The Quark Model Lecture 6

Particle physics, such as it was, was becoming a more mature field in the 1930's. The neutron was discovered by James Chadwick in 1932, about 15 years after the discovery of the proton by Ernest Rutherford. The discovery of the neutron enabled the first quantitative understanding of atomic nuclei (and consequences from extracting energy from it). Scientists at the time noted that there were some very interesting similarities between the proton and the neutron. From the PDG, some properties of protons and neutrons are:

	proton	neutron
mass:	938 MeV	940 MeV
lifetime:	$> 2.1 \times 10^{29}$ yrs	~900 s
charge:	+1 e	0 e
spin:	1/2	1/2

Some of these things are eerily similar: both protons and neutrons are spin-1/2 particles (that is, described by the Dirac equation) and their masses differ by about 2 MeV (about 1 part in 500!). Some of these are not: to the best of our knowledge, the proton is stable; its decay half-life is at least about 10^{29} years, while the neutron decays in about 900 s, or 15 minutes. (When bound in a nucleus, the ~~proton~~ neutron is stable; when it is isolated it can decay.) These times differ by a factor of about 10^{34}! However, for times relevant for particle physics, 900 seconds is a <u>really</u> long time. For a neutron traveling near the speed of light, this corresponds to a distance of about 3×10^{11} meters, or about the same distance of the Earth to the Sun! So, for particle physics, 900 s is essentially an infinite amount of time.

The other thing that differs between protons and neutrons are their electric charges. This is of course very important for chemistry, but depending on the questions we ask in particle physics, this may be irrelevant. For now, let's ignore the difference of charges of protons and neutrons. Another way to say this, is imagine a universe where electromagnetism does not exist. In this universe, protons and neutrons would be identical. That is, we could change all protons to neutrons and vice-versa and everything would be the same. While we don't live in this universe, we approximately live in this universe. It will be useful to study this approximate symmetry between protons and neutrons.

With these observations after the discovery of the neutron, Werner Heisenberg proposed an approximate symmetry between protons and neutrons, later called isotopic spin, or isospin. (Isospin, despite the name, has nothing to do with spin.) Let's figure out what this isospin is. Protons and neutrons are described quantum mechanically as wavefunctions, let's label them with the bra-ket notation:

$|p\rangle$, for the proton, and $|n\rangle$ for the neutron.

These ket wavefunctions are normalized with the corresponding bra:

$$\langle p|p\rangle = \langle n|n\rangle = 1$$
$$\langle p|n\rangle = \langle n|p\rangle = 0$$

The symmetry between protons and neutrons means that any linear combination of them is indistinguishable. We can write this linear combination by a matrix U multiplying the vector of wavefunctions

$$\begin{pmatrix} |p\rangle \\ |n\rangle \end{pmatrix} \quad \text{as} \quad U\begin{pmatrix} |p\rangle \\ |n\rangle \end{pmatrix}$$

The matrix U implements the symmetry. What are its properties? Well, it must maintain normalization. The "bra" conjugate of the vector of wavefunctions is

$$(\langle p| \; \langle n|) \quad \text{or with the matrix } U, \quad (\langle p| \; \langle n|)\, U^\dagger,$$

where $U^\dagger$ is the Hermitian conjugate of U. By normalization condition, we must have

$$(\langle p| \; \langle n|) \begin{pmatrix} |p\rangle \\ |n\rangle \end{pmatrix} = (\langle p| \; \langle n|)\, U^\dagger U \begin{pmatrix} |p\rangle \\ |n\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

That is, U is a 2×2 matrix with $U^\dagger U = \mathbb{1}$. This defines the unitary group in two dimensions, or $U(2)$. As with rotations we discussed last time, we can restrict to those matrices that have unit determinant:

$$\det U = 1$$

and still have a group. This group is called the special unitary group in two dimensions or $SU(2)$.

