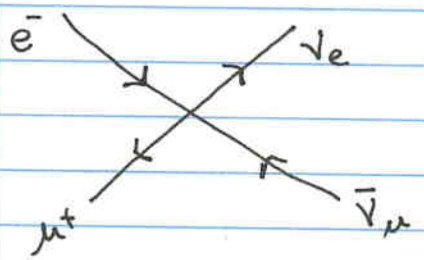
Spontaneous Symmetry Breaking Lecture 20

At the end of last lecture we were left with a puzzle. The V-A theory makes excellent predictions; last lecture, we used it to calculate the lifetime of the muon. On the other hand, this theory is severly lacking from the perspective of QCD or electromagnetism. In the V-A theory, we just postulate a four-fermion interaction, whose strength is controlled by the Fermi constant, G_F. In QCD and E+M, interactions of four fermions is mediated by a spin-1 boson; either the gluon (in QCD) or the photon (in E+M). If we are to understand this weak force at a fundamental level, we want it to have a force carrier, that, at most, communicates the force at the speed of light.

There is some other funny business with the V-A theory, as well. In the V-A theory, the interaction of, say, electrons, muons, and their neutrinos is governed by the interaction Lagrangian:

$$\mathcal{L}_{int} = \frac{4G_F}{\sqrt{2}}\left(\nu_{\mu L}^{\dagger}\bar{\sigma}^{\mu}\mu_L\right)\left(e_L^{\dagger}\bar{\sigma}_{\mu}\nu_{eL}\right)$$

Recall that the dimensionality of the Fermi constant is GeV^{-2}. With this observation, we can use dimensional analysis to estimate the rate for electron-muon collisions that produce neutrinos. The cross section for the process $e^- + \mu^+ \to \nu_e + \bar{\nu}_\mu$ must have dimensions of $[\text{energy}]^{-2}$ (because it is an area). The Feynman diagram for this scattering in the V-A theory is proportional to G_F:



$\propto G_F$, and so the squared matrix element and the cross section must be proportional to G_F^2. The rest of the dimensions in the cross section must be made up by factors of the center-of-mass collision energy, as that is the only other energy scale around. Therefore, the cross section must scale like:

$$\sigma(e^- + \mu^+ \to \nu_e + \bar{\nu}_\mu) \sim E_{cm}^2 G_F^2 .$$

This is a bit weird: the cross-section for electron-muon scattering diverges as the center-of-mass energy gets large? This doesn't make physical sense. At higher energies the electron and muon have smaller de Broglie wavelengths, and so it should be less likely for them to overlap and therefore collide. This physical picture is consistent with our calculation of electron-muon scattering via a photon. Recall that the cross-section for the process $e^+e^- \to \mu^+\mu^-$ is

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3E_{cm}^2} .$$

As expected, this vanishes as $E_{cm} \to \infty$.

Okay, perhaps this is weird, but let's stomach it, and soldier on. Let's just assume that indeed there is some force carrier that is responsible for the interactions in the V-A theory. In analogy with E+M + QCD, we would expect this force carrier to have spin-1 (like the photon and gluon) and perhaps, if it is like the gluon, carry some of the charge to which it communicates. Somehow this spin-1 force carrier needs to be responsible for the particular value of G_F. More importantly, it needs to introduce the appropriate dimensions of G_F. Very weirdly, this means that the force carrier must be massive!

We can interpret G_F as some sensitivity to a mass scale, according to its dimensionality. That is, we can write

$$G_F = \frac{1}{m_F^2}, \text{ for some mass } m_F,$$

that we might call the "Fermi Mass" (an instance of the Matthew effect). With $G_F \cong 1.17 \times 10^{-5}\ \mathrm{GeV}^{-2}$, the Fermi mass would be:

$$m_F = \frac{1}{\sqrt{G_F}} \cong 292\ \mathrm{GeV}.$$

So, somehow the force carrier of the V-A theory needs to introduce a mass scale of 292 GeV. This is problematic. When we discussed QCD, we argued that the gluon must be massless because a mass term in the Lagrangian was inconsistent with gauge invariance. If it is inconsistent with gauge invariance, then Noether says that the corresponding charge is not conserved. Yet, this seems to be what we need to do in the V-A theory to introduce a force carrier that is spin-1 and has a mass at the scale of the Fermi mass. As Winnie the Pooh might say, "Oh bother."

