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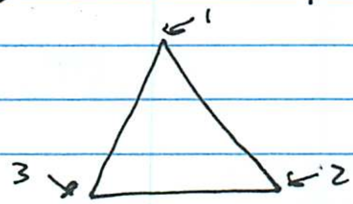
A Little Group Theory Lecture 5

Here is something I cut out of cardboard. You might recognize it; it's a triangle. There's nothing identifying about this triangle, but it is special: it is an equilateral triangle:



You might be wondering "What does a triangle have to do with particle physics," and rightly so. I'm holding this triangle with a tip pointed upward; call this vertex 1. You will have to remember this, I am forbidden from writing on the triangle. Going ~~[illegible]~~ clockwise (according to you), call the next vertex 2, and the final vertex 3. Again, you will have to remember this.

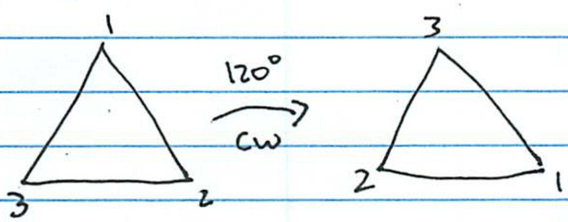
Now for the magic. In a second, I will ask you to close your eyes. When your eyes are closed, I will move the vertices around in some way. You will then open your eyes and I'll ask if you now know where vertex 1 is. Ready? Close your eyes. . . . Open them. Where is vertex 1? Can you tell? I'm holding the triangle in the same way as before you closed your eyes, and the triangle is equilateral, so there is no way that you could know (unless the triangle ~~[illegible]~~ had identifying marks otherwise). Anything that I did while your eyes were closed just permuted the vertices. Operations that leave an object or system unchanged, like permutation of ~~[illegible]~~ vertices of an equilateral triangle, is called a symmetry. Why are symmetries special? A symmetry of a physical system means that that system has a conservation law, by Noether's theorem.

Remember Noether's theorem? In words, it is that if the action that describes a system has a symmetry, then that system has a corresponding conservation law, and vice-versa. As a diagram, we might write:

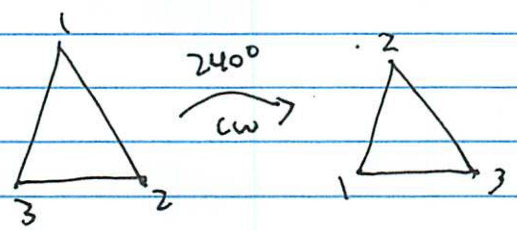
Symmetry $\Leftrightarrow$ Conservation law

So, what are these symmetries, and can we characterize them mathematically? Let's go back to the triangle and think about what we could do to it that makes it look the same before and after acting with a symmetry transformation.

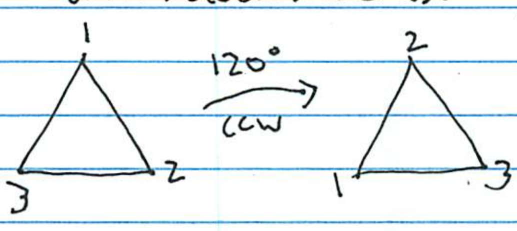
One thing we can do is, in fact, nothing at all. Another is rotation by 120° clockwise:



Another is rotation by 240°:



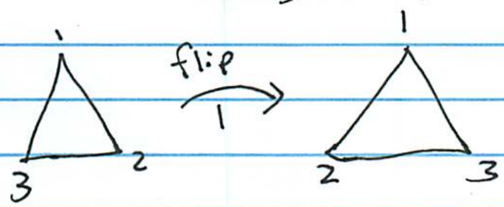
We could also rotate by 360°, but this is the same as not doing anything. Additionally, we could rotate by 120° or 240° counterclockwise as:



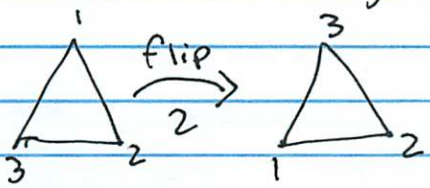
but, this is identical to the 240° cw rotation. Therefore, there are only 3 possible rotations. Is that all we can do?

(The answer to every rhetorical yes/no question is "No"; see Hinchliffe's rule.)

