- contours?
- HW
- Chocolate bribe

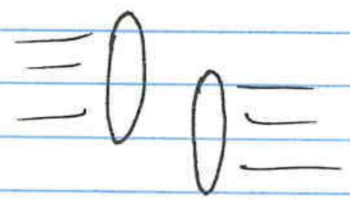
Fermi's Golden Rule Lecture 7

Golden Rule #1: "Do unto others as you would have them do unto you." - The foundation of all civilization.

Golden Rule #2: "The transition rate from an initial to final state in quantum mechanics is the matrix element squared of the Hamiltonian times the density of states."
- The foundation of predictions in particle physics.

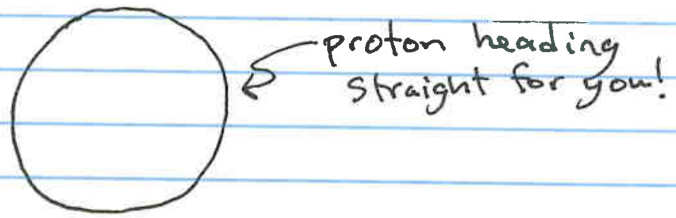
Over the last few weeks, we have reviewed tools necessary for speaking the language of particle physics: units, relativity, group theory, etc. Starting this week, and for the rest of the class, we apply this new language to understand and predict experiments in particle physics. The first step of this endeavor is to figure out exactly what is done in particle physics experiments and how to map our calculations onto it.

Let's consider what happens at the Large Hadron Collider. Protons are collided at high energies and we observe what comes out. This is a quantum mechanical process, so really all we can measure is not exactly what will come out of each proton collision, rather just the probability that protons will interact and produce a particular final state. So, we want to define a unit that is a measure of this scattering probability. Let's imagine watching the protons collide. It would look something like this:



Protons are Lorentz contracted into pancake shapes from their spherical shape at rest.

What property of the proton controls the probability of interaction? Well, this drawing makes <u>it clear that</u> the size of the proton is clearly important. As we discussed earlier, the radius of the proton is hard to define; we can't just measure it with a ruler. I mentioned one way to define the size of the proton in class, as (effectively) the radius at which the electric field of the proton is largest ("charge radius"; c.f. to a spherical shell of charge). Another way to define the size/radius of a proton is through its interaction rate. If you were a proton traveling toward another proton for collision, you would see something like



That is, you would see a cross-sectional area of the proton. The larger this area is, the more likely it is that you will interact with it. The smaller the area, the less likely to interact.

In particle physics, collision or interaction rates are expressed in effective cross-sectional areas, or cross sections. How do we measure cross sections; i.e., what unit do we use? Interestingly, we do not, by practice, express cross sections in natural units.

If we wanted to define cross sections at human scales, we would express them in square meters, or perhaps square centimeters. For example, the cross section of a baseball is <u>about</u> 45 cm², while the "meat" of a bat is about 150 cm², which are nice numbers.

Square centimeters aren't so useful for particle physics. For the proton with a radius of about 10^{-15} m (or 10^{-13} cm) it has a cross section of about 10^{-26} cm^2. Another tiny exponent that can be annoying to lug around.

The standard unit of cross section in particle physics is called the "barn", and its origins are quite amusing. During WWII, when scientists were working on the atomic bomb, it was very important to understand the cross section of uranium (i.e., the probability that uranium interacted with itself), and scientists wanted to name an appropriate unit for this cross sectional area. The history is accounted in M.G. Holloway, C.P. Baker "How the Barn was Born", Physics Today, p. 9 July 1972:

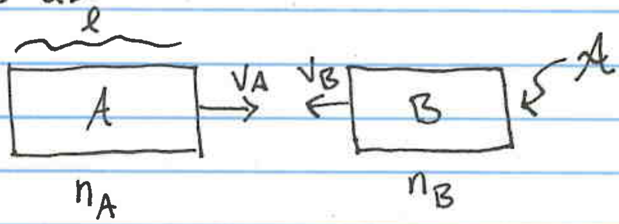
> The tradition of naming a unit after some great man closely associated with the field ran into difficulties since no such person could be brought to mind. Failing this, the names Oppenheimer and Bethe were tried, since these men had suggested and made possible the work on the problem with which the Purdue project was concerned. The "Oppenheimer" was discarded because of its length, though in retrospect "Oppy" or "Oppie" would seem to be short enough. The "Bethe" was thought to lend itself to confusion because of the widespread use of the Greek letter. Since John Manley was directing the work at Purdue, his name was tried, but the "Manley" was thought to be too long. The "John" was considered, but was discarded because of the use of the term for purposes other than as the name of a person. The rural background of one of the authors then led to the bridging of

