The W and Z Bosons Lecture 21

In the previous lecture, we introduced spontaneous symmetry breaking as the feature of the ground state of a system to break a symmetry of the theory. Applying spontaneous symmetry breaking to a gauge theory, coupling a scalar whose vacuum breaks the symmetry, is the Higgs mechanism. A gauge boson can aqquire a mass if it is coupled to a scalar field that acquires a vacuum expectation value, or vev.

Last lecture, we argued that if the weak force were to have a spin-1 boson force carrier then those force carriers necessarily are massive. This is a consequence of the fact that Fermi's constant is dimensionful. Massive force carriers are unfamiliar to our experience with electromagnetism or QCD, but with the Higgs mechanism, we are equipped to ~~westan~~ understand how this could work. In this lecture, we will use the Higgs mechanism to understand what the weak force carriers are.

To begin, we need to enumerate what the properties of the force carrier(s) of the weak force are and are not. Recall that decay of the neutron corresponds to the process:

$n \to p^+ + e^- + \bar{\nu}_e$ or, in terms of the fundamental quarks:

$d \to u + e^- + \bar{\nu}_e$. The interaction that describes this decay in the V-A model is:

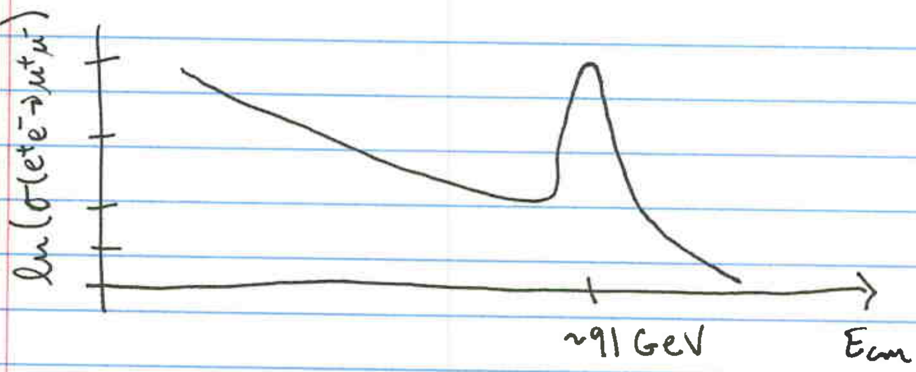
$$\mathcal{L}_{int} \supset \frac{4G_F}{\sqrt{2}}\left(\nu_L^\dagger \bar{\sigma}^\mu e_L\right)\left(d_L^\dagger \bar{\sigma}_\mu u_L\right)$$

This interaction tells us many things. First, it tells us that the force carrier must be colourless: electrons and neutrinos have no colour, and the two factors in the interaction are just multiplied. So, the colour of the up and down quarks does not flow outside of their interaction. Next, the force carrier must have electric charge. The sum of the charges of the neutrino and electron is -1 (in terms of the fundamental charge e) and the charge of the up and (anti) down quark is $+1$. (Thus the charge of this interaction is 0, which ensures that it is electromagnetically gauge invariant and conserves charge.) Therefore, whatever particle is mediating this interaction must have electric charge ± 1. We also already know that this mediating particle must be spin-1 and have a mass comparable to $1/\sqrt{G_F}$. For future reference, we will refer to this particle (until we give it a proper name) as the "charged current".

But wait, there's more. Let's go back to our analysis of e^+e^- scattering, say $e^+e^- \to \mu^+\mu^-$. At low energies, say $E^2_{cm}: Q^2 \simeq (1\,\text{GeV})^2$, this scattering is extremely well described by the interaction mediated by electromagnetism. The masslessness of the photon means that the energy dependence of the cross section is:

$$\sigma(e^+e^- \to \mu^+\mu^-) \propto \frac{1}{E^2_{cm}}.$$

However, we know that as we increase the center-of-mass collision energy this $1/E^2_{cm}$ dependence changes, and we observe a peak, or a resonance, in the cross section, as a function of E_{cm}. That is, we see something like:



This bump in the cross section clearly cannot be ascribed to the photon and electromagnetism, because there is nothing special about 91 GeV in Maxwell's equations. Thus, we call it a new particle, the Z <u>boson</u>. However, for now, we will refer to it as the "<u>neutral current</u>", as it must be neutral (like the photon) and is massive.

