-HW
-Typos

Relativistic Wave Equations Lecture 4

Last lecture, we introduced relativistic wave equations, with a motivation from the Schrödinger equation. From the relativistic energy-momentum relation

$$E^2 - \vec{p}^{\,2} - m^2 = 0$$

and making the quantum mechanical identifications

$$E \leftrightarrow +i\hbar\frac{\partial}{\partial t}, \quad \vec{p} \leftrightarrow -i\hbar\vec{\nabla}, \quad \text{(with explicit } \hbar\text{)}$$

we find the following wave equation for ϕ:

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi = 0 \quad (\text{with } \hbar = c = 1)$$

This wave equation is invariant to Lorentz transformations and has the solution

$$\phi(\vec{x},t) = e^{-ip\cdot x}, \text{ where}$$

$$p\cdot x = Et - \vec{p}\cdot\vec{x}, \text{ the Lorentz-invariant four-vector}$$

dot product of momentum and position. This is indeed Lorentz-invariant and E must satisfy

$$E^2 = m^2 + \vec{p}^{\,2}. \text{ There are two solutions}$$

$$E = \pm\sqrt{m^2 + \vec{p}^{\,2}}, \text{ because the wave equation is second-order.}$$

The existence of negative energy solutions is weird, and unfamiliar for free-particles from the Schrödinger equation. We'll address this in a bit.

Now, I want to discuss a powerful and standard formulation of particle physics via a Lagrangian. Let's look back at the Klein-Gordon equation, re-written in a suggestive form:

$$\frac{\partial^2}{\partial t^2}\phi = \vec{\nabla}^2\phi - m^2\phi \qquad (*)$$

This is a second-order in time differential equation for a field ϕ. As a field, $\phi(\vec{x},t)$ permeates space and time and its configuration must satisfy $(*)$. Contrast this with a particle: a particle is defined by a position $x(t)$; that is, a particle is a a unique location x at time t. In classical mechanics, the equation that governs $x(t)$ would be Newton's second law:

$$\frac{d^2}{dt^2}x(t) = -\frac{\partial U}{\partial x},$$

where $U(x)$ is the potential energy. Equation $(*)$ looks just like this, though for a field! We have the second time derivative piece:

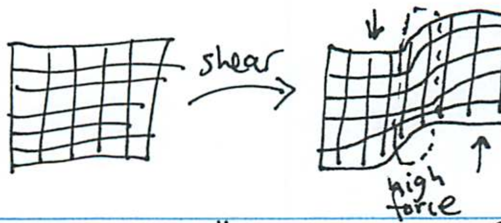
$$\frac{\partial^2}{\partial t^2}\phi,$$ which is like the acceleration for a particle.

Now look at the right side of $(*)$: $\vec{\nabla}^2\phi - m^2\phi$. The effective "force" from the mass:

$$-m^2\phi$$

is just like the restoring force for a harmonic oscillator with spring constant $k = m^2$. The gradient term

$$\vec{\nabla}^2\phi$$



is a "shear force": if the difference between values of ϕ at nearby spatial points are large, then effectively there is a large force. This makes sense: If you shear fabric, then there is a force, that is, the fabric becomes warped.

Okay, we now have an intuition for what this wave equation is telling us: it is just Newton's law for a field that experiences shear forces in a harmonic oscillator potential! We can integrate the force to determine the potential energy:

$$-\frac{\partial U}{\partial \phi} = \vec{\nabla}^2\phi - m^2\phi \Rightarrow U = +\frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{m^2}{2}\phi^2,$$

where I have ignored constant-energy contributions as they can be eliminated by appropriate relabeling. So that's the potential energy. The kinetic energy is found from the generalization of

$$\frac{1}{2}\dot{x}^2 \quad \text{to fields.}$$

This ~~is~~ is: $K = \frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2$.

