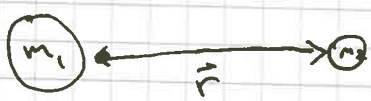
Physics 411: Classical Mechanics II

General Relativity

General Relativity is the geometric formulation of gravity. It is the extension of Newtonian gravitation that applies universally, but is especially required to describe phenomena when gravitational fields are strong, particles move near the speed of light, or have no intrinsic mass. This class will be an introduction to general relativity, or GR, through physical expectations, the mathematics of manifolds, and applications to phenomena like black holes, cosmology, and gravitational waves.

To begin, let's remind ourselves about Newtonian gravity, which you may not have thought about since freshman year. The gravitational force between two objects of mass m_1 and m_2 separated by a distance $\vec{r}$ is:



$$\Rightarrow \quad \vec{F}_G = -\frac{G_N m_1 m_2}{|\vec{r}|^2}\hat{r}$$

where we have expressed the force as acting on m_2. There are several features of this force law. First, Newtonian gravity postulates the weak equivalence principle: the gravitational mass and inertial mass are identical. This enables us to use Newton's second law to determine the dynamics of masses due to the force of gravity.

For instance, Newton's second law for mass m_2 is:

$$\vec{F}_G = -\frac{G_N m_1 m_2}{|\vec{r}|^2}\hat{r} = m_2 \vec{a} = m_2 \frac{d^2\vec{r}}{dt^2}.$$

The weak equivalence principle states that the m_2 on both sides of this equation are equal, so they can be cancelled. That is, the acceleration of mass m_2 is just:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\frac{G_N m_1}{|\vec{r}|^2}\hat{r},$$

which only depends on properties of mass m_1. To quantify this feature, we define the gravitational fied $\vec{G}$ of a mass m to be:

$$\vec{G} = -\frac{G_N m}{|\vec{r}|^2}\hat{r}$$

This is exactly analogous to electric field in electromagnetism. An electric field $\vec{E}$ is defined as the force per unit charge:

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|^2}\hat{r},$$ where Q is the electric charge of the object of interest. Thus, by analogy, mass is the "charge" of gravity: the force on an object is proportional to its mass. The weak equivalence principle ensures that this gravitational charge is inertial mass.

Another feature of gravitational force or gravitational field is the inverse-square law. The inverse square law is also a feature of electric field, and ultimately comes from the fact that we live in three-dimensions. This is captured in electromagnetism by Gauss's Law.

Recall Gauss's law for electric field:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}, \quad \text{where } d\vec{A} \text{ is an area element}$$

on the surface of a Gaussian surface, and Q_{enc} is the total electric charge within the surface. For a spherically-symmetric charge distribution, the Gaussian surface can be chosen to be a sphere centered on the charge. Then, the electric field is purely in the radial direction and the area of the sphere is $A = 4\pi|\vec{r}|^2$. Gauss's law then implies:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{enc}}{|\vec{r}|^2}\hat{r}, \text{ in 3 dimensions.} \qquad (\star)$$

The surface area of an n-dimensional sphere is

$$A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}|\vec{r}|^{n-1}, \text{ where } \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

is the Euler gamma function. So, Gauss's law for electromagnetism in n-dimensions would imply that the electric field is:

$$\vec{E} = \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}\varepsilon_0} \frac{Q}{|\vec{r}|^{n-1}}\hat{r}.$$

For $n=3$, $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$, and this reduces to $(\star)$.

Gauss's law in electromagnetism requires the <u>principle of superposition</u>: the electric field from a collection of charges is just the sum of electric fields from each charge individually. This is why the right side of Gauss's law is Q_{enc}, the total (sum) of the charges in the Gaussian sphere.

Assuming superposition for Newtonian gravity (as we usually do) implies the same Gauss's law for the gravitational field, $\vec{G}$:

$$\oint \vec{G} \cdot d\vec{A} = -4\pi G_N M_{enc},$$

where now M_{enc} is the enclosed mass in the Gaussian surface. This integral form is useful for situations with high symmetry, but the differential form is more useful generally. Using the divergence theorem, Gauss's law for Newtonian Gravity is:

$$\boxed{\vec{\nabla} \cdot \vec{G} = -4\pi G_N \rho}$$

Where ρ is the mass-density. This is exactly analogous to $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ where ρ is the charge density in electromagnetism.

We can go even further. The gravitational field $\vec{G}$ is curl free:

$$\vec{\nabla} \times \frac{\hat{r}}{|\vec{r}|^2} = 0,$$

so we can express it as the gradient of a potential Φ:

$$\vec{G} = -\vec{\nabla}\Phi$$

Gauss's law in Newtonian gravity then just becomes Poisson's equation:

$$\boxed{\nabla^2 \Phi = 4\pi G_N \rho}.$$

In terms of the gravitational potential Φ, the acceleration of a massive object that feels $\vec{G}$ is just

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi .$$

In some sense, this is everything in Newtonian gravity.

From the assumptions:

- Weak equivalence principle: gravitational mass = inertial mass
- Superposition: total field = Σ fields
- (- Gauss's law in 3D: inverse square law) implied

we are uniquely led to one gravitational potential describing everything in Newtonian gravity.

However, as you probably guessed, these assumptions aren't necessarily valid. They do not imply a universal speed of light, for example. Additionally, Newton's second law treats time special: it just marches along, unaffected by gravity. That is, Newtonian gravity violates special relativity by at least one of the assumptions above. Superposition turns out to be the problematic assumption, so we will need to figure out a consistent set of physically-motivated assumptions on which to build a new, relativistic, theory of gravity. But we'll get to special relativity next lecture.