Einstein's Equations

Last lecture, we introduced Einstein's equations which describe the curvature of space-time in the presence of energy density:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu}} \qquad (1)$$

We were also able to write an action for which Einstein's equations are a consequence of the principle of least action. This is:

$$\boxed{S = \int d^4x \sqrt{|g|} \left(\frac{c^4}{16\pi G_N} R + \mathcal{L}_{matter} \right)}, \qquad (2)$$

where $\mathcal{L}_{matter}$ is the Lagrangian for matter (not space-time; E&M, dust, planets, etc.). Eqs. (1) and (2) have the exact same physical content and define the general theory of relativity. In the rest of this class, we will study explicit consequences of Einstein's equations.

In this lecture we survey some properties of Einstein's equations, namely:

- non-linearity,
- cosmological constant, and
- restrictions on the stress-energy tensor

Newton's law of gravitation is linear. This means that the gravitational field of two masses m_1 and m_2 is just the superposition of their individual fields:

$$\vec{F}_G = \frac{G_N m_1 m}{r_1^2} \hat{r}_1 + \frac{G_N m_2 m}{r_2^2} \hat{r}_2$$

This linearity makes Newtonian gravity easy to study and enables exact solutions, even for complicated mass configurations. For example, we can directly solve for the orbit of a satellite around several large mass planets. Assuming that the mass of the satellite is small, Newton's equations are:

$$\sum_i \frac{G_N m_i}{r_i^2} \hat{r}_i = \ddot{r}$$

Even more simply, in Newtonian gravity, we can exactly solve for the motion of two bodies orbiting one another. They each orbit about their mutual center of mass, and because angular momentum is conserved, the orbit with the same angular velosity.

In general relativity, we lose linearity and superposition. Recall that the connection is:

$$\Gamma^{\rho}_{\mu\nu} = g^{\rho\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right),$$

which is already non-linear in the metric. The Riemann tensor is schematically

$R \sim \partial\Gamma + \Gamma\Gamma$, which contains terms that are

quatric in the metric. Actually, its worse than that. The inverse metric can be expressed as an infinite series of the metric:

$$g^{\mu\nu}(g_{\rho\sigma}) \overset{\sim}{=} \eta^{\mu\nu} \pm \eta^{\mu\rho}\eta^{\nu\sigma} g_{\rho\sigma} \pm \eta^{\mu\rho}\eta^{\sigma\lambda}\eta^{\alpha\nu} g_{\rho\sigma} g_{\lambda\alpha} + \dots$$

This can be verified by taylor expanding the metric $g_{\mu\nu}$ about flat space-time:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \dots$$

So, Einstein's equations are <u>very</u> non-linear, and actually consist of powers of metric to arbitrarily high orders! This means that if we find one solution $g_{1\mu\nu}$ for the metric from Einstein's equations, that can't be added to another solution $g_{2\mu\nu}$ to find a third solution. Superposition fails.

This means we need another plan of attack for solving Einstein's equations. The first solution we will study starting Wednesday, will exploit symmetries extensively. But I'm getting ahead of myself.

We argued for Einstein's equations from geometry (Riemann and Ricci) and physics (stress-energy). The divergence-free nature of the Einstein tensor $G_{\mu\nu}$ was guaranteed by the Bianchi identity of the Riemann tensor. However, we know of another object that, by assumption, has no divergence: the metric. There's no principle, mathematically or physically that prohibits the addition of a term into Einstein's equations that is propotional to the metric:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu},$$

Λ is called the <u>cosmological constant</u> and is just some number. It is also called the <u>vacuum energy</u> because it exists (or can) even when $T_{\mu\nu}=0$. The history of such a term is that Einstein needed to add it to be able to describe an infinite, static universe, which was his belief. However, with observations by Hubble that the universe was expanding, Einstein called it his "greatest blunder." Nevertheless, we will see that a cosmological constant is necessary to describe our visible universe.