the gap between the "John" and the "barn". This immediately seemed good and further it was pointed out that a cross section of $10^{-24}\,cm^2$ for nuclear processes was really as big as a barn. Such was the birth of the "barn".

So the barn, which is about the cross-sectional area of a uranium nucleus, is $10^{-24}\,cm^2$. In these units, the cross sectional area of a proton is about a hundredth of a barn, or so. In elementary particle physics, the cross sections that we consider are typically much smaller than a barn, so we often use nanobarns (10^{-9}), picobarns (10^{-12}), femtobarns (10^{-15}), or even attobarns (10^{-18}).

We'll discuss how to measure interaction/scattering probabilities as cross sections next week. For the rest of this week, we will discuss how to calculate and predict these scattering rates and cross sections.

For concreteness, let's consider colliding protons at the LHC. The way this is done is by colliding bunches of protons; let's call one bunch A and the other bunch B. We can visualize this as:



The velocity of bunch A is v_A, and similarly for B. The number density of protons in bunch A is n_A and similarly for B. The length of the bunches is ℓ and the cross sectional area of both bunches is script $\mathcal{A}$. Note that this cross sectional area is the size of the bunch, and not the size of an individual proton. For colliding protons at the LHC, the relevant quantity is the number of collision events per second, as this will tell us how often protons will collide over the time that we run the machine. Let's see how to define this.

Note that as the bunches pass through one another, every proton of bunch B travels through a region of length ℓ where there are protons (and similarly for bunch A). The total number of protons that a single proton in bunch B sees in one unit of time is:

$$N_A/t = n_A \cdot A\,|v_A - v_B|, \text{ where } |v_A - v_B| \text{ is the relative}$$

speed of the bunches. The total number of protons in bunch B that could possibly interact with the protons of bunch A are:

$$N_B^{\text{eff}} = n_B \cdot \ell \cdot \sigma$$

That is, the number of B protons that can possibly interact with the protons of bunch A is given by the number density n_B times the total volume of pure protons in bunch B. This volume is just the length of the bunch times the cross sectional area of each proton, denoted as σ. The total number of scattering events per unit time is then the product of these two:

$$\text{events}/\text{time} = \frac{N_A N_B^{\text{eff}}}{t} = n_A n_B A \ell\,|v_A - v_B|\,\sigma.$$

The prefactor $n_A n_B A \ell\,|v_A - v_B|$ is called the flux factor, and depends on the precise parameters of the LHC accelerator. The proton scattering cross section σ is intrinsic to individual proton-proton interactions, and is the same for protons scattering in the LHC as in the center of the galaxy. Note indeed that it has units of an area.

For the rest of this lecture and next lecture we will work to define and learn how to calculate the proton scattering cross section, σ. A careful and detailed derivation of the form of the cross section σ requires quantum field theory and its axioms; here, we will justify the form of the cross section by its expected properties.

Break //

When we scatter and collide protons, there are two things that we need to define. We need to define what the initial state is; that is, what momentum the protons have. We also need to define what final state we are considering: what particles are produced in the collision, what their momenta are, etc. Then, we want to determine the overlap of the wavefunction of the initial state:

$$|p_A p_B\rangle_{in}$$

with the wavefunction of the final state,

$$_{out}\langle p_1 p_2 \cdots p_n|.$$

Here, p_A and p_B are the momentum of the protons in the A and B bunches that collide, and $p_1, \ldots, p_n$ are the momentum of the n particles produced in the collision. For example, we might be interested in the collision of protons that produces two positrons:

$$pp \to e^+ e^+$$

or, two positrons and three photons:

$$pp \to e^+ e^+ \gamma\gamma\gamma$$

or whatever. Note that charge is conserved in these reactions. The probability that two protons collided to produce n final state particles with momenta $p_1, \ldots, p_n$ is then

$$\text{Prob of scattering of protons to } p_1, \ldots, p_n = $$
$$= \left| {}_{\text{out}}\langle p_1 p_2 \cdots p_n | p_A p_B \rangle_{\text{in}} \right|^2$$

Here, I am using an implicit notation where all integrals are suppressed. For compactness in what follows, we will denote the final state as f (but we'll consider explicit final states in a bit).