The proton and the neutron form what is called a two-dimensional representation of $SU(2)$ isospin. We also say that they form a doublet of $SU(2)$. Last lecture we discussed the rotation group in 3 dimensions $SO(3)$, and I mentioned that this is identical (well, nearly) to $SU(2)$ as groups:

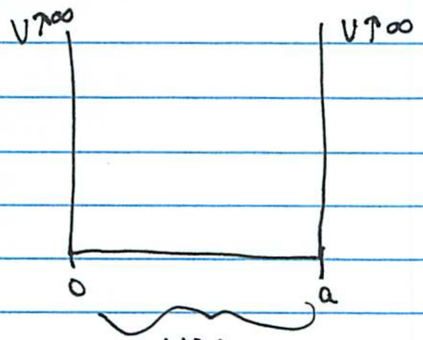
$$SO(3) \cong SU(2)$$

(The difference between them is irrelevant for this class, but has to do with the topology of the groups as abstract manifolds.) That is, the proton and neutron are doublets of $SO(3)$ also, in the sense that the isospin matrices U satisfy the multiplication laws of $SO(3)$ as a group.

Recall that the symmetry corresponding to rotations in three dimensions (the group $SO(3)$) means that angular momentum is conserved, by Noether's theorem. If you remember angular momentum addition in quantum mechanics, it was all studying properties of the representations of the rotation group $SO(3)$! Clebsch-Gordon and all that (which I'll remind about soon) also exists for isospin because $SU(2) \cong SO(3)$, hence the name "isospin". If you haven't seen angular momentum addition in quantum mechanics, don't fret; we'll discuss details of how this works.

Another thing to note at this stage: we said that, at least to good approximation ignoring electromagnetism, isospin is a symmetry. This means that there is a conservation law: isospin is (approximatly) conserved in reactions. We'll discuss this later.

To see the power of working with symmetries, I want to define what we mean by a particle. Let's recall what the possible outcomes of experiment are in quantum mechanics. For concreteness, let's consider a particle in an ∞-D well:



The well has width a and perfectly rigid walls. The wavefunction for the particle in the well, $\psi(x)$, can be expressed as a linear combination of all possible energy eigenstates:

$$\psi(x) = \sum_{i=0}^{\infty} \alpha_i \psi_i(x)$$

where $\psi_i(x)$ is the wave function for the i^{th} energy level and α_i is the probability amplitude. While the wave function $\psi(x)$ is allowed to be completely arbitrary, the outcomes of experiment are not. If we measure the energy of the particle, we only find energies that correspond to the eigen-energies of the well. In general, the outcome of any quantum mechanical experiment is an eigenvalue of the Hermitian (= real) operator that corresponds to that experiment.

In the case of measuring energies of the system, we only can measure the eigenvalues of the Hamiltonian of the infinite square well.

Because particle physics is quantum mechanical, we should define a particle in a way that respects this. That is, the outcomes of experiments are only eigenvalues of Hermitian operators, so we should only define a particle by what we can measure. What are the Hermitian operators and their eigenvalues for particle physics? It's all related to symmetries!

Let's see this concretely before working through consequences for isospin. Let's consider the electron. What are the properties of the electron? Well,

mass: 511 KeV
spin: 1/2
charge: −1 e

You tell me these things, and I know it is an electron. What does this have to do with symmetries? These numbers tell us the eigenvalues of symmetries for the electron!

Let's denote the wave function of the electron in the bra-ket notation as $|e\rangle$. Then, there are mass, spin, and charge operators for which the electron has these eigenvalues. I won't discuss these in detail, but just make some observations. The mass operator just returns the mass (squared), which is just the four-vector squared! So, if we act on the electron wavefunction with the four-vector squared operator this just returns the mass:

$P_\mu P^\mu |e\rangle = m^2 |e\rangle$, where m^2 is the mass of the electron squared. The operator P_μ is → in position space

$P_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) = \partial_\mu$, which is just the quantum mechanical identification! Note that

$P_\mu P^\mu |e\rangle = m^2 |e\rangle$ is just the Klein-Gordon equation!

Similar things hold for the other properties (which I won't discuss in detail). for spin, we have

$S^2 |e\rangle = s(s+1)|e\rangle = \frac{3}{4}|e\rangle$, where S^2 is the spin operator and $s=1/2$ is the spin of the electron. For charge operator G we have

$G|e\rangle = -e|e\rangle$, where e is the fundamental charge, $e = 1.6\times10^{-19}$ Coulombs.