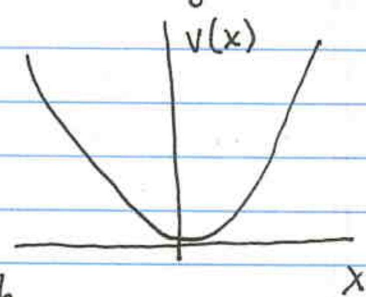
How do we ever hope to get out of this conundrum? In the 1940's, 50's, and 60's, many people including

Ernst Stueckelberg, Yoichiro Nambu, Philip Anderson, Tom Kibble, Robert Brout, François Englert, Gerry Guralnik, Carl Hagen, Gerardus 't Hooft, Peter Higgs, and others provided the insight and solution to all of these seemingly insurmountable problems. It's typically called the Higgs Mechanism and predicts the existence of the Higgs boson, which was discovered at the LHC in 2012. Higgs and Englert won the Nobel Prize for their work in 2013.

In this lecture, I will discuss the Higgs mechanism in a simple theory, and then next lecture we will introduce the group structure and force carriers of the weak force, using ideas developed today.

To motivate the Higgs mechanism, let's go back to quantum mechanics, and think about what our goals are. The goal of our understanding of a quantum system is to diagonalize the Hamiltonian; that is, we want to find the eigenenergies of a quantum system. with these eigenenergies, we can then fully classify the system, and calculate the time-evolution of an arbitrary state. I argue that the simplest, interesting quantum system is the harmonic oscillator. This is a quantum system with a quadratic potential:

$$V = \frac{k}{2}x^2 \quad \text{that looks like:}$$



Importantly, note that the minimum of the potential is at $x=0$. Therefore, to study this system, we can start with the wavefunction localized around $x=0$, and then perturb it to identify the energy levels.

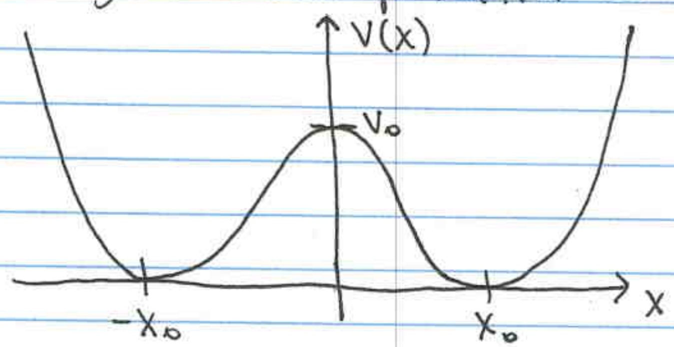
The way in which you may be familiar with doing this is by use of the raising and lowering ladder operators. Anyway you do it, you find that the eigenenergies are:

$$E_n = \hbar\omega(n+\tfrac{1}{2}),$$ where ω is the characteristic frequency of the harmonic oscillator, $\omega = \sqrt{\frac{k}{m}}$.

Okay, that was easy; let's make the system a bit more challenging. Let's consider the system with the potential:

$$V(x) = \frac{V_0}{x_0^4}\left(x^2 - x_0^2\right)^2$$

Here, V_0 is the value of the potential when $x=0$, and x_0 is some characteristic length/distance of the system. This potential looks like:



This is called a "double well" potential, for the obvious reasons. Note that, like the harmonic oscillator, it is symmetric in $x \to -x$: $V(x) = V(-x)$, and unlike the harmonic oscillator, has minima at $x = \pm x_0$, away from $x=0$. Now, to work to diagonalize the Hamiltonian, it doesn't make any sense to consider a wave function localized about $x=0$ and then perturb. Any wave function that is localized about $x=0$ (or, any ball placed ~~at~~ at or near $x=0$) will just roll down the potential, and land in one of the potential wells located at $x = x_0$ or $-x_0$. That is, the true ground state is described by a wavefunction localized at either ~~x=x~~ $x=x_0$ or $x=-x_0$, and not about $x=0$.