There's nothing that distinguishes the back from the front of the triangle, so we can also flip the triangle about a vertex and nothing changes! Let's flip about vertex 1:



What if we flip about 1 again? This goes back to the original orientation; "nothing" was done. We can flip about any vertex:



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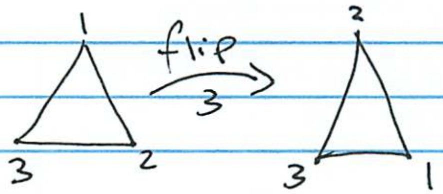
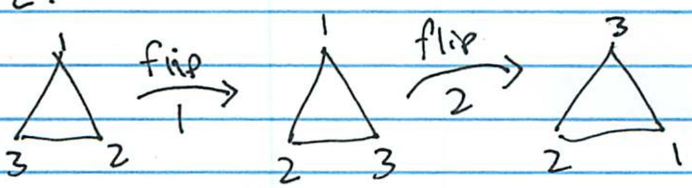


Is there anything else we can do? What if we do two different flips consecutively? Let's flip about 1 and then 2:



But this is just identical to a 120° cw rotation! One can try this with any two operations we have listed. The composition (subsequent application) of two symmetries of a triangle is still a symmetry. These 6 symmetries that we identified are everything.

You're welcome to play with this more, but I think this is sufficient to illustrate that the action of these symmetries has a rich mathematical structure. In fact, the action of these symmetries on the triangle form a mathematical object called a <u>group</u>. A group is a set of actions (like $120°$ rotation) that multiply (or compose) with an operation denoted by $\cdot$. The set of these actions satisfies the four properties:

1) In the group, there is an identity element, 1. For any element a in the group,

$$a \cdot 1 = 1 \cdot a = a$$

2) Every element of the group has an inverse. For an element a, we denote its inverse as a^{-1} and it satisfies

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

3) The group is closed. For any two elements a, b in the group, their product c is in the group:

$$a \cdot b = c \text{ is in the group.}$$

4) The multiplication/composition operation is associative. For three elements in the group a, b, c, the association of terms in a product is irrelevant:

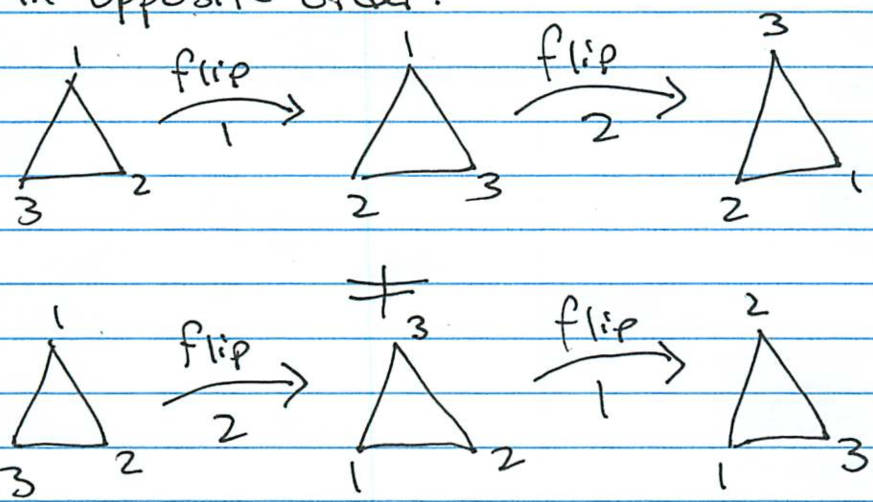
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

These four properties define a group. It's a very useful exercise to verify that the symmetries of the equilateral triangle form a group, though we won't do that here.

One thing I want to note here is that while these properties seem familiar from typical multiplication of numbers (which forms a group), there is a very important property that is <u>not</u> implied by the group requirements. It is not required that a group be commutative: for two elements of the group a, b we do not require

$$a \cdot b = b \cdot a$$

For groups in which all elements do commute (like multiplication with numbers) we say that the group is Abelian. For groups where this is not true, the group is called non-Abelian. The symmetries of the triangle are non-Abelian. Two different flips result in a different orientation if they are applied in opposite order:



While we will consider and study Abelian groups in this class, most symmetries in particle physics are implemented by non-Abelian groups. To get a better intuition for groups, in the rest of this lecture, we will study the group of rotations in two- and three dimensions. We've already seen this group in our earlier discussion of the construction of the dot product.