Additionally, the fact that both the photon and the neutral current mediate the scattering $e^+e^-\to\mu^+\mu^-$ suggests that they are related somehow. Because the photon and the neutral current are intermediate particles in $e^+e^-\to\mu^+\mu^-$ collisions, we must sum their contribution to the cross section at the matrix element or amplitude level. (There is, in principle, no measurement we can do to distinguish them.) So, apparently, the photon and neutral current mix into one another quantum mechanically. To do this, they must share (some) quantum numbers, and if they do, are somehow intimately related. More on this in a second.

There's another thing we can look for in an e^+e^- collider and that is the process $e^+e^-\to\nu_e\bar{\nu}_e$. In our detector, because it is incredibly hard to detect neutrinos, we would see electrons and positrons colliding and producing <u>nothing</u>. We can still measure the cross section for the observed process of $e^+e^-\to$ nothing as a function of the center of mass energy, E_{cm}.

This $e^+e^- \to \nu_e\bar{\nu}_e$ scattering can be mediated by the charged current. The interaction would be:

$$\mathcal{L} \supset \frac{4G_F}{\sqrt{2}}\left(\nu_L^\dagger \bar{\sigma}^\mu e_L\right)\left(\nu_L^\dagger \bar{\sigma}_\mu e_L\right), \text{ for example.}$$

That is, the charged current turns electrons into neutrinos. This interaction cannot be mediated by electromagnetism, because the neutrino is neutral. However, it may be mediated by the neutral current, if the neutral current couples to neutrinos. Indeed, one can measure the cross section for $e^+e^- \to$ nothing as a function of E_{cm}, and finds, as with $e^+e^- \to \mu^+\mu^-$, that there is a peak when $E_{cm} \sim 91$ GeV. This then suggests that the neutral and charged currents are related to one another!

So, these observations have told us that four bosons: the photon, the neutral current, the +1 charged current and the -1 charged current are related to one another. Apparently, three of these bosons (the neutral and charged currents) are massive, while the photon is massless. So, we need to develop a theory in which these bosons are related and three of them get mass via the Higgs mechanism. This fundamental theory is called the unified electroweak force, as it describes electromagnetism and the weak force in a unified manner. Let's see how this is done in the rest of this lecture.

Let's follow our noses to see if we can construct a sensible theory of this unified electroweak force. Because it has worked in the past (for electromagnetism and QCD), let's try to figure out what the gauge symmetry of this theory is. To do this, requires identifying the relationship between the gauge group and the number of force-carrying bosons. In our discussion of QCD, we identified the SU(3) colour symmetry. We argued that SU(3) has 8 basis matrices T^a, for $a=1,\ldots,8$. We determined this by considering how many free parameters there were in an SU(3) matrix. Again, a general 3x3 complex matrices has 18 real parameters. Enforcing the unitary constraint $U^\dagger U = \mathbb{1}$ fixes 9 parameters. Further enforcing the unit determinant is one more constraint. Therefore, there are 8 basis matrices of SU(3). For the group SU(N), there are in general N^2-1 basis matrices, by extension of this argument.

So, for the group SU(N), there are in general N^2-1 force-carrying bosons. To describe the electroweak force, we need 4. N^2-1 for integer N cannot be 4, so our approach that we used for QCD must be generalized. Note, however, that if we do not impose the unit determinant constraint, then we have the group U(N), which has a total of N^2 basis elements. $2^2=4$, and so the simplest group that has four gauge bosons is U(2), the group of unitary 2x2 matrices. This group can be equivalently expressed as $SU(2)\otimes U(1) \cong U(2)$, where $\otimes$ means direct product of the groups SU(2) and U(1). So, this is what we will work with as the electroweak gauge group. We call the SU(2) part "weak isospin" and the U(1) part "hypercharge". Let's now see how the weak isospin and hypercharge combine to produce the charged and neutral currents, and the photon.

