Geodesics of Schwarzschild

The geodesic equation in general relativity is the analogy of Newton's law of gravitation: it describes how particles move when exclusively under the influence of gravity. So, if we want to determine how particles move under the influence of gravity as defined by the Schwarzschild metric, we need to calculate geodesics.

Unfortunately, the geodesics of even the Schwarzschild metric are god-awful mess, so we won't even attempt to solve them. Instead we will approach the problem from a different direction. Let's denote a geodesic path as $x^\mu(\lambda)$, for some parameter λ, and its tangent vector is therefore $dx^\mu/d\lambda$.

Along the geodesic, there are a number of conserved quantities that will make our lives much easier. First, the projection of any Killing vector on the geodesic is conserved (constant):

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

To prove this, we just take the derivative projected along the direction of the geodesic:

$$\frac{dx^\nu}{d\lambda}\partial_\nu\left(K_\mu \frac{dx^\mu}{d\lambda}\right) = \frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda}\left(\nabla_\nu K_\mu\right) + K_\mu \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda}$$

We started with the partial derivative because $K_\mu dx^\mu/d\lambda$ is a scalar, and then using the product rule, we needed covariant derivatives. Recall that, for a Killing vector,

$\nabla_{(\nu}K_{\mu)} = 0$, and so it follows that

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\nabla_\mu K_\nu = 0.$$

Correspondingly, if $x^\mu(\lambda)$ is a geodesic, then

$$\frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0,$$

is the geodesic equation. Therefore, along a geodesic $K_\mu \, dx^\mu/d\lambda$ is indeed conserved.

Further, the length of the tangent vector along a geodesic is conserved:

$$\varepsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}.$$

While I won't go into the details, the argument is essentially the same as for the Killing vector, exploiting the geodesic equation and metric compatibility. In this lecture, we'll exclusively study time-like geodesics for which $\varepsilon = 1$, but one can also study light-like geodesics for which $\varepsilon = -1$. For the schwarzschild metric, because it is diagonal, we therefore have:

$$1 = -g_{tt}\left(\frac{dt}{d\lambda}\right)^2 - g_{rr}\left(\frac{dr}{d\lambda}\right)^2 - g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 - g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2$$

We would like to use the Killing vectors of the Schwarzschild metric to rewrite this in a nice form.

We have 4 Killing vectors at our disposal: time translation, and three rotations.

With two of the rotations, we are able to rotate our coordinates so that the geodesic lies exclusively in the equitorial plane, $\theta = \pi/2$. Note that the geodesic <u>must</u> lie in a plane because of the spherical symmetry. With this assumption, we then have

$$\frac{d\theta}{d\lambda} = 0 \quad \text{so that}$$

$$1 = -g_{tt}\left(\frac{dt}{d\lambda}\right)^2 - g_{rr}\left(\frac{dr}{d\lambda}\right)^2 - g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2$$

Now, we'll use two other Killing vectors. A time translation corresponds to a killing vector of

$$K_t^{\mu} = (1,0,0,0)^{\mu} \quad \text{or} \quad K_{t\mu} = g_{\mu\nu} K_t^{\nu} = (g_{tt}, 0, 0, 0)_{\mu}$$

Therefore, one conserved quantity is:

$$K_{t\mu} \frac{dx^{\mu}}{d\lambda} = g_{tt} \frac{dt}{d\lambda} = -E, \text{ which we call the "energy".}$$

The remaining unused killing vector generates translations of the azimuthal coordinate ϕ:

$$K_{\phi}^{\mu} = (0,0,0,1)^{\mu} \quad \text{or} \quad K_{\phi\mu} = g_{\mu\nu} K_{\phi}^{\nu} = (0,0,0,\underbrace{r^2 \sin^2\theta}_{= g_{\phi\phi}})_{\mu}$$

Another conserved quantity is then

$$K_{\phi\mu} \frac{dx^{\mu}}{d\lambda} = g_{\phi\phi} \frac{d\phi}{d\lambda} = r^2 \frac{d\phi}{d\lambda} = L, \text{ which we call}$$

"angular momentum". We can plug these quantities into the tangent vector length equation and find:

$$1 = -g_{tt}\left(\frac{E}{g_{tt}}\right)^2 - g_{rr}\left(\frac{dr}{d\lambda}\right)^2 - g_{\phi\phi}\left(\frac{L}{g_{\phi\phi}}\right)^2$$

With the explicit expressions for the metric components:

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right), \quad g_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1}, \quad g_{\phi\phi} = r^2$$

we find

$$\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 = -1 + \left(1 - \frac{2GM}{r}\right)^{-1} E^2 - \frac{L^2}{r^2}$$

or that

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 = -\frac{1}{2} + \frac{G_N M}{r} + \frac{1}{2}E^2 - \frac{L^2}{2r^2} + \frac{GM L^2}{r^3}}$$