The eigenvalues of Hermitian operators acting on a particle are called its "quantum numbers": the quantum numbers of a particle uniquely define it.

Break

So, how does this relate to isospin?

Isospin can help us organize possible combinations of protons and neutrons in nuclei! That is, we can define particular combinations of protons and neutrons by their isospin eigenvalues; i.e., as distinct particles. Let's see how this works in an example.

Ex Let's consider the system that consists of two nucleons (protons or neutrons), which we denote generically as N. We will denote the wave function of two nucleons as $|NN\rangle$. We want to construct the smallest linear combinations of pairs of nucleons that are closed under action by the SU(2) isospin symmetry. These minimal combinations are called "irreducible representations" and will correspond to distinct eigenvalues and therefore distinct particles.

So, what are we working with? The possible nucleon combinations are:

$$|pp\rangle,\ |pn\rangle,\ |np\rangle,\ \text{and } |nn\rangle,$$

where the first entry is nucleon 1 and the second is nucleon 2, SU(2) isospin symmetry acts to replace a nucleon with a linear combination of protons and neutrons, maintaining normalization. That is, under action of isospin, the proton wave function becomes:

$$|p\rangle \to |(\alpha p + \beta n)\rangle = \alpha|p\rangle + \beta|n\rangle, \text{ with } |\alpha|^2 + |\beta|^2 = 1.$$

The matrix U that implements this transformation acts on the vector $\begin{pmatrix}|p\rangle\\|n\rangle\end{pmatrix}$ as

$$U\begin{pmatrix}|p\rangle\\|n\rangle\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}|p\rangle\\|n\rangle\end{pmatrix} = \begin{pmatrix}\alpha|p\rangle + \beta|n\rangle\\ \gamma|p\rangle + \delta|n\rangle\end{pmatrix}$$

where the entries of $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. we also restrict $\det U = 1 = \alpha\delta - \beta\gamma$.

Let's see what happens if we isospin transform the linear combination:

$$a|pn\rangle + b|np\rangle$$

This transforms according to U as:

$$a|pn\rangle + b|np\rangle \rightarrow a|(\alpha p + \beta n)(\gamma p + \delta n)\rangle + b|(\gamma p + \delta n)(\alpha p + \beta n)\rangle$$

$$= \alpha\gamma(a+b)|pp\rangle + \beta\delta(a+b)|nn\rangle$$

$$+ (a\beta\gamma + b\alpha\delta)|pn\rangle + (a\alpha\delta + b\beta\gamma)|np\rangle$$

For $a|pn\rangle + b|np\rangle$ to be unchanged from this transformation, we must forbid contributions from $|pp\rangle$ and $|nn\rangle$; this enforces $b = -a$. With this choice, we then have:

$$a(|pn\rangle - |np\rangle) \rightarrow -a(\alpha\delta - \beta\gamma)(|pn\rangle - |np\rangle)$$

But, we enforce $\det U = 1$ or $\alpha\delta - \beta\gamma = 1$ and so

$$a(|pn\rangle - |np\rangle) \rightarrow -a(|pn\rangle - |np\rangle) \text{ under an } SU(2)$$

isospin transformation. The fact that there is a "$-$" sign after the transformation means we refer to this antisymmetric combination as "odd". The constant a can be determined by demanding normalization:

$$(\langle pn| - \langle np|)a^* a(|pn\rangle - |np\rangle) = 2a^*a \Rightarrow a = \frac{1}{\sqrt{2}}$$

So, the antisymmetric state

$$\frac{1}{\sqrt{2}}\left(|pn\rangle - |np\rangle\right) \quad \text{tranforms to (minus) itself}$$

under an isospin transformation. Because it transforms to itself, it is a one-dimensional representation, also called a "singlet".

There is one other linear combination of $|pn\rangle$ and $|np\rangle$ that can be formed that is orthogonal to the singlet. It is:

$$\frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right)$$

Let's see how this transforms under an isospin transformation:

$$\frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right) \rightarrow \sqrt{2}\,\alpha\gamma|pp\rangle + \sqrt{2}\,\beta\delta|nn\rangle + \frac{\alpha\delta+\beta\gamma}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right)$$

While this doesn't transform into itself, it does transform to the combinations:

$$|pp\rangle,\ |nn\rangle,\ \text{and}\ \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right)$$

Under ~~one~~ any isospin ~~ato~~ transformation these three combinations transform into linear combinations of each other. That is, they are closed under isospin and form another representation. Because there are three combinations

$$|pp\rangle,\ |nn\rangle,\ \text{and}\ \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right)$$

this is a three-dimensional representation, or a "triplet". Note that this is symmetric under exchange of nucleons 1 and 2.

