Gravitation

We are now in the position to combine our physics knowledge and interpretation of gravity with the insights we have gained over the past few weeks about manifolds. We would like to directly construct an equation that immediately, explicitly states the physics insight ve have about gravity, geometry, and energy. That is, we want an equation that says:

$$\text{Geometry} = \text{Energy Density.}$$

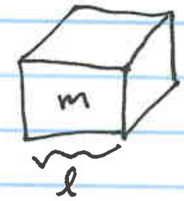
The shape of spacetime at one point is determined by the energy density at that point.

We have a lot of guiding principles. First, we want this to be a tensorial equation, so that it is independent of coordinate system used. If that's the case, then the honest tensor that describes energy density is the stress-energy tensor: $T_{\mu\nu}$. Properly, only the 00 component is the energy density:

$T_{00} = \dfrac{E}{Vol}$, and other components correspond to momentum density, shear, and stresses. Under Lorentz transformations in flat space-time (or coordinate transformations in general) energy turns into momentum, or vice-versa, so only the total tensor $T_{\mu\nu}$ has a physical interpretation. For example, for a particle at rest, its energy density would be

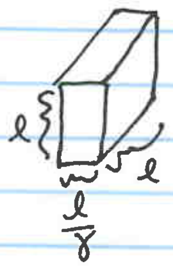
$T_{00} = \dfrac{mc^2}{Vol}$, while after a Lorentz boost, this would additionally contribute to momentum.

It's useful to think about transformations of the energy density to motivate the form of the two-index stress energy tensor. Let's consider a block of mass m with each dimension of size ℓ:



At rest, the energy density of this block is: mc^2/ℓ^3.

Now, if we Lorentz boost with velocity $\vec{v}$, the block is Lorentz contracted in the direction of motion:



$\longrightarrow \vec{v}$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

So, the volume of the boosted block is

$$\text{Vol} \rightarrow \frac{\ell^3}{\gamma} = \frac{\text{Vol}}{\gamma}$$

The energy of the boosted block is γmc^2 and so the energy density transforms as

$$\frac{E}{\text{Vol}} \rightarrow \frac{\gamma mc^2}{\left(\frac{\text{Vol}}{\gamma}\right)} = \gamma^2 \frac{mc^2}{\text{Vol}}$$

This is consistent with the stress energy tensor being a two index tensor. Under a coordinate transformation, $T_{\mu\nu}$ transforms as

$$T_{\mu\nu} \rightarrow \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta},$$ and for a Lorentz

transformation, the Jacobian matrices are just some collection of boost factors γ and velocities. The fact that there are two Jacobian matrices corresponds to the two factors of γ we identified above.

The Stress energy tensor is two-index, symmetric: $T_{\mu\nu} = T_{\nu\mu}$, and conserved. Conservation of stress-energy in flat space means:

$\partial_\mu T^{\mu\nu} = 0$, which you proved in electromagnetism

in HW3. In a general curved space-time manifold, this conservation law is not a tensor relationship. To make it a tensor, in curved space, we need the covariant derivative:

$\nabla_\mu T^{\mu\nu} = 0$. This is what we mean by "conserved" in general relativity.

So, if we are to write down such a geometry = energy equation, we need geometry to be represented by a symmetric, two-index tensor that is conserved. We know just the thing from a few lectures ago. The Einstein tensor, $G_{\mu\nu}$ is

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is symmetric, $G_{\mu\nu} = G_{\nu\mu}$, and is conserved:

$$\nabla_\mu G^{\mu\nu} = 0.$$

So, this motivates the equation:

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu}$, where k is

some constant. To determine the constant, we will get <u>most</u> of it from dimensional analysis here; you book provides another derivation of K. For dimensional analysis, the units of the stress-energy tensor are:

$$[T_{\mu\nu}] = \frac{[\text{energy}]}{[\text{Vol}]} = ML^{-1}T^{-2},$$

as it represents energy density. On the geometry side, the Ricci tensor $R_{\mu\nu}$ is formed from a contraction of the Riemann tensor:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}.$$

So, the units of the Ricci ~~scalar~~ tensor are

$[R_{\mu\nu}] = [\partial][\Gamma]$. Recall that the connection

is: $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$

and so the units of the connection are:

$$[\Gamma] = [g^{-1}][\partial][g] = [\partial]$$

Remember that the metric with upstairs and downstairs indices has inverse units because:

$$g_{\mu\lambda} g^{\lambda\nu} = \delta^{\nu}_{\mu}, \text{ a pure number } (0 \text{ or } 1).$$

Therefore, the units of the Ricci tensor are

$[R_{\mu\nu}] = [\partial]^2 = L^{-2}$. A partial derivative

has units of inverse length because we are always allowed to use Cartesian coordinates in which

$\partial_x = \frac{\partial}{\partial x}$, for example. (The metric might be

nasty; cf. the sphere with (x,y) coordinates.)

