The Metric of Maximally Symmetric Space-Time

In this lecture, we'll work to calculate a metric that solves Einstein's equations in vacuum. That is, we set $T_{\mu\nu}=0$. In that case, Einstein's Equations are:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0$$

Let's take the trace of this equation by contracting indices with $g^{\mu\nu}$:

$$g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=R-\frac{D}{2}R=0$$

Recall that $g_{\mu\nu}g^{\mu\nu}=D$, the dimension of space-time. Therefore, $R=0$ as long as $D>2$. So, Einstein's equations in vacuum reduce to:

$$R_{\mu\nu}=0.$$

We will solve this equation by postulating a form of the metric, and then identifying the equations for each component of the metric. We want to study the metric for a maximally symmtric space; In four dimensions, this means 10 killing vectors. We can write this metric in any coordinates that we want and we know of two coordinate systems that we understand and manifest (= make obvious) the 10 symmetry transformations Those are: Cartesian coordinates and spherical coordinates. We could use Cartesian, but it's not easy writing down a metric that is the most general possible that exhibits 10 symmetries. Spherical coordinates are much nicer.

To see this, let's first focus on the spatial part of the metric:

$$ds^2 = e^{2\beta(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2.$$

This is the most general form of a metric that exibits spherical symmetry. The angular coordinates have a rotational invariance for any fixed radius r. As we move in r, $dr \neq 0$, the radial metric component can change, in a very analogous way to how the electric field of a point charge is spherically symmetric, but decreases in strength as r increases. We have expressed this function in the form $e^{2\beta(r)}$, which we can do wlog. This will make calculating Christoffel symbols easier.

Now, for the time component of the metric. For now, we will assume that we are considering static space-times: that is, no dependence on time. Then, with that assumption and spherical symmetry, the time component of the metric can only depend on the radial coordinate r:

$$ds^2 = -e^{2\alpha(r)} dt^2.$$

For compactness here and in what follows, we set the speed of light $c=1$.

So, the metric we are studying is:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2.$$

Extremely nicely, this metric is diagonal!

So, we can use the results from a couple homeworks ago to evaluate the Cristoffel connections. Recall that, for a diagonal metric:

$$\Gamma^\lambda_{\mu\nu} = 0, \quad \Gamma^\lambda_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}$$

$$\Gamma^\lambda_{\mu\lambda} = \partial_\mu \ln\sqrt{|g_{\lambda\lambda}|}, \quad \Gamma^\lambda_{\lambda\lambda} = \partial_\lambda \ln\sqrt{|g_{\lambda\lambda}|}$$

where μ, ν, λ are all distinct. Let's identify the possible connections of the form $\Gamma^\lambda_{\mu\mu}$. In this connection we take a derivative wrt λ. The only coordinate dependence in the metric is in r and θ. So, $\lambda = r$ or θ. So, the non-zero connections are:

$$\Gamma^r_{tt}, \quad \Gamma^r_{\theta\theta}, \quad \Gamma^r_{\phi\phi}, \quad \Gamma^\theta_{\phi\phi}$$

We find: $\Gamma^r_{tt} = +\frac{1}{2} e^{-2\beta(r)} \partial_r e^{2\alpha(r)} = e^{2(\alpha-\beta)} \partial_r \alpha$

$$\Gamma^r_{\theta\theta} = -\frac{1}{2} e^{-2\beta(r)} \partial_r r^2 = -r e^{-2\beta}$$

$$\Gamma^r_{\phi\phi} = -\frac{1}{2} e^{-2\beta} \partial_r r^2 \sin^2\theta = -r e^{-2\beta} \sin^2\theta$$

$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2} r^{-2} \partial_\theta r^2 \sin^2\theta = -\frac{1}{2} r^{-2}\left(\frac{\partial}{\partial\theta}\right) r^2 \sin^2\theta = -\sin\theta\cos\theta$$

Now, onto the $\Gamma^\lambda_{\mu\lambda}$ connections. This involves a μ derivative, so as before, $\mu = r$ or θ. The non-zero connections are:

$$\Gamma^t_{rt}, \ \Gamma^\theta_{r\theta}, \ \Gamma^\phi_{r\phi}, \ \Gamma^\phi_{\theta\phi}.$$

We then find:

$$\Gamma^{t}_{rt} = \partial_r \ln\sqrt{e^{2\alpha}} = \partial_r \alpha,$$

$$\Gamma^{\theta}_{r\theta} = \partial_r \ln\sqrt{r^2} = \frac{1}{r}$$

$$\Gamma^{\phi}_{r\phi} = \partial_r \ln\sqrt{r^2\sin^2\theta} = \frac{1}{r}$$

$$\Gamma^{\phi}_{\theta\phi} = \partial_\theta \ln\sqrt{r^2\sin^2\theta} = \cot\theta$$

Finally, we consider those connections with all the same index, $\Gamma^\lambda_{\lambda\lambda}$ The only possibility is if $\lambda = r$ in which

$$\Gamma^{r}_{rr} = \partial_r \ln\sqrt{e^{2\beta}} = \partial_r\beta.$$

We know all of the Christoffel symbols! Let's now calculate the Riemann tensor. As a function of Christoffel symbols, recall that the Riemann Tensor is:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