Our first goal will be to spontaneously break this gauge symmetry down to just electromagnetism using the Higgs mechanism. We will call the gauge bosons of the weak isospin $SU(2)_W$ A^1_μ, A^2_μ, and A^3_μ, while the gauge boson of hypercharge $U(1)_Y$ will be B_μ. For the Higgs mechanism, we need to introduce the Higgs field that can do the symmetry breaking. This Higgs field must couple to all of the electroweak gauge bosons, and so must carry weak isospin and hypercharge. The simplest possibility is to have the Higgs field be in the fundamental representation of $SU(2)_W$, which is a two-component vector:

$$\vec{\varphi} = \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix},$$ where φ^+, and φ_0 are complex scalar fields. This Higgs field therefore has four degrees of freedom. It transforms under an $SU(2)_W \otimes U(1)_Y$ gauge transformation as:

$$\vec{\varphi} = \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix} \rightarrow e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} e^{i\beta/2} \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix},$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli sigma matrices (the basis matrices of the fundamental representation of $SU(2)$), and $\vec{\alpha}$, and β are parameters that determine the gauge transformation. The hypercharge of the Higgs is $+1/2$.

Just as we did with QCD, we need to introduce a covariant derivative that enables gauge-invariant interactions between the Higgs field and the electroweak gauge bosons A^i_μ and B_μ. This covariant derivative is:

$$D_\mu = \partial_\mu - i g_W \frac{\sigma_a}{2} A^a_\mu - i g_Y Y B_\mu.$$

In this expression, g_W and g_Y are the coupling constants of the weak isospin and the hypercharge, respectively, while Y is called the hypercharge.

It then follows that the gauge-invariant Lagrangian for the interactions of the Higgs boson with the weak isospin and hypercharge is:

$$\mathcal{L} \supset \left(D_\mu \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix}\right)^{\dagger} \left(D^\mu \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix}\right).$$

Now that we know the interactions of the Higgs field with the electroweak bosons, we can work to spontaneously break the symmetry. To do this we need to give the Higgs field a vev, which can be accomplishd by writing down a gauge-invariant potential. With the vacuum expectation value of $\frac{v}{\sqrt{2}}$ the potential is:

$$V(\vec{\varphi}) = \lambda\left(|\vec{\varphi}|^2 - \frac{v^2}{2}\right)^2.$$

Note that $|\vec{\varphi}|^2$ is gauge invariant. This is:

$$|\vec{\varphi}|^2 = \left((\varphi^+)^* \ (\varphi_0)^*\right)\begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix} = |\varphi^+|^2 + |\varphi_0|^2.$$

A gauge transformation can rotate the field components φ^+ and φ_0, but not change its magnitude.

So, we want to expand the potential about its minimum. To do this, we can write:

$$\vec{\varphi} = \begin{pmatrix} \pi^+(x) \\ \left(\frac{v}{\sqrt{2}} + \frac{h(x)}{\sqrt{2}}\right) e^{i\phi(x)} \end{pmatrix}$$

In this expression, I have expanded the upper component of the field $\vec{\varphi}$ about 0, while the lower component is expanded about $v/\sqrt{2}$. Just like we did last lecture, we can perform some $SU(2)_w \otimes U(1)_Y$ gauge transformation to eliminate the three real fields corresponding to $\pi^+(x)$ (which is complex, and so has two real components) and $\phi(x)$. By eliminating three of these fields, this will have the effect of giving mass to three linear combinations of the weak isospin and hypercharge gauge bosons. The one scalar field that is left over, $h(x)$, is called the Higgs boson.

Let's see how these gauge bosons get a mass. With the gauge choices that eliminate $\pi^+(x)$ and $\phi(x)$, the Higgs field about the vacuum is:

$$\vec{\varphi} \rightarrow \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}$$

To determine the masses of the gauge bosons, it is sufficient to set $h(x)=0$ for now. Then, the masses induced by the covariant derivative are:

$$\mathcal{L}_{M^2} \simeq \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} \end{pmatrix} \left(-ig_w \frac{\sigma_a}{2} A^a_\mu - i\frac{g_Y}{2} B_\mu \right) \left(ig_w \frac{\sigma_b}{2} A^{\mu b} + i\frac{g_Y}{2} B^\mu \right) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

