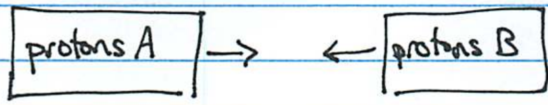
Feynman Diagrams Lecture 8

Last lecture, we discussed the master formula for the description of the number of collision events at the Large Hadron Collider. We considered the collision of bunches of protons:



and found the number of collision events per unit time to be:

$$\text{events}/\text{time} = n_A n_B \, \text{Vol} \, |v_A - v_B| \, \sigma,$$

where n_A, n_B are the densities of protons in the bunches, Vol is the volume of a bunch of protons, and $|v_A - v_B|$ is the relative velocity of the bunches. σ is called the cross section, and represents the probability for protons to collide and produce other particles. At the very end of last lecture, we wrote this as:

$$\sigma = \frac{1}{2E_A} \frac{1}{2E_B} \frac{1}{|v_A - v_B|} \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} 2\pi \delta(p_i^2 - m_i^2) \left| \mathcal{M}(A+B \to 1+\cdots+n) \right|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_{i=1}^{n} p_i\right)$$

The integrals

$$\int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} 2\pi \delta(p_i^2 - m_i^2)$$

is called n-body Lorentz-invariant phase space and represents the total volume that n relativistic particles occupy with allowed four-momentum. This is like the density of states in statistical physics: it counts how many configurations of n on-shell particles' momenta there are. The δ-function

$$(2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_{i=1}^{n} p_i\right)$$

enforces energy and momentum conservation. This is actually four δ-functions:

$$\delta^{(4)}\left(p_A + p_B - \sum_{i=1}^{n} p_i\right) = \delta\left(E_A + E_B - \sum_{i=1}^{n} E_i\right)\delta\left(p_A^x + p_B^x - \sum_{i=1}^{n} p_i^x\right)\delta\left(p_A^y + p_B^y - \sum_{i=1}^{n} p_i^y\right)$$
$$\times\, \delta\left(p_A^z + p_B^z - \sum_{i=1}^{n} p_i^z\right)$$

The last piece is the Lorentz-invariant matrix element (squared in the cross section)

$$\mathcal{M}(A+B \to 1+2+\cdots+n),$$

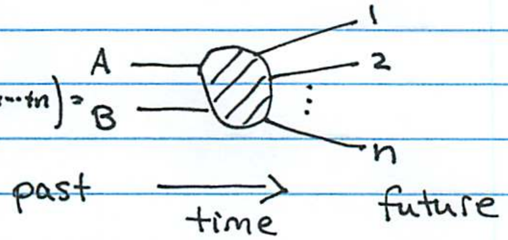
which is the ~~probability~~ amplitude for protons A and B to collide and produce particles 1, 2, 3, ..., n after collision. It represents the overlap of the wavefunction of the initial- and final states:

$$\mathcal{M}(A+B \to 1+2+\cdots+n) = {}_{out}\langle p_1 p_2 \cdots p_n | p_A p_B \rangle_{in},$$

where the momentum of all final particles $p_1, p_2, \ldots, p_n$ and the momentum of the initial particles p_A, p_B are specified. The object $\mathcal{M}$ must be Lorentz invariant for the cross section to have the correct properties. It is called a "matrix element" because it represents an entry of the scattering matrix, or S-matrix. The S-matrix of a quantum system encodes <u>all</u> possible transitions from an initial to a final state.

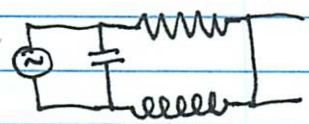
This is all still incredibly abstract. What we want is to know how to calculate the Lorentz invariant matrix element. It will be exceptionally convenient to introduce graphs or diagrams to represent these transitions. For the $A+B \to 1+2+\cdots+n$ process, or, compactly, "2 to n" scattering, we can draw the diagram:

$\mathcal{M}(A+B \to 1+\cdots+n) =$



Time in this diagram runs left to right: in the (far) past, protons A + B were accelerated and then collided. After collision, n final state particles were then produced, and subsequently measured. The blob in the middle represents the magic that happened at collision. We would like to unpack that blob and have a way of calculating it explicitly. Somehow, that blob should be able to be represented by lines and vertices that mean something.