There's actually something weird about this, too. Unlike the harmonic oscillator, this potential has two, degenerate (= equal energy) minima. Expanding about either of these minima is an equally valid description of the (low) energy eigenvalues. Which one we pick (or is picked for us) is random. Note, however, that while the potential $V(x)$ is symmetric for $x \to -x$, if we pick, say, the minima located at $x=x_0$ to expand about, our description of the system is no longer symmetric in $x \to -x$. This is a manifestation of a phenomena called spontaneous symmetry breaking: our complete quantum system (in this case the potential $V(x)$) has a symmetry for $x \to -x$. However, our description of the ground state does not, because we happen to choose the minima at $x=x_0$ to expand about. By making this (random) choice, we have "spontaneously" broken the $x \to -x$ symmetry.

Let's see mathematically what this expansion about $x=x_0$ means. To expand about $x=x_0$, we will write

$x = x_0 + \delta x$, for some fluctuation position δx. With this

expansion, the potential becomes:

$$V(x) = \frac{V_0}{x_0^4}\left((x_0+\delta x)^2 - x_0^2\right)^2 = \frac{V_0}{x_0^4}\left(2x_0\,\delta x + (\delta x)^2\right)^2.$$

This expression for the potential no longer obviously has the $x \to -x$ symmetry. In particle physics, we typically say that the symmetry is no longer "manifest". However, it's still there, we just have to translate the $x \to -x$ symmetry to the fluctuation δx. For

$\delta x \to \delta x - 2x_0$, the potential becomes:

$$V(x) \to \frac{V_0}{x_0^4}\left(2x_0(\delta x - 2x_0) + (\delta x - 2x_0)^2\right)^2$$

$$= \frac{V_0}{x_0^4}\left(-2x_0\delta x + (\delta x)^2\right)^2$$, which corresponds

to expanding about the minima at $x = -x_0$.

One more thing to note about this potential is its expansion for small δx. In the limit where $\delta x \ll x_0$, we have:

$$V(x) = \frac{V_0}{x_0^4}\left(2x_0\delta x + (\delta x)^2\right)^2 = 4\frac{V_0}{x_0^2}(\delta x)^2 + \ldots$$

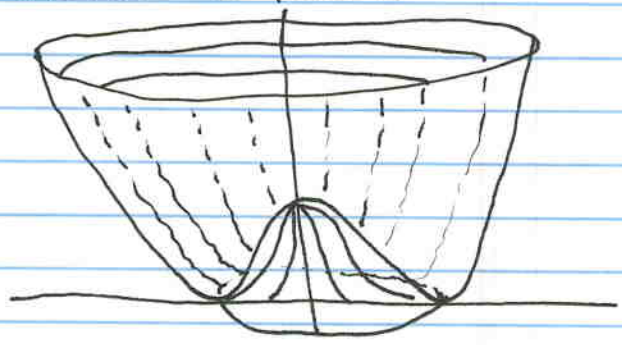
This is just a harmonic oscillator potential with $k = 8\frac{V_0}{x_0^2}$. The ground state energy in this minima is then:

$$E_0 = \frac{\hbar\omega}{2} = \frac{\hbar}{2}\frac{2\sqrt{2}}{x_0}\sqrt{\frac{V_0}{m}} = \frac{\sqrt{2}\hbar}{x_0}\sqrt{\frac{V_0}{m}}$$. Note that this is non-zero.

Okay, enough in 1D, let's move to 2D quantum systems. Let's take this double well potential in 1D and just rotate it about the vertical axis. In 2D this is

$$V(x,y) = \frac{V_0}{r_0^4}\left[(x^2+y^2) - r_0^2\right]^2$$

This looks like:



(Apologies for my drawing skills!)