The matrices of gauge bosons can be explicitly written in components as:

$$\frac{g_w}{2}\sigma_a A^a_\mu + \frac{g_Y}{2} B_\mu = \begin{pmatrix} \frac{g_w}{2}A^3_\mu + \frac{g_Y}{2}B_\mu & \frac{g_w}{2}A^1_\mu - i\frac{g_w}{2}A^2_\mu \\ \frac{g_w}{2}A^1_\mu + i\frac{g_w}{2}A^2_\mu & -\frac{g_w}{2}A^3_\mu + \frac{g_Y}{2}B_\mu \end{pmatrix}$$

To evaluate this, I have just expanded out the sigma matrix product $\sigma_a A^a_\mu$ and multiplied B_μ by a 2×2 unit matrix.

Then, multiplying out the matrices, we have:

$$\begin{pmatrix} 0 & \frac{v}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{g_W}{2} A_\mu^3 + \frac{g_Y}{2} B_\mu & \frac{g_W}{2}(A_\mu^1 - iA_\mu^2) \\ \frac{g_W}{2}(A_\mu^1 + iA_\mu^2) & -\frac{g_W}{2} A_\mu^3 + \frac{g_Y}{2} B_\mu \end{pmatrix}$$

$$= \begin{pmatrix} \frac{v}{\sqrt{2}}(A_\mu^1 + iA_\mu^2)\frac{g_W}{2} & \frac{v}{\sqrt{2}}\left(-\frac{g_W}{2}A_\mu^3 + \frac{g_Y}{2}B_\mu\right) \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{g_W}{2} A_\mu^3 + \frac{g_Y}{2} B_\mu & \frac{g_W}{2}(A_\mu^1 - iA_\mu^2) \\ \frac{g_W}{2}(A_\mu^1 + iA_\mu^2) & -\frac{g_W}{2} A_\mu^3 + \frac{g_Y}{2} B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{v}{\sqrt{2}} \frac{g_W}{2}(A_\mu^1 - iA_\mu^2) \\ \frac{v}{\sqrt{2}}\left(-\frac{g_W}{2}A_\mu^3 + \frac{g_Y}{2}B_\mu\right) \end{pmatrix}$$

So then, we have

$$\mathcal{L}_{m^2} = \begin{pmatrix} \frac{v}{\sqrt{2}}(A_\mu^1 + iA_\mu^2)\frac{g_W}{2} & \frac{v}{\sqrt{2}}\left(-\frac{g_W}{2}A_\mu^3 + \frac{g_Y}{2}B_\mu\right) \end{pmatrix} \begin{pmatrix} \frac{v}{\sqrt{2}} \frac{g_W}{2}(A_\mu^1 - iA_\mu^2) \\ \frac{v}{\sqrt{2}}\left(-\frac{g_W}{2}A_\mu^3 + \frac{g_Y}{2}B_\mu\right) \end{pmatrix}$$

$$= \frac{v^2}{2}\frac{g_W^2}{4}\left((A_\mu^1)^2 + (A_\mu^2)^2\right) + \frac{v^2}{2}\left(-\frac{g_W}{2}A_\mu^3 + \frac{g_Y}{2}B_\mu\right)^2$$

Awesome! This then tells us which particles have a mass! We call the linear combinations of the weak isospin fields:

$$A_\mu^1 + iA_\mu^2 \equiv W_\mu^+, \quad A_\mu^1 - iA_\mu^2 \equiv W_\mu^-,$$

the positive and negative charged W bosons. Their mass term in the Lagrangian is:

$$\mathcal{L}_{m^2} = \frac{1}{2}\left(\frac{g_W v}{2}\right)^2 W_\mu^+ W^{\mu -},$$ with a mass of

$$m_W = \frac{g_W v}{2}.$$

The other linear combination of weak isospin and hypercharge gauge fields that gets a non-zero mass is:

$$+\frac{g_W}{\sqrt{g_W^2+g_Y^2}} A_\mu^3 - \frac{g_Y}{\sqrt{g_W^2+g_Y^2}} B_\mu \equiv Z_\mu ,$$

which is called the Z boson. Its mass is

$$\mathcal{L}_{m^2} = \frac{1}{2}\frac{v^2}{4}(g_W^2+g_Y^2)\, Z_\mu Z^\mu \Rightarrow m_Z = \frac{v\sqrt{g_W^2+g_Y^2}}{2} .$$