(By the way, to see that $U > \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi)$, we can take a derivative by the finite difference method:

$$\frac{\partial U}{\partial \phi} = \lim_{\varepsilon \to 0} \frac{U(\phi+\varepsilon) - U(\phi)}{\varepsilon} = \frac{1}{2\varepsilon}\left(\vec{\nabla}(\phi+\varepsilon)\cdot\vec{\nabla}(\phi+\varepsilon) - \vec{\nabla}\phi\cdot\vec{\nabla}\phi\right)$$

$$= \frac{1}{\varepsilon}\left(\vec{\nabla}\varepsilon\,\vec{\nabla}\phi + \cancelto{0}{(\vec{\nabla}\varepsilon)(\vec{\nabla}\cdot\varepsilon)}\right) = \frac{1}{\varepsilon}\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi) - \vec{\nabla}^2\phi.$$

We can safely ignore the $\vec{\nabla}\cdot(\varepsilon\vec{\nabla}\phi)$ term, called a "total derivative", because we always must integrate over all space and this just contributes a constant.)

Then, the total energy of the field at the point (x,t) is:

$$K+U = \frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{m^2}{2}\phi^2$$

The total energy in the field ϕ requires integrating over all space; i.e., summing over all energy contributions at time t:

$$H = \int d\vec{x}\left[\frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{m^2}{2}\phi^2\right]$$

The integral extends over all space in the x, y, and z directions. The integration measure $d\vec{x} = dx\,dy\,dz$ I will often denote as $d^3x = d\vec{x} = dx\,dy\,dz$. With this total energy or Hamiltonian, we can formulate the relativistic field equations in a totally different, and ultimately more useful, way.

The classical mechanics of a point particle can be re-formulated with a Lagrangian and the principle of least action. The Lagrangian is the difference of the kinetic and potential energy of a particle with trajectory $x(t)$:

$$L = \frac{1}{2}\dot{x}^2 - U(x)$$

The action $S[x(t)]$ is defined as a function of the function of time $x(t)$, or a functional, of the Lagrangian integrated over time:

$$S[x(t)] = \int dt\left[\frac{1}{2}\dot{x}^2 - U(x)\right]$$

Newton's 2^{nd} law is a consequence of the principle of least action: classical trajectories $x(t)$ are those that minimize the action. Minimizing the action (taking the derivative of $S[x(t)]$ ~~w.r.t.~~ with respect to $x(t)$ and setting it to 0) one finds that $x(t)$ must satisfy:

$$\ddot{x} = -\frac{\partial U}{\partial x},$$ exactly Newton's second law.

We can formulate the Lagrangian and action for the relativistic field ϕ in an exactly analogous way. The Lagrangian L is the difference of the total kinetic and potential energies:

$$L = K - U = \int d^3x \left[\frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 - \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) - \frac{m^2}{2}\phi^2 \right]$$

The action is then the time integral of the Lagrangian:

$$S[\phi(x,t)] = \int dt\, d^3x \left[\frac{1}{2}\left(\frac{\partial}{\partial t}\phi\right)^2 - \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) - \frac{m^2}{2}\phi^2 \right]$$

The wave equation that we started with, the Klein-Gordon equation, can be found by minimizing this action with respect to $\phi(x,t)$.

We can massage this into a nice form. We will denote

$dt\, d^3x = d^4x$, the relativistically-invariant

integration measure. Further, we can form the four-vector

$\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$ from the time and space derivatives.

Then, we can nicely express the action as:

$$S[\phi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \right].$$

This is "manifestly" Lorentz invariant: all four-vectors are only present in dot products. The object

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2$$

is called the Lagrangian density, or often just the Lagrangian. Given this Lagrangian, we have all of the information of the wave equation (and actually more!).

While this was a long, roundabout way to get there, this now tells us <u>how</u> to construct in general descriptions of relativistic, quantum mechanical systems. We just need to be able to write down the Lorentz-invariant Lagrangian, and we are "done"! I put done in quotes because much of the rest of this class will be devoted to attempting to unpackage the information in the Lagrangian.