// Break

Recall that a matrix $\mathbb{M}$ that rotates a vector $\vec{v}$ leaves the identity matrix invariant:

$$\mathbb{M}^T \mathbb{1} \mathbb{M} = \mathbb{1}$$

The vector $\vec{v}$ is rotated as: $\vec{v} \to \mathbb{M}\vec{v}$. The set of matrices that leave the identity invariant form a group. To see this, we just check all four properties. First, note that the identity matrix $\mathbb{1}$ is an element of the group:

$$\mathbb{1}^T \mathbb{1} \mathbb{1} = \mathbb{1} \checkmark$$

The matrix $\mathbb{1}$ multiplied by any rotation matrix $\mathbb{M}$ is just $\mathbb{M}$.

The rotation matrices have an inverse: $\mathbb{M}^T \mathbb{M} = \mathbb{1}$ and so $\mathbb{M}^{-1} = \mathbb{M}^T$. Matrix multiplication is associative: for three matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}$, we have

$$\mathbb{A}(\mathbb{B}\mathbb{C}) = (\mathbb{A}\mathbb{B})\mathbb{C}$$

Finally, the group is closed. Let $\mathbb{M}$ and $\mathbb{N}$ be two rotation matrices:

$$\mathbb{M}^T \mathbb{1} \mathbb{M} = \mathbb{1}, \quad \mathbb{N}^T \mathbb{1} \mathbb{N} = \mathbb{1}$$

Then, consider the matrix $\mathbb{M}\mathbb{N}$. We have;

$$(\mathbb{M}\mathbb{N})^T = \mathbb{N}^T \mathbb{M}^T \text{ and so}$$

$$(\mathbb{M}\mathbb{N})^T \mathbb{1} (\mathbb{M}\mathbb{N}) = \mathbb{N}^T \mathbb{M}^T \mathbb{1} \mathbb{M} \mathbb{N} = \mathbb{N}^T \mathbb{1} \mathbb{N} = \mathbb{1} \checkmark$$

Let's see how this works in some explicit examples. First, let's consider rotations in two-dimensions; that is, those 2×2 matrices $\mathbb{M}$ that satisfy

$$\mathbb{M}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let's figure out some properties of this group. First, note that

$$\det(\mathbb{M}^T \mathbb{M}) = \det \mathbb{M}^T \cdot \det \mathbb{M} = 1, \text{ and so}$$

$\det \mathbb{M} = \pm 1$. What we typically think of rotations are matrices with determinant 1. A matrix with a determinant of -1 flips entries of a vector. For example, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ flips } v_1 \leftrightarrow v_2 \text{ when acting on } \vec{v},$$

while the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ turns $v_1 \to v_1$, $v_2 \to -v_2$.

We will discuss it in more detail later, but these matrices with determinant of -1 are examples of parity transformations: they flip entire axes, without rotation. Standard vector rotation is implemented by matrices that have $\det \mathbb{M} = 1$. Note that this restriction is a group: two matrices $\mathbb{M}$ and $\mathbb{N}$ each with determinant 1 have a product that also has determinant 1:

$$\det(\mathbb{M}\mathbb{N}) = \det \mathbb{M} \cdot \det \mathbb{N} = 1$$

For the rest of this lecture, we will restrict to studying matrices with unit determinant.

The set of all N dimensional matrices $\mathbb{M}$ with determinant 1 that satisfy

$$\mathbb{M}^T \mathbb{M} = \mathbb{1} \text{ is called the special orthogonal}$$

group of dimension N, denoted as $SO(N)$. We're studying $SO(2)$ now.

All 2x2 matrices in SO(2) can be written as a function of one angle θ:

$$\mathbb{M}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Note the product of two matrices with angles θ and ϕ:

$$\mathbb{M}(\theta)\,\mathbb{M}(\phi) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

Note that this result implies that the SO(2) group is Abelian:

$$\mathbb{M}(\theta)\,\mathbb{M}(\phi) = \mathbb{M}(\phi)\,\mathbb{M}(\theta) = \mathbb{M}(\theta+\phi)$$

This makes sense: SO(2) is the group of symmetries of a circle. We can rotate the circle by any angle θ and the circle looks unchanged. An additional rotation by an angle ϕ is just another rotation, with a total rotation angle $\theta+\phi$.