Again, like the double-well potential, the minima of this potential is not located at the origin $x=y=0$, but displaced from the origin, when $x^2+y^2=r_0^2$. So, just like the double-well potential, we should expand the potential about the minima. To do this, note that the potential is radially symmetric, so we can re-express coordinates x and y in terms of r, the distance from the origin, and ϕ, the angle about the origin. That is:

$x = r\cos\phi$ and $y = r\sin\phi$. Then, the potential is:

$V(r,\phi) = \frac{V_0}{r_0^4}\left(r^2-r_0^2\right)^2$, with no dependence on ϕ!

One can then study this radial potential, expand about $r=r_0$, determine the ground state energy, etc., just like for the double-well. We won't do that exercise here. Instead, we will focus on the ϕ coordinate. In re-writing the potential in terms of r and ϕ, we had to choose a value of ϕ. Any choice of $\phi \in [0,2\pi]$ is equally valid in which to calculate energies, etc., but we picked a particular value. This is like what we did for the double-well: we had to pick (arbitrarily) one of the wells to expand in.

However, in this case, the angle coordinate is continuous: by choosing <u>some</u> ϕ, we spontaneously broke the continuous rotational symmetry of the system. This has extremely cool consequences. Let's consider the Schröding equation for the wave function in the r,ϕ coordinates, $\psi(r,\phi)$. The Schrödinger equation for this system is:

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)\psi + \frac{V_0}{r_0^4}\left(r^2-r_0^2\right)^2\psi = E\psi.$$

This can be solved by separation of variables, where $\psi(r,\phi) = \psi_r(r)\psi_\phi(\phi)$. Doing this, we find the two differential equations

$$-\frac{\hbar^2}{2m} r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)\psi_r + r^2\left(V(r) - E\right)\psi_r = \alpha \psi_r,$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \phi^2}\psi_\phi = +\alpha \psi_\phi. \qquad (*)$$

Again, the equation for ψ_r is what it is, and can be (approximatly) solved by expanding about the minimum of the potential $V(r)$. The equation for ψ_ϕ is much more interesting. In these expressions, α is some constant from separation of variables. In the ψ_ϕ equation, α has the interpretation as the energy of the state corresponding to ψ_ϕ! We have the boundary condition that $\psi_\phi(\phi) = \psi_\phi(\phi + 2\pi)$, and so the solutions to $(*)$ can be written as:

$$\psi_\phi(\phi) = \frac{1}{\sqrt{\pi}}\sin(n\phi), \text{ for } n = 1, 2, 3, \ldots$$
$$\text{and } \alpha = \frac{n^2\hbar^2}{2m}.$$

For $n > 0$, these wavefunctions correspond to an "energy" α that is non-zero; that is, it takes finite energy to excite the system to these states. However, because the angle $\phi \in [0, 2\pi]$, there is a normalizable solution for $\alpha = 0$. It is the true ground state:

$$\psi_{\phi_0}(\phi) = \frac{1}{\sqrt{2\pi}}.$$ This state therefore has 0 energy, and as such is "always" there. No amount of energy is needed to get to this state. This is a manifestation of a phenomena called Goldstone's Theorem.

Goldstone's theorem states that when a continuous symmetry is spontaneously broken (in this case, rotations about the origin of the potential) there exists 0-energy states in the spectrum of the Hamiltonian. This exactly corresponds to the $n=0$ state of the angular Hamiltonian. Note also the requirement of a continuous symmetry; this is why we didn't see 0 energy states in the double-well case.

