Lorentz-invariant. The right side looks almost like a dot product:

$$E^2 - \vec{p}^{\,2} = E \cdot E - \vec{p} \cdot \vec{p},$$

though with a weird $-$ sign. Ignoring the sign, let's keep plugging on. Let's make a "vector" of energy and momentum

$$p = (E, \vec{p}) \quad \text{(The notation is unfortunate and standard.)}$$

This is called a four-vector (because it has 4 components) and we can just define by fiat the dot product to be:

$$p \cdot p = E^2 - \vec{p}^{\,2} = m^2.$$

This dot product is Lorentz invariant by construction, and the four-vector p will transform under Lorentz transformations. But more next week... like how to define all Lorentz transformations!