The W and Z bosons correspond to the charged and neutral currents, respectively. The linear combination of field that does not get a mass via the Higgs mechanism is

$$\frac{g_Y}{\sqrt{g_W^2+g_Y^2}} A_\mu^3 + \frac{g_W}{\sqrt{g_W^2+g_Y^2}} B_\mu \equiv A_\mu ,$$

which is called the photon of electromagnetism! Extremely importantly, note that because the photon is still massless, it has a corresponding manifest gauge symmetry. This in turn implies that electric charge is (still!) conserved.

This theory with W, Z, and photon bosons is called the "Broken" theory, as the W and Z bosons are massive. Interactions between particles of the standard model with the electroweak bosons are still implemented via the covariant derivative (just like in QCD or electromagnetism). However, it is useful to express the covariant derivative in terms of the W, Z, and photons by just inverting their linear combinations. The weak isospin and hypercharge gauge bosons are thus:

$$A^1_\mu = \frac{W^+_\mu + W^-_\mu}{2}, \quad A^2_\mu = \frac{-iW^+_\mu + iW^-_\mu}{2}$$

$$A^3_\mu = \frac{g_W}{\sqrt{g_W^2+g_Y^2}} Z_\mu + \frac{g_Y}{\sqrt{g_W^2+g_Y^2}} A_\mu$$

$$B_\mu = -\frac{g_Y}{\sqrt{g_W^2+g_Y^2}} Z_\mu + \frac{g_W}{\sqrt{g_W^2+g_Y^2}} A_\mu$$

The covariant derivative is therefore:

$$D_\mu = \partial_\mu - i\frac{g_W}{2}\left(\frac{\sigma_1 - i\sigma_2}{2}\right)W^+_\mu - i\frac{g_W}{2}\left(\frac{\sigma_1 + i\sigma_2}{2}\right)W^-_\mu$$

$$- i\left(I_3 \frac{g_W^2}{\sqrt{g_W^2+g_Y^2}} - Y\frac{g_Y^2}{\sqrt{g_W^2+g_Y^2}}\right)Z_\mu$$

$$- i\left(I_3 + Y\right)\frac{g_W g_Y}{\sqrt{g_W^2+g_Y^2}} A_\mu$$

If A_μ is to be the photon, then the coefficient in the covariant derivative must be the electric charge of the particle to which it couples. The fundamental unit of electric charge e is:

$$e = \frac{g_W g_Y}{\sqrt{g_W^2+g_Y^2}},$$ while the charge of the particle is

$Q = I_3 + Y$, where I_3 is the third component of weak isospin of the particle, and Y is the hypercharge.

To summarize, we have explained the four electroweak boson's relations and developed a method for 3 of them to get masses. This $SU(2)_w \otimes U(1)_Y$ electroweak theory has three parameters: the value of the Higgs vev v, and the two coupling constants g_w and g_Y. There are extensive relations between them. Explicitly, we know of 4 relations already that over constrain the three parameters of the electroweak theory. Namely:

$$m_W = \frac{g_w v}{2}, \quad m_Z = \frac{v\sqrt{g_w^2+g_Y^2}}{2}$$

$$e = \frac{g_w g_Y}{\sqrt{g_w^2+g_Y^2}}, \quad G_F = \frac{\sqrt{2}\, g_w^2}{8 m_W^2} = \frac{1}{\sqrt{2}\, v^2}.$$

Note that the relationship with the Fermi constant follows from identifying the W boson as the charged current. From these relations it follows that:

$$v = 246 \text{ GeV}, \quad g_w = 0.65, \quad g_Y = 0.35.$$

The fact that these over constrained relationships yield a consistent result is a huge test for the electroweak theory.

By the way, the mixing between the A_μ^3 and B_μ bosons to produce the Z_μ and A_μ bosons is typically characterised by the sine of the weak mixing angle:

$$\sin\theta_W = \frac{g_Y}{\sqrt{g_w^2+g_Y^2}} = 0.47.$$