Break //

~~While I won't go into as much detail, we know of another physical system that is relativistically invariant: electromagnetism. I'll just sketch the construction of the Lagrangian of E&M here; more details are in the textbook. The Lagrangian of E&M will to be the basis for our discussion of any of the fundamental forces of the Standard Model, so I want to provide some baseline here.~~

With the Lagrangian for the Klein-Gordon equation, I want to discuss two more relativistic systems that will be the basis for our study in this class. One, electromagnetism, I will very briefly remind about; more is in the textbook. The second, that we will discuss now, is the Dirac equation. The Klein-Gordon equation is a second-order, relativistically invariant wave equation. Is it possible to construct a first order relativistically invariant wave equation? Indeed it is, and is necessary to describe spin 1/2 particles like the electron.

Let's assume that there exists a linear wave equation in time and spatial derivatives, for a field ψ. Let's write this as

$$\left(\alpha \frac{\partial}{\partial t} + \vec{\beta} \cdot \vec{\nabla} + \gamma m\right) \psi = 0$$

for some constants α, γ and a constant vector $\vec{\beta}$. If this is relativistically-invariant, it must imply the Klein-Gordon equation. How can it do this? We can act from the left by the complex conjugate of the differential operator:

$$\left(\alpha^* \frac{\partial}{\partial t} + \vec{\beta}^* \cdot \vec{\nabla} + \gamma^* m\right)$$

Why the complex conjugate? The Klein-Gordon equation (or the relativistic energy-momentum relation) is quadratic and has − signs in it. When square-roots are taken, this can produce imaginary numbers, and we want to make sure that the equation is real. Then, we consider

$$\left(\alpha^* \frac{\partial}{\partial t} + \vec{\beta}^* \cdot \vec{\nabla} + \gamma^* m\right)\left(\alpha \frac{\partial}{\partial t} + \vec{\beta} \cdot \vec{\nabla} + \gamma m\right) \psi = 0$$

Or, expanded out

$$\left[\alpha^*\alpha\frac{\partial^2}{\partial t^2}+\alpha^*\vec{\beta}\cdot\vec{\nabla}\frac{\partial}{\partial t}+\vec{\beta}^*\cdot\vec{\nabla}\alpha\frac{\partial}{\partial t}+(\vec{\beta}^*\cdot\vec{\nabla})(\vec{\beta}\cdot\vec{\nabla})\right.$$
$$\left.+\left(\alpha^*\frac{\partial}{\partial t}+\vec{\beta}^*\cdot\vec{\nabla}\right)\gamma m+\gamma^* m\left(\alpha\frac{\partial}{\partial t}+\vec{\beta}\cdot\vec{\nabla}\right)+\gamma^*\gamma m^2\right]\psi=0$$

For this to produce the Klein-Gordon equation,

$$\left[\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2+m^2\right]\psi=0,$$

we have: $\alpha^*\alpha=1$, $\gamma^*\gamma=1$, $\beta_i^*\beta_j=-\delta_{ij}$

$$\alpha=-\alpha^*,\quad \vec{\beta}=-\vec{\beta}^*$$

and $\alpha^*\beta_i+\beta_i^*\alpha=0=\alpha\beta_i+\beta_i\alpha$

The requirements $\alpha^*\alpha=1$, $\alpha=\alpha^*$, $\vec{\beta}=-\vec{\beta}^*$ are simple, an can be satisfied by complex numbers. The requirement

$\beta_i^*\beta_j=-\delta_{ij}$ is weird, and cannot be satisfied

by the β_i being just numbers! Similarly, $\alpha\beta_i+\beta_i\alpha=0$ cannot be satisfied by numbers. However, let's just keep going and see where it takes us.