All of this background was to prepare you for the situation in quantum field theory, and how this can be exploited to provide mass to spin-1 bosons. The analogous system of the 2D radial potential in quantum field theory is a complex spin-0 scalar field $\phi(x)$, with the Lagrangian:

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - \lambda(|\phi|^2 - v^2)^2$$

Here, the scalar potential is $V(\phi) = \lambda(|\phi|^2 - v^2)^2$. The value v is the minimum of the potential; that is, when $|\phi| = v$, where the potential is 0. v is called the "vacuum expectation value" or vev, because it is the state that the field ϕ will assume in the vacuum; that is, with no energy input. λ is the quartic coupling, and should be positive so that energy is always non-negative.

This Lagrangian as it stands is pretty weird. Let's expand out the potential:

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) + 2\lambda v^2 |\phi|^2 - \lambda |\phi|^4 - \lambda v^4.$$

Don't worry about the $|\phi|^4$ or λv^4 term for now; just focus on the $|\phi|^2$ term and the kinetic term. Varying this Lagrangian with respect to ϕ^* yields the Klein-Gordon equation:

$$\left(\partial_\mu \partial^\mu - 2\lambda v^2\right)\phi = 0$$, which has the corresponding

Energy-momentum relation in special relativity of:

$$E^2 - |\vec{p}|^2 = -2\lambda v^2 < 0!$$ Huh? A negative mass?

This is not good! As we argued in quantum mechanics this just means that you're expanding about the wrong point. We don't want to expand about $\phi = 0$, but rather about $|\phi| = v$.

To do this, let's write: $\phi(x) = \left(v + \frac{r(x)}{\sqrt{2}}\right) e^{i\frac{\theta(x)}{\sqrt{2}v}}$ for two real fields $r(x)$ and $\theta(x)$. Note that the conjugate field is:

$$\phi^*(x) = \left(v + \frac{r(x)}{\sqrt{2}}\right) e^{-i\frac{\theta(x)}{\sqrt{2}v}}.$$

Plugging these into the Lagrangian and simplifying, we find:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu r)(\partial^\mu r) + \tfrac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \frac{r}{\sqrt{2}v}(\partial_\mu \theta)(\partial^\mu \theta)$$

$$+ \frac{r^2}{4v^2}(\partial_\mu \theta)(\partial^\mu \theta) - \frac{\lambda}{4}\left(2\sqrt{2}vr + r^2\right)^2$$

Now, the potential is purely a function of the field r, with no θ field. Expanding in the small r field limit, where $r \ll v$, this Lagrangian reduces to:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu r)(\partial^\mu r) - 2\lambda v^2 r^2 + \tfrac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) + \ldots$$

That is, the field $r(x)$ has a mass of $m_r = 2\sqrt{\lambda}\, v_? > 0$, while the field ~~$\phi$~~ θ is massless! This is another manifestation of Goldstone's Theorem. That the field $\theta(x)$ is massless is a consequence of the fact that we had to spontaneously choose a value of θ, which broke the rotational symmetry. This is a continuous symmetry, and so we find that the field $\theta(x)$ is massless. That is, there is no minimum energy required to excite particles from the θ field. This is exactly like what we found in the quantum mechanical example. In this system, we would call the $\theta(x)$ field the "Goldstone Boson".

Okay, getting closer. Now for the main event. Let's see how we can use this property of spontaneous symmetry breaking to give a spin-1 boson a mass. In this lecture, we'll consider the simple case of giving the field ϕ and electric charge; therefore, we will show how to give the photon a mass. Next lecture, we'll then be able to apply this insight to the W and Z bosons.

To give the boson ϕ a charge and couple it to the photon, we just replace the partial derivative by the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \text{ , where } e \text{ is the electric}$$

charge and A_μ is the photon field. The Lagrangian then becomes:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)(D^\mu \phi^*)^* - \lambda\left(|\phi|^2 - v^2\right)^2 .$$

Recall that the field strength for electromagnetism is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

This Lagrangian is invariant under the $U(1)$ gauge transformations:

$$\phi \to e^{i\alpha(x)}\phi, \quad \phi^* \to e^{-i\alpha(x)}\phi^*, \quad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x).$$

Here, $\alpha(x)$ is an arbitrary function of spacetime coordinate x.

