Black Holes

We've seen some weird features of the Schwarzschild solution:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

For an observer falling toward the mass M along a timelike, radial geodesic we discussed how someone stationary outside would never see the person actually make it to the radius $R_s = 2GM$. They would approach it more and more slowly and get more and more redshifted in doing so. However, we also argued that the weirdness with the metric at $r=2GM$ is a coordinate artifact and not physical. No coordinate invariant quantity has any singularity at this radius, so the person falling toward the mass wouldn't know when they passed through $r=2GM$. Spacetime would appear smooth there.

Nevertheless, something weird does happen at $r=2GM$. The Schwarzschild solution is static/time-independent, so there exists a Killing vector corresponding to time translations:

$$K^\mu = \partial_t = (1,0,0,0)^\mu$$

Outside the radius $r=2GM$, this is a time-like vector:

$$K\cdot K = g_{\mu\nu}K^\mu K^\nu = g_{tt}K^t K^t = -\left(1-\frac{2GM}{r}\right) < 0$$

The interpretation of this is as follows. ∂_t generates time translations; that is it is the vector that enables you to move forward in time. That it is timelike, means that each "step" in time has a non-zero size. To be concrete, let's consider a function $f(t,r)$ evaluated at some time t. To go forward one time step $t \to t+\Delta t$, we have:

$$f(t+\Delta t, r) \cong f(t,r) + \Delta t\, \partial_t f(t,r)$$

$$\cong f(t,r) + \Delta x^\mu K_\mu f(t,r),$$

where Δx^μ is the coordinate step size and K^μ is the killing vector. The size of the time step is controlled by the radius:

$$\Delta x^\mu K_\mu = g_{\mu\nu} \Delta x^\mu K^\nu = -\Delta t \left(1 - \frac{2GM}{r}\right),$$ which

is something we have seen before. Closer to the mass M (smaller r), time moves more slowly.

At $r = 2GM$, however, time stops completely. The time step $\Delta t\, \partial_t \to 0$ and the killing vector K^μ becomes null:

$$K \cdot K \big|_{r=R_s} = g_{\mu\nu} K^\mu K^\nu = -\left(1 - \frac{2GM}{R_s}\right) = 0,$$

where $R_s = 2GM$. The interpretation of the fact that the time translation killing vector becomes null at the Schwarzschild radius is that following. If you are at the Schwarzschild radius, time does not move forward, so there is no possible way for you to move to a different radius. You are stuck.

The point at which $r = R_s = 2GM$ is thus called an *event horizon* and is defined by a set of normal vectors to its surface. These normal vectors are null, and reppesent the set of geodesics at the surface. Because the event horizon is defined by a fixed radius condition $r=2GM$, its normal vectors can only have time or radial components. ~~Because the radius is constant, the tangent vectors to~~ The only way for the normal vectors to be null is if they only have a time component. That is, the Schwarzschild solution ~~is defined~~ event horizon is defined by the point at which

$$K^\mu = \partial_t = (1,0,0,0)^\mu \quad \text{becomes null.}$$

This is also a killing vector, so the Schwarzschild event horizon is also a killing horizon.

Here

We'll see more examples of event horizons later, but let's study the Schwarzschild solution more to see if we can make sense of everything. Our goal will be to construct good coordinates that don't exhibit singularities at the event horizon.

I just want to note that the killing vector $K^\mu = \partial_t$ is only a geodesic at the event horizon. As a killing vector, it satisfies:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0.$$

Dotting with K^μ, we have

$$K^\mu(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = K^\mu \nabla_\mu K_\nu + K^\mu \nabla_\nu K_\mu$$

However, by the product rule note that

$$\nabla_\nu \underbrace{k \cdot k}_{\substack{\shortparallel \\ 0 \text{ at horizon}}} = 0 = \nabla_\nu g_{\alpha\beta} k^\alpha k^\beta = g_{\alpha\beta} k^\alpha \nabla_\nu k^\beta + g_{\alpha\beta} k^\beta \nabla_\nu k^\alpha$$

$$= 2 k^\mu \nabla_\nu k_\mu$$

Therefore, when $k^\mu = \partial_t$ is null, it satisfies

$k^\mu \nabla_\mu k_\nu = 0$, which is the geodesic equation.