As discussed earlier, the proton scattering cross section is proportional to this probability:

$$\sigma \propto \left| {}_{\text{out}}\langle f | p_A p_B \rangle_{\text{in}} \right|^2$$

Probabilities are dimensionless and cross sections have units of area, so we still need a bit more to determine the cross section. This is determined by thinking about how the protons can scatter, which is sensitive to their de Broglie wavelength.

The de Broglie wavelength of a particle is the distance over which that particle is coherent;

That is, the distance over which the particle's position can be. The de Broglie wavelength is inversely proportional to the momentum of a particle:

$$\lambda = \frac{\hbar}{p}$$

That is, higher momentum/energy particles are localized in a smaller region of space, while low momentum particles are delocalized (can be "anywhere"). It is harder for protons with short de Broglie wavelengths to collide because their position is a small region of space, while it is much easier for protons with long de Broglie wavelengths to scatter. Therefore, we expect the cross section to be proportional to the product of the de Broglie wavelengths of the two protons:

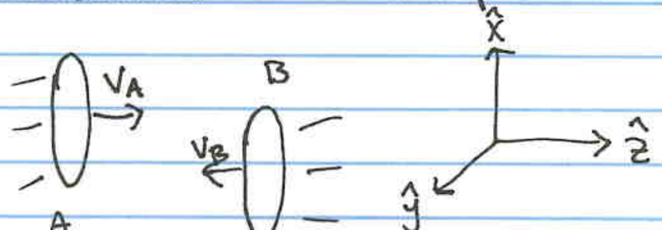
$$\sigma \propto \lambda_A \lambda_B \left| {}_{out}\langle f | p_A p_B \rangle_{in} \right|^2$$

The de Broglie wavelength is inversely proportional to the energy of the particle (in natural units) and so we can also express this as

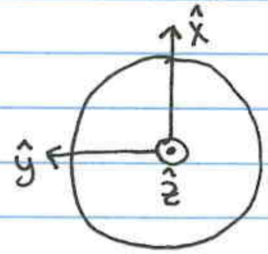
$$\sigma \propto \frac{1}{2E_A} \frac{1}{2E_B} \left| {}_{out}\langle f | p_A p_B \rangle_{in} \right|^2$$

The factors of $\frac{1}{2}$ come from careful normalization.

Okay, we're almost there. As a cross section, σ should have particular properties under Lorentz transformations. Let's remind ourselves about the picture of proton collision:



Let's call the axis along ~~###~~ which the scattering occurs $\hat{z}$. The cross-sectional areas of the protons ~~###~~ then lie in the plane of the $\hat{x}$ and $\hat{y}$ axes. Looking at a proton in a head on collision we see:



This is clearly rotationally-invariant about the $\hat{z}$ axis, and it is Lorentz-boost invariant about the $\hat{z}$ axis. That is, the area that you see does not change with these transformations. We need to make sure that this is true for the expression for the cross section. The probability

$$|{}_{out}\langle f | p_A p_B \rangle_{in}|^2$$

is fully Lorentz invariant (essentially by definition), so it remains unchanged under rotations and boosts. (We'll define this in a second.) The overall factors of energy are rotationally invariant, but change if the system is boosted along the $\hat{z}$ axis! So, we need to fix this.