Now, like we did in the case when the scalar field was uncharged, we need to expand ϕ about the minimum of the potential, to ensure that all fields have non-negative mass-squared. Again, we write

$$\phi(x) = \left(v + \frac{h(x)}{\sqrt{2}}\right) e^{i\frac{\theta(x)}{\sqrt{2}v}}.$$

Unlike in the previous case, we can simplify the expression of $\phi(x)$, by exploiting gauge transformations! The field $\theta(x)$, the Goldstone boson, can be eliminated at the expense of picking a gauge; that is choosing a particular $\alpha(x)$. If we choose:

$$\alpha(x) = -\frac{\theta(x)}{\sqrt{2}v}, \text{ then we have}$$

$$\phi(x) = \left(v + \frac{h(x)}{\sqrt{2}}\right) e^{i\frac{\theta(x)}{\sqrt{2}v}} \to \left(v + \frac{h(x)}{\sqrt{2}}\right) e^{i\frac{\theta(x)}{\sqrt{2}v}} \cdot e^{-i\frac{\theta(x)}{\sqrt{2}v}}$$

$$= v + \frac{h(x)}{\sqrt{2}}$$

We have eliminated the Goldstone boson! However, by choosing and fixing a gauge, we seem to have broken the symmetries of electromagnetism. However, this breaking was just spontaneous, which means that the symmetries of E&M are still there, just not manifest. Thus, charge is still conserved, just not manifestly so when expanding in this potential well.

In this gauge, let's now evaluate the Lagrangian. We have:

$$\phi(x) = \phi^*(x) = v + \frac{h(x)}{\sqrt{2}} \quad \text{and so}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[\left(\partial_\mu - ieA_\mu\right)\left(v + \frac{h(x)}{\sqrt{2}}\right)\right]\left[\left(\partial^\mu + ieA^\mu\right)\left(v + \frac{h(x)}{\sqrt{2}}\right)\right]$$

$$- \lambda\left(\left(v + \frac{h(x)}{\sqrt{2}}\right)^2 - v^2\right)^2$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + 2e^2 v A_\mu A^\mu h + e^2 v^2 A_\mu A^\mu$$

$$-\frac{\lambda}{4}\left(2\sqrt{2}vh + h^2\right)^2$$

This Lagrangian is insane. We call the field h the Higgs field. Expanding its interactions to quadratic order, we have:

$$\mathcal{L}_h = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - 2\lambda v^2 h + \ldots \quad , \text{ and so the Higgs}$$

field has a mass of $m_h = 2v\sqrt{\lambda}$. The weirder part of this Lagrangian is that which involves the photon. Expanding the photon part to quadratic order, we find:

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2 v^2 A_\mu A^\mu + \ldots$$

The term $e^2 v^2 A_\mu A^\mu$ is a mass for the photon! What!?!

Apparently, the process of spontaneous symmetry breaking effectively gives a mass to the photon! This mass is:

$m_A = \sqrt{2}\, e^2 v^2$, and so is controlled by the electric charge and the vev.

Note also that this Lagrangian describes a photon with three degrees of freedom, that is, there are three components of A_μ that propagate. This is different from the case of a massless photon that only has left- and right-handed helicity. Where did this third degree of freedom ~~come~~ come from? It was the Goldstone field, $\theta(x)$! Because the photon acquired mass through the existence of the Goldstone field $\theta(x)$, we say that the photon "ate the Goldstone boson and became fat", that is, massive.

This is called the Higgs mechanism, and this particular application describes superconductivity in the BCS theory (if you are familiar with that). I emphasize that the Lagrangian with the massive Higgs boson field is still gauge invariant and so electric charge is still conserved. However, in the expansion about the vev this gauge symmetry is not manifest, and so the states of the system will not appear to conserve charge (i.e., the photon will have a non-zero mass).

So, we know how to give a spin 1 boson a mass that is consistent with the gauge invariance of the system! We will use this knowledge to provide the carriers of the weak force with a mass, that can account for all of the subtleties we identified with the V-A theory.