As an example of another instance in physics where we use diagrams (and to motivate the diagrams in particle physics), remember circuit diagrams? For a circuit diagram, we define different lines to represent different circuit elements, and are often asked to determine a voltage across some external wires. For example, consider the following circuit diagram:



(This is meaningless; it's just an illustration.) The circuit is sourced by an AC voltage denoted by (~). The different symbols:

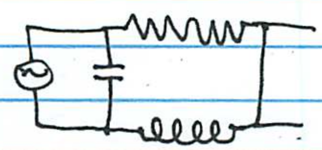
denote different circuit elements; capacitors, resistors, and inductors. Really, they represent different functions for the voltage across that element. The voltage across a capacitor is $V = CQ$, where C is the capacitance and Q is the total charge on the capacitor. The other voltages are:

$$\Rightarrow V_c = CQ, \quad \Rightarrow V_r = R\frac{dQ}{dt}, \quad \Rightarrow V_I = L\frac{d^2Q}{dt^2}$$

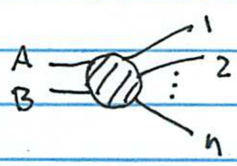
So, we can compactly denote voltage/charge relationships with symbols in circuit diagrams. Additionally, we need information about what happens at nodes in the circuit:

Of course, we have Kirchoff's rules, which are just the statements of energy and charge conservation applied to circuits. Kirchoff's first law is just the statement that the charge flowing into a node is equal to the charge flowing out of a node. Kirchoff's second law is the identical statement, but applied to energy. (A consequence of this is that the net voltage around any closed loop in a circuit is 0.)

So, given mathematical definitions of the symbols of lines and Kirchoff's rules, we can uniquely determine the voltage across the two wires on the right in the diagram:



We want to develop similar diagrammatics for particle physics to calculate the probability that an initial collection of particles turns into a final collection of particles:



Just like circuit diagrams, these particle physics diagrams should have the following properties:

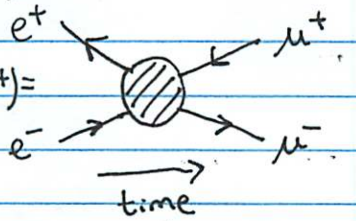
1) Different particles are represented by different shapes of lines, just like circuit elements.

2) At nodes (or vertices) in the diagram energy is conserved. Actually, because particle physics is relativistic, we want the complete four-vector of energy and momentum to be conserved.

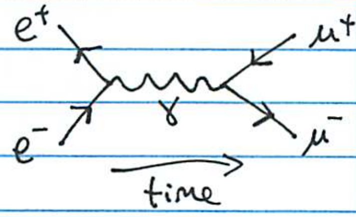
3) The charge flowing into a vertex is equal to the charge flowing out of a vertex.

Unlike in circuit diagrams, in particle physics we will also need to specify what happens at vertices. But, it is best to see how this works in an example.

For concreteness, let's consider one of the simplest processes in particle physics: the collision of an electron and positron that annihilate and produce a muon and anti-muon, $e^+e^- \to \mu^+\mu^-$. We can visualize this as

$\mathcal{M}(e^-e^+ \to \mu^-\mu^+) =$ 

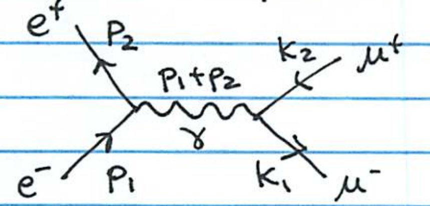
The electron and muon are electrically charged, and so interact via electromagnetism. The simplest way they can interact is by exchange of a photon, the force carrier of electromagnetism. We can denote a photon by a wavy line: 〰 called a propagator. Then, we can draw this scattering process as:



I have also denoted particles with arrows that point with time (e^-, μ^-) and anti-particles with arrows that point against the flow of time (e^+, μ^+). Such a diagram is called a "space-time diagram" or a Feynman diagram after Richard Feynman who introduced them in the 1940's.