A circle can be represented in many ways, and this enables us to express its symmetries in many ways. One way to represent a circle is on the complex plane. The circle with radius r are those points (x,y) such that

$$|z|^2 = x^2 + y^2 = r^2$$

A complex number z can be expressed in terms of a magnitude r and angle ϕ:

$$z = re^{i\phi}$$

The absolute value squared is then $|z|^2 = re^{i\phi}\cdot re^{-i\phi} = r^2$

This is unchanged if the angle ϕ is rotated by any angle θ. That is is $\phi \to \phi+\theta$, then

$$z \to re^{i(\phi+\theta)} \quad \text{but} \quad |z|^2 \to |z|^2 = r^2.$$

That is, a rotation of the circle in the complex plane is accomplished by multiplying complex numbers by $e^{i\theta}$. This angle $\theta \in [0, 2\pi]$, because angles larger than 2π or smaller than 0 can be mapped onto $[0, 2\pi]$. Note also that the product of these factors, with angles of rotation of θ and ϕ are

$$e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$$

This is the exact same multiplication law as we found for SO(2)! These rotations of a circle in the complex plane are called the unitary group of one dimension, or U(1). What we have shown is

$$SO(2) \cong U(1)$$

where $\cong$ means that these two groups are identical as groups. U(1) is called unitary because an element times its complex conjugate is unity:

$$\left(e^{i\theta}\right)\left(e^{i\theta}\right)^* = e^{i\theta} e^{-i\theta} = 1$$

We say that the set of rotations implemented by $e^{i\theta}$ are a unitary representation of SO(2). That is, they are unitary and satisfy ("represent") the multiplication law of SO(2).

Okay, enough about SO(2). What about SO(3), the symmetries of a two-dimensional sphere? The matrices that are elements of SO(3) satisfy:

$$M^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \det M = 1.$$

I won't discuss the details of SO(3) as much as SO(2), but just describe some of its features. First, unlike SO(2), SO(3) is non-Abelian. I'll try to draw this; well, I'm a bad artist, so I won't draw it. But we can visualize it by rotation of a three-dimensional object, like my coffee cup. Let's consider rotation of the handle, and then about the cup; and vice-versa. We find something different depending on the order of the rotations!

These rotations can be expressed by the three Euler angles, which might be familiar from classical mechanics.

As with SO(2), we want to find unitary representations of SO(3). Why unitary representations? Well, in quantum mechanics we study wave functions, for example, like that for describing an electron with spin. Call this wavefunction ψ. Under a rotation, the wave function will change because the spin will rotate somehow. However, we want to maintain probabilities; that is, we want to make sure that probabilities are unchanged with a rotation.

Let's call the matrix that implements the rotation $\mathbb{M}$. Then, under a rotation, the spin of the wavefunction changes to:

$\psi \to \mathbb{M}\psi$, The probability is the absolute value squared of the wavefunction: $P = \psi^{\dagger}\psi$. Under a rotation this becomes:

$$\psi^{\dagger}\psi \to (\mathbb{M}\psi)^{\dagger}\mathbb{M}\psi = \psi^{\dagger}\mathbb{M}^{\dagger}\mathbb{M}\psi$$

If this is to remain unchanged, we must enforce

$\mathbb{M}^{\dagger}\mathbb{M} = \mathbb{1}$, which is the requirement of unitarity.

Recall that $\mathbb{M}^\dagger$ is the Hermitian conjugate of $\mathbb{M}$: we transpose $\mathbb{M}$ and then complex conjugate all entries.

Enforcing Unitarity is actually quite easy. Remember what we did for SO(2): we expressed the "matrix" as a complex exponential. Well, let's just do the same thing for SO(3). Let's write a matrix in the group SO(3) $\mathbb{M}$ as:

$\mathbb{M} = e^{iT}$, where T is a matrix. Exponentiating a matrix may look weird to you, but this can be defined by the Taylor expansion:

$$e^{iT} = \mathbb{1} + iT - \frac{T^2}{2} - \ldots$$

These T matrices (or an appropriate basis of them) are called "generators" of the group. The most familiar generator is $\hat{p} = -i\hbar\vec{\nabla}$.

Let's now see what unitarity means:

$$\mathbb{M}^\dagger \mathbb{M} = e^{-iT^\dagger} e^{iT} = \mathbb{1} = e^{i(T-T^\dagger)}$$

Therefore, for $\mathbb{M}$ to be unitary, we must require that the exponentiated matrix T is Hermitian: (That is, the T are measurement operators in QM!)

$T = T^\dagger$. The eigenvalues of Hermitian matrices are all real. You'll explore more in homework some unitary representations of SO(3). In particular, you will study its two-dimensional representation, where the exponentiated matrices T can be written in terms of the Pauli spin matrices. In fact, this shows that

$SO(3) \cong SU(2)$, the special unitary group in two-dimensions. This might ring a bell with angular momentum addition, Clebsch-Gordon coefficients, etc. This will be useful next lecture.