Therefore, we have shown that the ~~product~~ state that consists of two nucleons $|NN\rangle$ decomposes into the singlet and triplet irreducible representations:

singlet	triplet
	$|pp\rangle$
$\frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle)$	$\frac{1}{\sqrt{2}}(|pn\rangle + |np\rangle)$
	$|nn\rangle$

By the way, deuterium (hydrogen with a proton and a neutron in the nucleus) corresponds to either the singlet or the symmetric combination of $|pn\rangle$ and $|np\rangle$ (depending on the spin of the deuterium nucleus).

The factors of $\pm\frac{1}{\sqrt{2}}$ that appear places are Clebsch-Gordon coefficients, that tell you what linear combinations of states appear in the decomposition of product states into irreducible representations, like $|NN\rangle$. You'll play with Clebsch-Gordon coefficients a bit more in homework.

In the first homework, you estimated the mass of the pion. The textbook tells the story of discovery of the pion in the mountains of Bolivia in the 1950s. Actually, there are three pions, denoted as π^+, π^-, π^0, with $+e$, $-e$, and 0 electric charge, respectively. Like the proton and neutron, these pions have eerily close masses:

$$m_{\pi^+} = 139 \text{ MeV}, \quad m_{\pi^-} = 139 \text{ MeV}, \quad m_{\pi^0} = 135 \text{ MeV}$$

Their charge does distinguish them, but we can imagine a universe where electricity and magnetism are turned off. This is suggestive of the pions forming a triplet of isospin, like we saw with pairs of protons and neutrons. However, because the mass of protons and neutrons are much larger than the pions, the pions cannot consist of pairs of nucleons. Therefore, they must consist of more fundamental constituents!

Another thing to note: the triplet representation (three-dimensional) of isospin SU(2) is <u>not</u> the simplest representation (which would be two-dimensional). This is necessarily constructed out of states that are products of two-dimensional representations.

In the 1950s and 1960s more and more of these particles, called hadrons were discovered and interesting relationships between them were identified. Beyond SU(2) isospin, it was realized that there was a larger SU(3) group structure to the pattern of hadrons. This symmetry is called SU(3) flavour. Murray Gell-Mann (though not known to him at the time) had organized the measured hadrons in what he called "The Eightfold Way", which corresponded to the representations of SU(3) flavour. For example, one group of hadrons is called the baryon octet and consists of:

(mass in MeV)

	(939) n		(938) p
(1197) Σ^-		(1192) Σ^0 Λ (1115)	(1189) Σ^+
	Ξ^- (1321)		Ξ^0 (1314)

This octet is an 8 dimensional representation of flavour SU(3). Like the pion triplet, it is not the simplest representation (which would be 3 dimensional). In fact, representations corresponding to groups of hadrons of all stripes were observed (decuplets, ...), but <u>not</u> the simplest, called the fundamental, representation. With this and other evidence, Gell-Mann and George Zwieg theorized that there were fundamental particles, whose different combinations produced the zoo of hadrons. Because the observed flavor group was $SU(3)$, they predicted that there were just 3 fundamental particles that were responsible for everything. They called them quarks, after the poem in James Joyce's novel "Finnegans Wake":

- Three quarks for Muster Mark!
  Sure he hasn't got much of a bark
  And sure any he has it's all beside the mark.

These three quarks were named up (u), down (d), and strange (s). The pions, for example, consist of up and down quarks in the following combinations:

$$|\pi^+\rangle = |u\bar{d}\rangle, \quad |\pi^-\rangle = |\bar{u}d\rangle, \quad |\pi^0\rangle = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right)$$

where $\bar{u}$ is the anti-up quark. A beautiful application of fundamental mathematics leading to discovery of new physics!

Now, we know of 6 quarks, which we'll discuss later.