This equation is exact for the Schwarzschild metric and is the generalization of conservation of energy in Newtonian gravity. In Newtonian gravity, we instead would find:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{G_N M m}{r}$$

The "E" here is honest energy, $\frac{1}{2}m\dot{r}^2$ is the radial kinetic energy, $L^2/2mr^2$ is the angular kinetic energy, and $-G_N M m/r$ is the gravitational potential energy. In general relativity, everything is expressed per unit mass, λ parametrizes the trajectory instead of time t, ~~and there are tw~~ and the conserved "energy" is $\frac{1}{2}E^2$, which is still just some constant. There are two other terms that are important: the "$-\frac{1}{2}$" is just a constant offset of the potential, but the really new term:

$$+\frac{G_N M L^2}{r^3} \quad \text{is responsible for } GR \neq \text{Newton.}$$

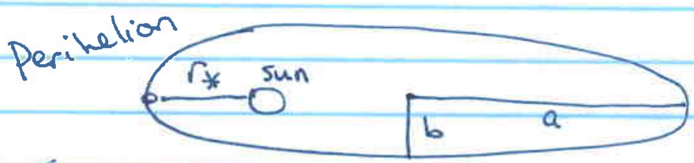
Especially at small r, this term becomes important.

There are many analyses that we can do from this equation; in the rest of this lecture, we will study the first verified prediction of general relativity. By multiplying by $\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}$, we can derive

an differential equation for r in terms of ϕ:

$$\left(\frac{dr}{d\phi}\right)^2 = -\frac{r^4}{L^2} + \frac{2G_N M}{L^2} r^3 + \frac{r^4 E^2}{L^2} - r^2 + 2GMr$$

This equation will allow us to study the angular dependence of the point of closest approach of an orbit. The point of closest approach in an elliptical orbit about the sun is called the perihelion. An exactly elliptical orbit is like:



~~Let's call the radius of the perihelion r_*. Then for an exactly elliptical orbit~~

$$\cancel{\frac{dr}{d\phi} =}$$

After orbiting through an angle of 2π, the planet would get back to exactly the same perihelion point. We can in general express this angular dependence as

$$r(\phi) = r_0 \left(1 + e\cos\phi\right)^{-1},$$ where e is called

the eccentricity, $e = 1 - \frac{b^2}{a^2}$. Newton's or Kepler's laws predict exactly elliptical orbits for two bodies. By contrast, general relativity has that extra term which will destroy perfect ellipses. In this case, the perihelion will precess; that is, the point of closest approach will itself orbit around the sun: The rate of this precession within general relativity can be calculated as the deviation from the 2π period exact ellipse. Let's do it.

The first step will be to consider the new variable

$$x = \frac{L^2}{G_N M r}, \quad \text{or} \quad r = \frac{L^2}{G_N M x}, \quad \text{Then,} \quad \frac{dr}{dx} = -\frac{L^2}{G_N M x^2}$$

This produces the differential equation:

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G_N^2 M^2} - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 = \cancel{E}\,\frac{E^2L^2}{G_N^2M^2}$$

Now, differentiate with respect to ϕ again; we find

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G_N^2M^2}{L^2}x^2$$

Everything on the left is what Newton would predict; that on the right is GR.

To solve this, we use the method of successive approximations. We will write

$x = x_0 + x_1 + \ldots$ and the procedure is the following.

First plug in $x = x_0$ on the left, and $x = 0$ on the right. Then with x_0, plug x_1 on the left and x_0 on the right, etc.

Doing this for x_0, we have

$$\frac{d^2x_0}{d\phi^2} - 1 + x_0 = 0$$, which has a solution of

$$x_0 = 1 + e\cos\phi$$, which is just the expected

elliptical orbit. Now, for x_1 we have

$$\frac{d^2x_1}{d\phi^2} + x_1 = \frac{3G_N^2M^2}{L^2}(1+e\cos\phi)^2$$

This can be solved and one finds

$$x_1 = \frac{3G_N^2M^2}{L^2}\left[\left(1+\frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right]$$

To study the precession of the perihelion, we're only interested in the one term that does not return to itself when $\phi \to 2\pi$. This is the middle term. So, the approximate solution x contains:

$$x \supset 1 + e\cos\phi + e\frac{3G_N^2M^2}{L^2}\phi\sin\phi \cong 1 + e\cos\left[\left(1 - \frac{3G_N^2M^2}{L^2}\right)\phi\right]$$

When $\phi \to 2\pi$, this does not return to itself. The rate of precession can be then calculated from the angular momentum L and the mass of the sun. For Mercury, the GR effect of the rate of precession is:

$$\Delta\phi = 43.0''/\text{century}$$, which was verified as

the missing precession rate by Arthur Eddington in an expedition to the island of Principe for the solar eclipse of 1919. Needless to say, Einstein was famous after that.