For this diagram to be useful it should a) tell us a physical picture of what is happening and b) assist in calculation. The beauty (some may say curse) of Feynman diagrams is their immediate physical interpretation. From this diagram, an electron and positron collide and annihilate into a photon. The photon travels some distance and then transmogrifies into a $\mu^+ \mu^-$ pair that sails off into the sunset. This is a nice picture, but I emphasize that it is not what is actually happening, for reasons we will discuss shortly. Feynman diagrams are our mathematical representation of a particle physics process.

To make the Feynman diagram useful, we need to define what all of its parts are. These are called the Feynman rules. First, as we mentioned earlier, as this represents a relativistic process, energy and momentum should be conserved at each vertex. To impose this, let's go back to our diagram and write the momentum four vectors for each particle:



with $p_1+p_2 = k_1+k_2$. Note also that the electrons and muons are external particles and as such are on-shell; for example, $p_1^2 = m_e^2$, the mass of the electron. (More on this in a second.) The photon is an internal particle: its ends are stuck on other particles. The momentum of the photon is also fixed to be the sum of the electron and positron momenta. Because of this, it is impossible for the photon to be on-shell!

To see this, let's fix ourselves to scattering e^+e^- along the $\hat{z}$ axis, with equal energy. Then, the four-vectors of the electron and positron can be written as:

$$p_1 = \left(\sqrt{p^2+m_e^2}, 0, 0, p\right), \quad p_2 = \left(\sqrt{p^2+m_e^2}, 0, 0, -p\right)$$

The four-vector of the photon is

$$p_\gamma = p_1 + p_2 = \left(2\sqrt{p^2+m_e^2}, 0, 0, 0\right)$$

Note that $p_\gamma^2 = 4(p^2+m_e^2) \neq m_\gamma^2 = 0$! This can never equal 0, and therefore, the internal photon is not on-shell. Particles that are not on-shell are said to be off-shell or virtual.

Virtual particles are <u>not</u> real: they can never be measured directly. The reason why they cannot be measured directly is because the p_γ^2 of the photon does <u>not</u> correspond to the eigenvalue of any Hermitian operator. For a <u>real</u> photon described by a wave function $|\gamma\rangle$, the Hermitian operator $P_\mu P^\mu$ has eigenvalue:

$$P_\mu P^\mu |\gamma\rangle = 0 |\gamma\rangle$$

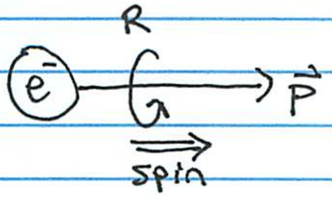
This is why I say that Feynman diagrams are useful mathematical tools, but as physical descriptions of the scattering process, shouldn't be taken too literally. Internal particles are virtual, and cannot be directly measured.

Okay, what else is going on here? The external particles e^+, e^-, μ^+, and μ^- are on-shell and all are spin-1/2 fermions. Therefore, they are described by solutions of the Dirac equation. As spin-1/2 particles, their spin can be up or down, or some combination of up and down. Recall that we could express the solution ψ to the Dirac equation as: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

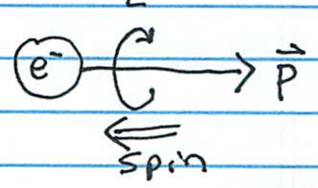
$$\psi = u^s(p)\, e^{-ip\cdot x} \quad \text{and} \quad v^s(p)\, e^{ip\cdot x}$$

where $p = (E_p, \vec{p})$ with $E_p = \sqrt{\vec{p}^{\,2} + m^2}$. $u^s(p)$ and $v^s(p)$ are the four component spinors that describe the spin of external particles (u) and anti-particles (v). They are complex (as they are wavefunctions) and so have conjugates $u^\dagger$ and $v^\dagger$, appropriately. The spinors $u(p)$ and $v^\dagger(p)$ describe the spin of an initial spin-1/2 particle, or anti-particle, respectively, while $u^\dagger(p)$ and $v(p)$ describe the spin of a final spin-1/2 particle. The superscript s in u^s denotes the spin state (up or down), which is defined by the direction of the momentum p.