We can choose $\gamma=1$ and let's denote $\alpha=i\gamma_0$, $\beta_i=i\gamma_i$, for $i=1,2,3$ and some things $\gamma_0,\gamma_1,\gamma_2,\gamma_3$. Then, these things satisfy:

$$\gamma_0\gamma_0=1,\quad \gamma_i\gamma_j=-\delta_{ij},\quad \gamma_0\gamma_i+\gamma_i\gamma_0=0$$

That is, $$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=\begin{pmatrix}2&0&0&0\\0&-2&0&0\\0&0&-2&0\\0&0&0&-2\end{pmatrix}=2\eta_{\mu\nu}$$

We can denote this with curly braces: $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.
The curly braces $\{,\}$ denote the anti-commutator. With this notation, our linear equation is

$$\left(i\gamma_0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = 0$$, or, with the derivative

four-vector ∂_μ, $\quad \left(i\gamma_\mu \partial^\mu - m\right)\psi = 0.$

Then, the Klein-Gordon equation is

$$\left(-i\gamma_\mu\partial^\mu - m\right)\left(i\gamma_\mu\partial^\mu - m\right)\psi = -\left(\partial\cdot\partial + m^2\right)\psi = 0 \checkmark$$

So, what are these γ_μ things? It turns out that the γ_μ are matrices; the smallest matrices that satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ are 4×4. One set of matrices that satisfy this requirement is

$$\gamma_0 = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & -1 & 0 \\ 0 & & 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where σ_i are the Pauli spinor matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Because this linear equations have Pauli spin matrices, this linear equation describes spin 1/2 particles, like the electron. This is called the Dirac equation. Because the γ_μ are matrices, the solution ψ will be a vector; properly, it is a four-component spinor.

Let's look at the solution of the Dirac equation and its interpretation. To solve the Dirac equation we write the solution as

$$\psi = u\, e^{-ip\cdot x}, \text{ where } u \text{ is a four-compentent spinor.}$$

We would like to find the u that works. Plugging this solution into the Dirac equation, we have

$$(i\gamma_\mu \partial^\mu - m)\psi = (i\gamma_\mu(-ip^\mu) - m)\, u e^{-ip\cdot x} = 0, \text{ or}$$

that $(\gamma\cdot p - m)\, u = 0.$

To solve this, we can go to a frame (for $m\neq 0$) where $p=(E,0,0,0)$. Then, the matrix

$$\gamma\cdot p + m = \begin{pmatrix} E & & & \\ & E & & \\ & & -E & \\ & & & -E \end{pmatrix} - \begin{pmatrix} m & & & \\ & m & & \\ & & m & \\ & & & m \end{pmatrix} = \begin{pmatrix} E-m & & & \\ & E-m & & \\ & & -(E+m) & \\ & & & -(E+m) \end{pmatrix}$$

Then, u satisfies

$$\begin{pmatrix} E-m & & & \\ & E-m & & 0 \\ 0 & & -(E+m) & \\ & & & -(E+m) \end{pmatrix} u = 0$$

There are two possibilities: Either $E=m$ or $E=-m$.
If $E=m$, then

$$\psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix} e^{-imt}, \text{ where } \xi \text{ is a two-component spinor.}$$

If $E=-m$, then $\psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix} e^{imt}$, where η is a two-component spinor.

These two-component spinors ξ and η describe the spin of an electron, or whatever spin ½ particle we're considering. The positive energy solution describes a spin ½ particle, like an electron. The negative energy solution describes an anti-particle, like the positron. Anti-particles have opposite charges from one another (an electron has charge -1, while a positron has charge +1) and opposite spin. When Dirac discovered his equation, he found these two ~~equations~~ solutions and therefore predicted the existence of the positron. Paul Dirac, a British physicist, found his equation in 1928 and the positron, a particle with spin-½, positive charge and mass of 511 KeV was discovered in 1932 by Carl Anderson.

Paul Dirac was perhaps the oddest of theoretical physicists, a group that is synonymous with aloof, absent-minded, and out of touch. Dirac was famously awkward in conversation. He married Margit Wigner, the sister of Eugene Wigner, another famous theoretical physicist. He would sometimes introduce his wife to guests at his house saying, "This is Wigner's sister." When he met Feynman at a conference, his first words were, "I have an equation. Do you have one too?" Dirac was also an atheist and, when he spoke, often critical of religion, especially for his comments at the 1927 Solvay conference. Wolfgang Pauli's thoughts on Dirac's comments summarize his personality well: Pauli said "There is no God and Paul Dirac is his prophet."