So the units on the two sides of the equation we identified are:

$$[R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R] = L^{-2} = [k][T_{\mu\nu}] = [k] M L^{-1} T^{-2}$$

or that $[k] = M^{-1} L^{-1} T^{2}$.

What could produce these units for k? Special relativity is the flat-space limit of general relativity, so the speed of light c should be present. Additionally, Newton's universal gravitation is the weak-fied/low curvature/small energy density limit so G_N should appear. The units of these terms are:

$$[c] = L T^{-1} \quad , \quad [G_N] = M^{-1} L^{3} T^{-2},$$

where we use Newton's second law, $\frac{G_N M}{r^2} = \frac{d^2 r}{dt^2}$ to determine units of G_N. So, k is some general product of c and G_N raised to some powers:

$$[k] = [c]^{\alpha}[G_N]^{\beta} = L^{\alpha} T^{-\alpha} M^{-\beta} L^{3\beta} T^{-2\beta}$$

$$= L^{\alpha+3\beta} M^{-\beta} T^{-\alpha-2\beta} = M^{-1} L^{-1} T^{+2}$$

Then, $\beta = 1$, by matching mass dependence, so that

$$L^{\alpha+3} M^{-1} T^{-\alpha-2} = M^{-1} L^{-1} T^{+2}$$

It then follows that $\alpha = -4$ so that

$[k] = \left[\frac{G_N}{c^4}\right]$. This gets us nearly there;

there are possible pure number factors that we can't get this way. It is described how to get it in the book, but I won't discuss it here. Finally, we have:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G_N}{c^4} T_{\mu\nu}}$$

This system of equations is called Einstein's equation, and it appears, among other places in the animated film "Triplets of Belleville".

The equations in this form are similar to Newton's second law. On the left, the Ricci tensor and scalar consist of up to second derivatives of the metric and $T_{\mu\nu}$ is a source for space-time curvature. However, like Newton's second Law, we can equivalently express the content of this equation as an action with the principle of least action. Let's first imagine the vacuum, $T_{\mu\nu} = 0$. Then, the action exclusively generates, by the Euler-Lagrange equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

The Lagrangian must be a scalar (invariant to coordinate transformations) and the integration measure must also be coordinate-invariant. We know how to do the second thing:

$$S[g_{\mu\nu}] = \int d^4x \sqrt{|g|}\, \mathcal{L},$$

where $\mathcal{L}$ is the Lagrangian. What is the simplest, most obvious Lagrangian? Just the Ricci scalar! The action:

$$\boxed{S[g_{\mu\nu}] = \frac{c^4}{16\pi G_N} \int d^4x \sqrt{|g|}\, R}$$

is called the Einstein-Hilbert action. Let's see how this produces Einstein's equation.

As we are used to by now, the variation of the action wrt the metric is

$$\frac{\delta S}{\delta g_{\mu\nu}} = \lim_{\varepsilon_{\mu\nu}\to 0} \frac{S[g_{\mu\nu}+\varepsilon_{\mu\nu}] - S[g_{\mu\nu}]}{\varepsilon_{\mu\nu}} = 0$$

We've already calculated that

$$\sqrt{|g+\varepsilon|} = \sqrt{|g|}\left(1 - \frac{g^{\mu\nu}}{2}\varepsilon_{\mu\nu} + \dots\right)$$

and $R = g^{\mu\nu}R_{\mu\nu} \to (g^{\mu\nu}+\varepsilon^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}R_{\mu\nu}(g+\varepsilon)$,

where the final expression is the Ricci tensor evaluated with a modified metric. That is, we have:

$$S[g_{\mu\nu}+\varepsilon_{\mu\nu}] - S[g_{\mu\nu}] = \frac{c^4}{16\pi G_N}\int d^4x \sqrt{|g|}\, \varepsilon^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \delta R_{\mu\nu}\right),$$

where $\delta R_{\mu\nu}$ is the expansion of $R_{\mu\nu}(g+\varepsilon)$ to linear order in $\varepsilon_{\mu\nu}$. This is 0 as it can be written as a total derivative. This is a result of the fact that the Riemann tensor is a commutator of derivatives:

$$R^{\rho}{}_{\mu\nu} \sim [\nabla_\mu, \nabla_\nu]$$

The variation of this term, as it has two covariant derivatives, is proportional to a covariant derivative acting on some tensor, so we can safely ignore it. Therefore $\delta R_{\mu\nu} = 0$ and this implies that

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \text{ as promised.}$$

To include energy density is trivial. You showed in HW 3 that the variation of a Lagrangian wrt the metric is the stress-energy tensor:

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{1}{2}\int d^4x \sqrt{|g|}\, T^{\mu\nu}.$$

Therefore, we just need to specify what sources energy (electromagnetism, dust, etc.), write its Lagrangian, and add the Einstein-Hilbert action. That is:

$$S = \int d^4x \sqrt{|g|}\left(\frac{c^4}{16\pi G_N} R + \mathcal{L}\right) \quad \text{and}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu} \quad \text{contain the same}$$

physics.