In the case when the mass of the spin-1/2 particle is zero (or in the extreme relativistic limit $|\vec{p}| \gg m$), the spin is described by the helicity of the particle. The helicity is the projection of spin along the direction of motion. Because massless particles must travel at the speed of light, helicity is Lorentz invariant. We call helicity right- or left-handed based on the direction of the spin. For example, a (massless) electron with right handed helicity would be:



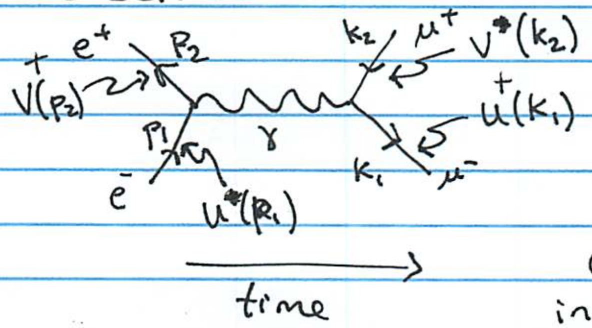
while left handed would be



To understand these names, curl your right or left hand about the spin and look at where your thumb points. Helicity will be very useful for defining spin going forward.

The textbook goes into detail about how to use these results to calculate matrix elements. We'll return to this in a few lectures; for now I want to continue to discuss the physical foundation of the Feynman diagrams.

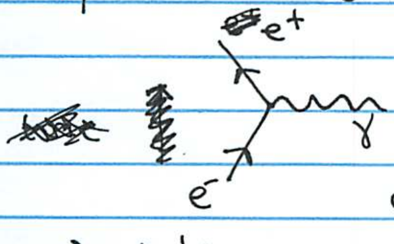
Because they describe on-shell fermions, we identify the external lines of the Feynman diagram with the spinors we defined:



The relative direction of the fermion arrow tells us if it's a particle or an anti-particle, and whether it is initial or final state tells us how we should treat it. The rules are:

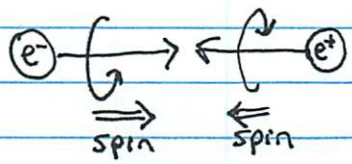
Final particle $\Rightarrow u^\dagger(p)$, Final anti-particle $\Rightarrow v^*(p)$
Initial particle $\Rightarrow u^*(p)$, Initial anti-particle $\Rightarrow v^\dagger(p)$

Getting closer! We just need to figure out what the squiggle means and what a vertex is.
First, let's focus on the vertex:

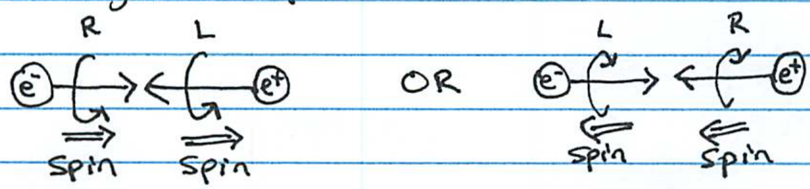


Look at what this is symbolizing: it's the emission of a photon off of an electron (with the appropriate arrow of time). The strength of this interaction is just controlled by the charge e of the electron (in natural units). This vertex also must conserve <u>angular momentum</u>. The electron and positron are spin-$1/2$ and the photon is spin-1. So, this vertex must align the spins of e^+ and e^- into spin 1.