There's a lot more going on here, so I'll just illustrate some examples and then write the non-zero components. Consider, for example the component

$$R^{r}{}_{trt} = \partial_r \Gamma^{r}_{tt} - \partial_t \cancelto{0}{\Gamma^{r}_{rt}} + \Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{tt} - \Gamma^{r}_{t\lambda}\Gamma^{\lambda}_{rt}$$

$$= \partial_r \Gamma^{r}_{tt} + \Gamma^{r}_{rr}\Gamma^{r}_{tt} - \cancelto{0}{\Gamma^{r}_{tr}\Gamma^{r}_{tt}} - \Gamma^{r}_{tt}\Gamma^{t}_{rt}$$

$$= \partial_r\left(e^{2(\alpha-\beta)}\partial_r\alpha\right) + (\partial_r\beta)\left(e^{2(\alpha-\beta)}\partial_r\alpha\right) - \left(e^{2(\alpha-\beta)}\partial_r\alpha\right)(\partial_r\alpha)$$

$$= e^{2(\alpha-\beta)}\left(2(\partial_r\alpha)^2 - 2(\partial_r\alpha)(\partial_r\beta) + \partial_r^2\alpha + (\partial_r\alpha)(\partial_r\beta) - (\partial_r\alpha)^2\right)$$

$$= e^{2(\alpha-\beta)}\left((\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta) + \partial_r^2\alpha\right)$$

Let's see what other Riemann tensors would be:

$$R^t{}_{ttt} = \partial_t \Gamma^t_{tt} - \partial_t \Gamma^t_{tt} + \Gamma^t_{t\lambda}\Gamma^\lambda_{tt} - \Gamma^t_{t\lambda}\Gamma^\lambda_{tt} = 0$$

$$R^\theta{}_{t\theta t} = \cancelto{0}{\partial_\theta \Gamma^\theta_{tt}} - \cancelto{0}{\partial_t \Gamma^\theta_{t\theta}} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{tt} - \cancelto{0}{\Gamma^\theta_{t\lambda}\Gamma^\lambda_{\theta t}}$$

$$= \Gamma^\theta_{\theta r}\Gamma^r_{tt} = \frac{1}{r}\cdot e^{2(\alpha-\beta)}\partial_r \alpha = e^{2(\alpha-\beta)}\left(\frac{1}{r}\partial_r \alpha\right)$$

$$R^\phi{}_{t\phi t} = \cancelto{0}{\partial_\phi \Gamma^\phi_{tt}} - \cancelto{0}{\partial_t \Gamma^\phi_{t\phi}} + \Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{tt} - \cancelto{0}{\Gamma^\phi_{t\lambda}\Gamma^\lambda_{\phi t}}$$

$$= \Gamma^\phi_{\phi r}\Gamma^r_{tt} = \left(\frac{1}{r}\right) e^{2(\alpha-\beta)}\partial_r \alpha = e^{2(\alpha-\beta)}\left(\frac{1}{r}\partial_r \alpha\right)$$

Combining all of these terms, we can find the Ricci tensor tt component:

$$R_{tt} = R^r{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t} =$$

$$= e^{2(\alpha-\beta)}\left[(\partial_r \alpha)^2 + \partial_r^2 \alpha - (\partial_r \alpha)(\partial_r \beta) + \frac{2}{r}\partial_r \alpha\right]$$

While I won't do it here, the other components of the Ricci tensor can also be found in this manner. They are:

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha\, \partial_r \beta + \frac{2}{r}\partial_r \beta$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r \beta - \partial_r \alpha) - 1\right] + 1$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

Now, we set each of these components to 0 and solve for α and β.

The simplest way forward to solve for α and β is to combine R_{tt} and R_{rr}. Note that

$$R_{tt}e^{-2(\alpha-\beta)} + R_{rr} = 0 = \frac{2}{r}\left(\partial_r\alpha + \partial_r\beta\right), \text{ or that}$$

$$\beta = -\alpha + c, \text{ where } c \text{ is some constant.}$$

The constant c can be set to 0 by appropriate relabeling of time, so we have $\beta = -\alpha$.

Now, applying this to the $R_{\theta\theta}$ component, we have

$$e^{2\alpha}\left[2r\,\partial_r\alpha + 1\right] = 1 = \partial_r\left(r e^{2\alpha}\right) = 1$$

That is, if we let $f \equiv re^{2\alpha}$, then $\partial_r f = 1$ or that $f = r - R_s$, for some constant R_s. That is,

$$e^{2\alpha} = 1 - \frac{R_s}{r} \quad \text{and our metric is}$$

$$\boxed{ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2}$$

This is called the <u>Schwarzschild metric</u>. The <u>Schwarzschild radius</u> R_s can be found by matching to Newtonian gravity in the flat space ($r\to\infty$) limit. Doing this, we find that $R_s = 2GM$, or, with factors of c restored,

$$\boxed{R_s = \frac{2GM}{c^2}}$$

Something funny happens to this metric when $r \to R_s$...