A vacuum energy might seem odd, but you're actually quite familiar with it from quantum mechanics. For example consider the harmonic oscillator potential, $V \propto x^2$. The ground state energy, that is, the energy with no injection of energy, is non-zero:

$$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{k}{m}}, \text{ where } \omega \text{ is the frequency.}$$

This is like a vacuum energy: even with no excitation, the system has non-zero energy.

Can we estimate what this energy density is in general relativity? General relativity is a <u>classical field theory</u> in which $\hbar$ does not appear. We can estimate the distance l_p down to which we might expect GR to ~~re~~ remain classical. l_p is called the Planck length and we can find it from some combination of the constants G_N, c, and $\hbar$:

$$l_p = G_N^{\alpha} c^{\beta} \hbar^{\gamma}.$$ Recall that the units of

The things on the right side are:

$$[G_N] = M^{-1} L^3 T^{-2}$$
$$[c] = L T^{-1}$$
$$[\hbar] = M L^2 T^{-1}$$

If this product is to have total dimensions of L, we must have:

$$\underbrace{-\alpha + \gamma = 0}_{M} \qquad \underbrace{3\alpha + \beta + 2\gamma = 1}_{L} \qquad \underbrace{-2\alpha - \beta - \gamma = 0}_{T}$$

This is solved by: $\alpha = 1/2$, $\beta = -3/2$, $\gamma = 1/2$.

That is, the planck length is:

$$l_p = G_N^{1/2} c^{-3/2} \hbar^{1/2} = \sqrt{\frac{\hbar G_N}{c^3}}$$

Plugging in numbers this is approximately: $l_p \sim 10^{-35}\,m$.

To find an energy density, we also need the Planck energy, E_p. Doing the same exercise with units, we find

$$E_p = G_N^{-1/2} c^{5/2} \hbar^{1/2} = \sqrt{\frac{c^5 \hbar}{G_N}} \cong 10^9 \text{ J}$$

This is an insane amount of energy! The corresponding Planck energy density is then

$$\frac{E_P}{\ell_P^3} \cong 10^9 \cdot 10^{+105} \cong 10^{115} \text{ J/m}^3$$

This is an unimaginable energy density! So unimaginable, that it is nowhere near what we actually observe in the universe. In fact it's off by about 120 orders of magnitude! We'll revisit this later. Unfortunately this "cosmological constant problem" has no known solution within physics; the best solution is an appeal to the anthropic principle: in order for the conditions in the universe to be such that life and eventually us could exist, we must observe a small cosmological constant.

Finally, we'll end today with a few comments on the stress-energy tensor. Observable energies are always non-negative; or, more precisely, always have a lower bound. A system without a lower bound on energy is unphysical. We might as well call that lower bound on energy 0. (A negative energy can just redefine the cosmological constant.) In terms of the stress energy tensor, the T_{00} component is the energy density, so we might think that requiring:

$T_{00} \geq 0$ is a good thing to do.

However, this is not a coordinate-invariant statement.

We can make it coordinate invariant by noting that

$T_{00} = T_{\mu\nu} t^\mu t^\nu$, where $t^\mu = (1,0,0,0)^\mu$. Such

a vector is <u>time-like</u> $t \cdot t < 0$, and this is a coordinate-invariant quality of a vector. Coordinate transformations turn t into other, arbitrary time-like vectors. So, the coordinate-invariant way to enforce positive energy density is the statement that:

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \text{ for } \underline{\text{all}} \text{ time-like vectors } t^\mu.$$

This constraint is called the <u>Weak Energy Condition</u>. It seems eminently reasonable, but there are actually several ways that we can implement such a restriction. In particular, gravity in general relativity doesn't only couple to energy, T_{00}, but to every component of the stress-energy tensor. So, for gravity to seem sensible, for example, to be universally attractive, we do not need to enforce the weak energy condition, because pressures, momentum, etc., all affect gravitation. For example, we could impose the <u>Null Energy Condition</u> which is the requirement that

$$T_{\mu\nu} l^\mu l^\nu \geq 0 \text{ for all light-like vectors } l^\mu.$$

This requirement allows for negative energy densities, but only if pressures are significantly large. You'll explore these restrictions for the electromagnetic stress-energy tensor in homework.