More Schwarzschild

Last lecture, we derived the Schwartzschild solution to Einstein's equation in vacuum. The metric we found was:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2.$$

As I hinted at at the end of that lecture, this metric appears to have some strange feature when $r = R_s$. At that point, the time and radial distances vanish, and when $r < R_s$, the sign of time and radius flip! This would, at least on the surface, suggest that the geometry of this spacetime is very strange around $r = R_s$. However, the metric $g_{\mu\nu}$ is not coordinate invariant, so its also possible that the weirdness is just a consequence of the coordinates that we use. To conclusively determine what happens we need to calculate curvature scalars; that is, objects that are independent of coordinates.

One of these curvature scalars is just the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$. We seemed to have a pretty tight argument that $R = 0$ from Einstein's equations and $T_{\mu\nu} = 0$. However, let's revisit this and carefully calculate R with our new insight of the Schwarzschild solution.

The Ricci scalar for Schwarzschild is:

$$R = g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi},$$ because

the metric is diagonal.

We had showed that $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$ and

$g^{\phi\phi} = \dfrac{g^{\theta\theta}}{\sin^2\theta} = \dfrac{1}{r^2\sin^2\theta}$, and so this simplifies

to

$$R = g^{tt}R_{tt} + g^{rr}R_{rr} + \frac{2}{r^2}R_{\theta\theta}.$$

Further, we had also shown that

$$R_{tt} = -\left(1-\frac{R_s}{r}\right)^2 R_{rr}.$$

Therefore, because $g^{tt} = -\left(1-\frac{R_s}{r}\right)^{-1}$, $g^{rr} = \left(1-\frac{R_s}{r}\right)$,

we have

$$\begin{aligned} g^{tt}R_{tt} + g^{rr}R_{rr} &= -\left(1-\frac{R_s}{r}\right)^{-1}R_{tt} + \left(1-\frac{R_s}{r}\right)R_{rr} \\ &= +\left(1-\frac{R_s}{r}\right)R_{rr} + \left(1-\frac{R_s}{r}\right)R_{rr} \\ &= 2\left(1-\frac{R_s}{r}\right)R_{rr} \end{aligned}$$

Then, the Ricci scalar is:

$$2\left(1-\frac{R_s}{r}\right)R_{rr} + \frac{2}{r^2}R_{\theta\theta} = R.$$

When we solved for the metric, we had written

$$e^{2\alpha} = \left(1-\frac{R_s}{r}\right) \quad \text{or that} \quad \alpha = \frac{1}{2}\ln\left(1-\frac{R_s}{r}\right).$$

In what follows, let's use this expression, but we'll actually eliminate our functional knowledge of r for now. So, we will write

$$\alpha = \frac{1}{2}\ln(1-f),$$

for some function $f \equiv f(r)$. With this form, we can evaluate the Ricci tensor components. We have:

$$R_{rr} = -\frac{1}{2}\partial_r^2 \ln(1-f) - \frac{1}{2}\left(\partial_r \ln(1-f)\right)^2 - \frac{1}{r}\partial_r\left(\ln(1-f)\right)$$

$$= \frac{1}{2}\partial_r \frac{f'}{1-f} - \frac{1}{2}\frac{(f')^2}{1-f} + \frac{1}{r}\frac{f'}{1-f}$$

$$= \frac{1}{2}\frac{f''(1-f) + \cancel{(f')^2}}{(1-f)^2} - \cancel{\frac{1}{2}\frac{(f')^2}{1-f}} + \frac{1}{r}\frac{f'}{1-f}$$

$$= \frac{1}{2}\frac{f''}{1-f} + \frac{1}{r}\frac{f'}{1-f} = \frac{1}{2(1-f)}\frac{1}{r^2}\partial_r\left(r^2\partial_r f\right)$$

That is, $2\left(1-\underbrace{\frac{R_s}{r}}_{=f}\right)R_{rr} = \frac{1}{r^2}\partial_r\left(r^2\partial_r f\right)$

Similarly, the θ-component of the Ricci tensor is:

$$R_{\theta\theta} = -(1-f)\left[+\frac{2r}{2}\partial_r \ln(1-f) + 1\right] + 1$$

$$= -(1-f)\left[-r\frac{f'}{1-f} + 1\right] + 1 = +rf' - 1 + f + 1$$

$$= rf' + f = \partial_r(rf)$$

Then, the Ricci scalar can be written as:

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r f\right) + \frac{2}{r^2}\partial_r(rf) = R$$

If we say that $R=0$ in vacuum, then $f = \frac{R_s}{r}$, as anticipated. However, we implicitly assume that $r>0$. When $r=0$, this differential equation is subtle. To understand it, let's go back to Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \quad (c=1)$$

and trace both sides:

$$R - 2R = 8\pi G_N T \quad \text{or that} \quad R = -8\pi G_N T$$

We aren't just setting $T=0$ yet. Our differential equation is:

$$\frac{1}{r^2}\partial_r\left(r^2 \partial_r f\right) + \frac{2}{r^2}\partial_r(rf) = -8\pi G_N T$$

The most general stress-energy tensor that is spherically symmetric can always be expressed with only a non-zero T_{00} component, in some coordinates. Let's integrate this equation in some "Einsteinian volume" which is just a ball of radius $R>0$:

$$\int_0^R dr\,(\cancel{r^2} \cdot 4\pi)\left[\frac{1}{\cancel{r^2}}\partial_r\left(r^2\partial_r f\right) + \frac{2}{\cancel{r^2}}\partial_r(rf)\right]$$

$$= 4\pi\left(R^2 \partial_r f\big|_{r=R} + 2R\, f(R)\right)$$

Knowing that $f = \frac{R_s}{r}$, this is:

$$4\pi\left(R^2\left(-\frac{R_s}{R^2}\right) + 2R_s\right) = 4\pi R_s = -32\pi^2 G_N \int_0^R dr\, r^2 T$$

or that $$\boxed{4\pi \int_0^R dr\, r^2 T = -\frac{R_s}{2G_N}}$$

The right-hand side is independent of integration radius R, so therefore must the left-hand side be. Clearly $T \neq 0$, and the only thing that works is:

$$T = \frac{1}{4\pi r^2}\delta(r)\cdot\left(-\frac{R_S}{2G_N}\right)$$

We had found that $R_s = 2G_N M$, where M is the mass of the object, so,

$T = -\frac{M}{4\pi r^2}\delta(r)$, which is the energy density of a point mass at the origin. Putting it all together, the Ricci scalar for Schwarzshild is:

$$\boxed{R = \frac{8\pi G_N M}{4\pi r^2}\delta(r) = \frac{R_S}{r^2}\delta(r).}$$

This is 0 for $r>0$ and divergent when $r=0$.

The interpretation of this is that the space-time manifold is actually sick at $r=0$, but everywhere else, is pefectly fine. Therefore, the weirdness we found for $r=R_s$ in the schwarzschild metric is simply a coordinate artifact: with different coordinates we would explicitly see that $r=R_s$ is regular.

One can calculate other scalars and find the same result: only $r=0$ is an honest singularity in the spacetime. For example, the square of the Riemann tensor is:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G_N^2M^2}{r^6},$$ which is smooth at $r=R_s$.

So what's happening at $r=R_s$? Let's imagine that you and a friend play Rock, Paper, Scissors (2-out-of-3) to see who will travel toward the black hole.

Your friend loses, so they untether from the spaceship and freely fall toward a spherically symmetric point mass. With no angular velocity, your friend just has a changing radius and coordinate time:

$$ds_f^2 = -\left(1 - \frac{2GM}{r_f}\right) dt_f^2 + \left(1 - \frac{2GM}{r_f}\right)^{-1} dr_f^2$$

You, on the other hand, stay at a fixed radius, so only your coordinate time marches on:

$$ds_y^2 = -\left(1 - \frac{2GM}{r_y}\right) dt_y^2$$

Now, you see something weird. Your friend must move along a timelike geodesic because they are massive and cannot travel the speed of light. However, as r approaches $R_s = 2G_N M$, their time coordinate t_f must tick faster and faster to ensure that $ds_f^2 < 0$. From your perspective, then, it appears like they move more and more slowly as they approach $r = R_s$. Actually, you would see them take an infinite amount of time to get to the schwarzschild radius!

Unfortunately, from your friends perspective, they would get closer and closer to the point mass, pass through the schwarzschild radius, and never be heard from again. But we'll pick that up next time...