Homework 3

Phys 411

September 13, 2018

Due: Friday, September 21

1. In this exercise, we will will study the Chern-Simons current, C. As a form, the Chern-Simons current can be expressed as

$$C = A \wedge dA\,. \tag{1}$$

   (a) Write the Chern-Simons current with explicit indices, $C_{\mu\nu\rho}$, in terms of the vector potential A_μ and partial derivatives, ∂_μ. It's a three-form, so there should be three anti-symmetrized indices. Completely expand out all anti-symmetrizations.

   (b) In three-dimensional spacetime, the Chern-Simons current can be a term in the Lagrangian. To construct the term in the Lagrangian, we contract the Chern-Simons three-form with the Levi-Civita tensor:

$$\mathcal{L} \supset \epsilon^{\mu\nu\rho} C_{\mu\nu\rho}\,. \tag{2}$$

   Express this Lagrangian in components. You should find

$$\mathcal{L} \supset \epsilon^{\mu\nu\rho} C_{\mu\nu\rho} = 6(A_0 F_{12} + A_1 F_{20} + A_2 F_{01})\,. \tag{3}$$

   (c) In three-dimensions, is the Chern-Simons term gauge invariant?

2. The action for electromagnetism in Minkowski space is

$$S[A_\mu] = \int d^4x\, \sqrt{|\eta|}\, \left(-\frac{1}{4}\eta_{\mu\alpha}\eta_{\nu\beta}F^{\mu\nu}F^{\alpha\beta}\right)\,. \tag{4}$$

Here, we've explicitly included factors of the metric. We have studied varying this action with respect to the vector potential, A_μ. What happens when we vary with respect to the metric, η? In this problem, we will consider taking the functional derivative with respect to the metric,

$$\frac{\delta S[A_\mu]}{\delta \eta_{\mu\nu}}\,.$$

To calculate this functional derivative, we add to the metric a small deviation $\eta_{\mu\nu} \to \eta_{\mu\nu} + \epsilon_{\mu\nu}$, and expand the action to first-order in $\epsilon_{\mu\nu}$.

(a) First consider the two Minkowski metrics that contract with the field strength tensors. Show that, to linear order in $\epsilon_{\mu\nu}$

$$-\frac{1}{4}(\eta_{\mu\alpha}+\epsilon_{\mu\alpha})(\eta_{\nu\beta}+\epsilon_{\nu\beta})F^{\mu\nu}F^{\alpha\beta} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\epsilon_{\mu\nu}}{2}F^{\mu\alpha}F^{\nu}{}_{\alpha} \,. \tag{5}$$

(b) Now, consider the determinant of the metric, $|\eta|$. Show that, to linear order in $\epsilon_{\mu\nu}$

$$\sqrt{|\eta+\epsilon|} = \sqrt{|\eta|} - \frac{\epsilon_{\mu\nu}}{2}\eta^{\mu\nu} \,, \tag{6}$$

where we assume that $\sqrt{|\eta|} = 1$. Recall that the determinant of the metric is defined as

$$|\eta| = \epsilon^{\mu\nu\rho\sigma}\eta_{0\mu}\eta_{1\nu}\eta_{2\rho}\eta_{3\sigma} \,. \tag{7}$$

(c) Combining these results, show that the functional derivative of the action with respect to the metric is

$$\frac{\delta S[A_\mu]}{\delta\eta_{\mu\nu}} = -\frac{1}{2}\int d^4x \left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{\eta^{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}\right] \,. \tag{8}$$

The tensor that is the integrand is called the stress-energy tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{\eta^{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta} \,. \tag{9}$$

Note that it is a symmetric tensor: $T^{\mu\nu} = T^{\nu\mu}$.

(d) What is the trace of the stress-energy tensor, $T^{\mu}{}_{\mu}$?

(e) Is the stress-energy tensor conserved? That is, what is its divergence

$$\partial_\mu T^{\mu\nu} = \,? \tag{10}$$

(f) Can you provide an interpretation for what it means for the stress-energy tensor to arise from the functional derivative of the action with respect to the metric? Justify this by analogy to electromagnetism.