This can be accomplished by including a factor of the relative velocity $|v_A - v_B|$ of the two protons. We can show that the product

$$E_A E_B |v_A - v_B| \text{ is invariant to boosts along}$$

the $\hat{z}$ axis. To see this, note that the velocity v_A is

$$v_A = \frac{p_A^z}{E_A} \quad \text{and so} \quad E_A E_B |v_A - v_B| = |p_A^z E_B - p_B^z E_A|.$$

Under a Lorentz boost along the $\hat{z}$ direction,

$$p^z \to \gamma(p^z + \beta E) \quad , \quad E \to \gamma(E + \beta p^z)$$

and so

$$|p_A^z E_B - p_B^z E_A| \to |\gamma(p_A^z + \beta E_A)\gamma(E_B + \beta p_B^z) - \gamma(p_B^z + \beta E_B)\gamma(E_A + \beta p_A^z)|$$

$$= \gamma^2 \underbrace{(1-\beta^2)}_{=1} |p_A^z E_B - p_B^z E_A| = |p_A^z E_B - p_B^z E_A| \quad \checkmark$$

Therefore, putting it all together, the expression for the proton-proton scattering cross section σ is:

$$\boxed{\sigma = \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} \left|{}_{out}\langle f | p_A p_B \rangle_{in}\right|^2}$$

This has the correct dimensions of a cross-sectional area (Energy^{-2} in natural units) and has the correct Lorentz transformation properties under rotations and boosts along/about the collision axis. Whew!

We've so far been a bit cavalier about what

$$\left|{}_{out}\langle f | p_A p_B \rangle_{in}\right|^2$$

actually is. Once we know how to calculate it, then we can predict how often protons will collide in our experiment! To define this, we need to think a bit about what we can measure in a particle physics experiment.

Experiments like ATLAS and CMS at the LHC are exceptionally good at measuring energy and momenta of particles, but exceptionally poor of measuring positions. So, in the calculation of the probability, we should work with momenta, and be completely ignorant about positions of particles. Then, according to the rules of quantum mechanics, we must sum over all possible final states consistent with our measurement. For example, let's measure two positrons in the final state. We then must sum over all possible final states. This includes two positrons with any momentum as well as any additional particles, like photons.

For concreteness, let's consider a final state with n particles:

$$\underbrace{p_A + p_B}_{\text{initial protons}} \rightarrow \underbrace{1+2+\cdots+n}_{\text{final particles}}$$

We will denote the Lorentz-invariant matrix element as

$$\mathcal{M}(A+B\rightarrow 1+2+\cdots+n)$$

which represents the amplitude for protons A and B to scatter into particles $1,\ldots,n$ with <u>particular momentum</u>. Four-momentum is conserved and so we can write the probability of scattering as:

$$\left|{}_{out}\langle 1+2+3\cdots+n|p_A\, p_B\rangle_{in}\right|^2 = \int d\Pi_n \left|\mathcal{M}(A+B\rightarrow 1+2+\cdots+n)\right|^2 (2\pi)^4\delta^{(4)}\Big(p_A+p_B-\sum_{i=1}^{n}p_i\Big)$$

In this expression, four-momentum conservation is imposed by the δ-function

$$(2\pi)^4\delta^{(4)}\Big(p_A+p_B-\sum_{i=1}^{n}p_i\Big)$$

which is only non-zero if the argument is zero in each component of energy and momentum. The integral

$$\int d\Pi_n$$

is called Lorentz-invariant phase space and is the integral over the four momentum of each of the n-final state particles. It can be expressed as

$$\int d\Pi_n = \int \frac{d^4p_1}{(2\pi)^4} 2\pi\delta(p_1^2-m_1^2) \int \frac{d^4p_2}{(2\pi)^4} 2\pi\delta(p_2^2-m_2^2) \cdots \int \frac{d^4p_n}{(2\pi)^4} 2\pi\delta(p_n^2-m_n^2)$$

This is "manifestly" Lorentz invariant. The integral for each particle is over all four momentum components (which is Lorentz invariant) and the δ-functions enforces that all final state particles are on-shell, which is also Lorentz invariant. The factors of 2π are necessary for proper normalization. Then, our final expression for the proton scattering cross section is:

$$\sigma = \frac{1}{2E_A}\frac{1}{2E_B}\frac{1}{|v_A-v_B|}\int \prod_{i=1}^{n} \frac{d^4p_i}{(2\pi)^4} 2\pi\delta(p_i^2-m_i^2) \left|\mathcal{M}(A+B\to 1+\cdots+n)\right|^2 \times (2\pi)^4 \delta^{(4)}\left(p_A+p_B-\sum_{i=1}^{n}p_i\right)$$

This result is called Fermi's Golden Rule. Next lecture we will discuss how to calculate the Lorentz-invariant matrix element, $\mathcal{M}(A+B\to 1+\cdots+n)$.