Let's again align the e^+e^- along the $\hat{z}$-axis and go to the massless limit. In the massless limit, we can specify the spin of the particles by their helicity; the projection of the spin along the direction of motion. The helicities of the electron must sum appropriately to spin-1. Consider right-handed e^+ and e^- colliding:



In this configuration, the spins of the e^+ and e^- are in opposite directions, which sum to 0 spin! This spin configuration is not allowed. Therefore, for production of a spin-1 photon we must align the spins of e^+ and e^-:



This can be accomplished by multiplying the spinors together with an appropriate spin matrix (in the massless case). For example, for the R-L configuration, this product is:

$v_L^\dagger \sigma^\mu u_R$, where $\sigma^\mu = (\mathbb{1}, \sigma^i)$. For the L-R configuration, we instead have:

$$v_R^\dagger \bar{\sigma}^\mu u_L, \text{ with } \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

Recall that the Pauli spin matrices appear in the γ-matrices of the Dirac equation. This configuration of matrices follows from the (massless) solutions of the Dirac Equation.

Okay, super close to getting everything. The last part of the Feynman diagram is the photon propagator. The photon expresses the momentum dependence of the strength of the electromagnetic field.

The ~~squiggle~~ propagator should also be Lorentz invariant; so the only momentum it can depend on is its four-momenta squared, which we will call q. That is, with momentum q flowing through the photon, the squiggle can only be a function of q^2:

$$\underset{q}{\longrightarrow} = f(q^2) \quad \text{What function?}$$

Remember that, although the internal photon is off-shell, $q^2 \neq 0$, we can think of q^2 as its effective mass. For e^+e^- collision in the center of mass (no net momentum) the four-vector q is

$$q = \left(2\sqrt{p^2+m_e^2}, 0, 0, 0\right); \text{ that is, the photon}$$

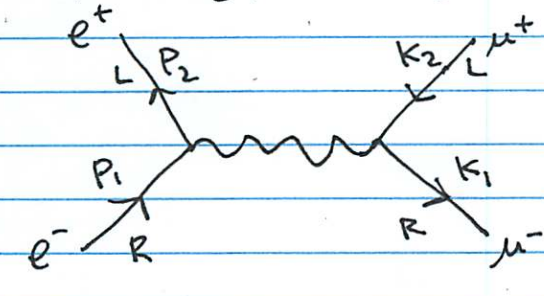
has "no" momentum; it is at rest! Then, the "size" of this photon at rest is determined by the Compton wavelength:

$$\lambda_{comp} = \frac{\hbar}{m} = \frac{\hbar}{\sqrt{q^2}}$$

If this wavelength is very small, then it is unlikely that the electron and muon interact; they have to be <u>very</u> close to know about one another. On the other hand, if the compton wave-length is large, then there is greater likelihood that the electron and muon interact; the photon can "see" both of them at the same time, even if they are widely separated. This motivates the ~~squiggle~~ propagator to be inversely proportional to q^2:

$$\underset{q}{\longrightarrow} \propto \frac{1}{q^2}$$

Putting this all together, we can express the mathematical content of the Feynman diagram! Explicitly, the Feynman diagram for $e^+e^- \to \mu^+\mu^-$ is:



$$= V_L^\dagger(p_2)\,\sigma^\mu\, u_R(p_1)\,\frac{e^2}{(p_1+p_2)^2}\,u_R^\dagger(k_1)\,\bar{\sigma}_\mu\, V_L(k_2)$$

$$= \mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)$$

The Feynman rules take us from the diagram to the mathematical expression!

We'll work with Feynman diagrams throughout the rest of this class, so you should become comfortable with what information they contain.

The Feynman rules we have discussed here (what $\mathcal{M}$ means, what happens at a vertex, etc.) can be directly derived from the Lagrangian density of the theory called Quantum Electrodynamics (QED):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\not{\partial} - m - e\gamma\cdot A\right)\psi,$$

where A_μ is the vector potential of the photon, and $F_{\mu\nu}$ is the field strength tensor of E&M:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

While we won't discuss QED in this class much, it is the most precise physical theory humanity has constructed. It makes predictions that agree to better than 1 part per billion with experiment.