Null geodesics are important to define the event horizon and where our original coordinates blow up. An infalling geodesic that is null is defined by

$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$, or that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}.$$

This clearly gets weird at $r = 2GM$, so let's define a new time coordinate v for which infalling null geodesics are independent of r:

$$\frac{dv}{dr} = 0.$$

Assuming that v is linear in t, $v = t + f(r)$, we find

$$\frac{dv}{dr} = \frac{\partial v}{\partial t}\frac{dt}{dr} + \frac{\partial v}{\partial f}\frac{\partial f}{\partial r} = -\left(1 - \frac{2GM}{r}\right)^{-1} + \frac{\partial f}{\partial r} = 0$$

Then, $\frac{df}{dr} = + \frac{1}{1 - \frac{2GM}{r}}$ or that

$$f = r + 2GM \ln\left(\frac{r}{2GM} - 1\right).$$

Replacing t with $v = t + r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$,

the Schwarzschild solution metric can be written as:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + (dv\,dr + dr\,dv) + r^2 d\Omega^2.$$

These are calle <u>Eddington-Finkelstein coordinates</u>. This metric is nonsingular everywhere (except $r=0$), so we have successfully removed the coordinate singularity that plagued the original coordinates. However, the sign of the "time" coordinate v does switch at $r = 2GM$, indicating that something is nevertheless interesting there.

While the book provides a bit more motivation, these coordinates are still incomplete. In writing v, we implicitly choose the "$-$" root of the derivative dt/dr. This "$-$" sign means we are traveling forward in time, but we could have traveld backward in time with the "+" root. Therefore, <u>Eddington-Finkelstein coordinates</u> treat forward and backward <u>in time differently</u>. As such, they don't "nicely" represent <u>all</u> of the Schwarzschild spacetime in one fell swoop. By contrast, the original coordinates (t, r, θ, ϕ), were symmetric in $t \Leftrightarrow -t$, but had the coordinate singularity. Can we eliminate the singularity and describe all of spacetime?

Yes! We can <u>introduce</u> Kruskal coordinates (T, R, θ, ϕ) <u>in which</u>:

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2,$$

with $T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$. $(*)$

This metric is symmetric in $T \leftrightarrow -T$ and so describes forward and backward temporal infinities on equal footing. Because T and R appear with equal coefficients, null radial curves correspond to

$T = \pm R + \text{constant}$, like in flat space. The

location of the event horizon corresponds to

$T = \pm R$, where $T^2 - R^2 = 0$, which is a finite ~~place~~

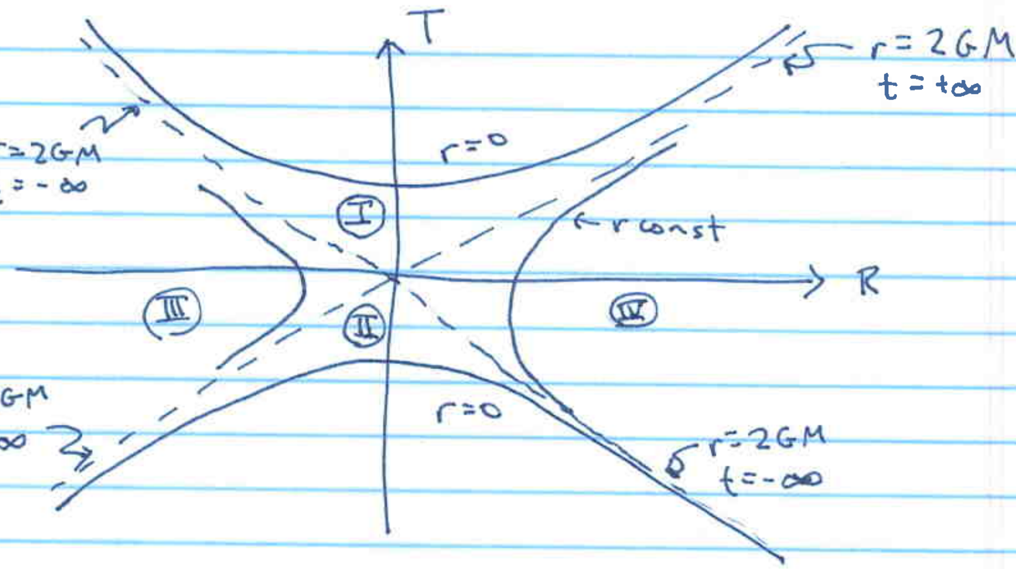
time! The range of R and T are:

$-\infty < R < \infty$, $T^2 < R^2 + 1$, which follows from $(*)$.

That is, the origin, the real singularity in the spacetime, where $r = 0$, is smeared on the hyperboloid $T^2 - R^2 = 1$. Constant time is defined by the relationship:

$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right)$, which

are just straight lines in the (T, R) plane. These observations motivate the Kruskal diagram which describes everything about schwarzschild:



Region I is the black hole: if you move forward in time toward it, you will get inevitably stuck. Region II is a white hole: if you went backward in time to it you would get sucked in. Forward in time, we would see this as the origin of everything, and the point to which we can never get to. We would live in region IV, where $r > 2GM$. Region III is our mirror image, $R \to -R$. It is connected to us by a wormhole. But, that's basically all I will say (for now).