Charged Black Holes

Last lecture, we defined precisely an event horizon and how a black hole sucks everything in. The Schwarzschild solution was our canonical black hole and the entry to these crazy phenomena; this lecture and next week, we will study more exotic black holes.

Today, we will add one complication beyond Schwarzschild: let's add an electric charge to the black hole. We'll keep the same spherical symmetry and time independence for simplicity. However, now, because there is an electric field from the charge, the stress-energy tensor is non-zero: we aren't in the vacuum anymore. Nevertheless, spherical symmetry greatly simplifies the calculation. First, the form of the metric with spherical symmetry still takes the form:

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2, \text{ where}$$

α and β are purely functions of radius r. The Einstein's equations we are working with is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu},$$

where $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}$ is the

electromagnetic field strength tensor. For a static, spherically-symmetric solution, the only possible non-zero component of the field-strength tensor is F_{tr}, which is just the radial electric field, E_r. Maxwell's equations in a curved spacetime are now

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0 \;, \quad \nabla_{[\mu}F_{\nu\rho]} = 0 \;,$$

in a region where there are no charges. We have to be careful with the metrics and Christoffel connections, but the solution to these equations with spherical symmetry is exactly the same as our flat space expectation:

$$E_r = -F_{tr} = \frac{Q}{4\pi r^2}\;, \text{ where } Q \text{ is}$$

the electric charge of the black hole. With this expression, let's calculate the stress tensor:

$$T_{\mu\nu} = g^{\rho\alpha}F_{\mu\rho}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}g^{\rho\alpha}g^{\sigma\beta}F_{\alpha\beta}$$

$$= g^{tt}F_{\mu t}F_{\nu t} + g^{rr}F_{\mu r}F_{\nu r} - \frac{2}{4}g_{\mu\nu}g^{tt}g^{rr}(F_{tr})^2$$

$$= -e^{-2\alpha}F_{\mu t}F_{\nu t} + e^{-2\beta}F_{\mu r}F_{\nu r} + \frac{1}{2}g_{\mu\nu}e^{-2(\alpha+\beta)}(F_{tr})^2$$

Because the metric is diagonal, the stress-energy tensor must also be diagonal. It's entries are then:

$$T_{tt} = \left(e^{-2\beta} - \frac{1}{2}e^{-2\beta}\right)(F_{tr})^2 = e^{-2\beta}\frac{Q^2}{32\pi^2 r^4}$$

$$T_{rr} = \left(-e^{-2\alpha} + \frac{1}{2}e^{-2\alpha}\right)(F_{tr})^2 = -e^{-2\alpha}\frac{Q^2}{32\pi^2 r^4}$$

$$T_{\theta\theta} = \frac{1}{2}r^2 e^{-2(\alpha+\beta)}\frac{Q^2}{16\pi^2 r^4} = e^{-2(\alpha+\beta)}\frac{Q^2}{32\pi^2 r^2}$$

$$T_{\phi\phi} = \sin^2\theta \, T_{\theta\theta}.$$

Importantly, the electromagnetic stress-energy tensor is traceless: $g^{\mu\nu}T_{\mu\nu} = 0$.

This observation enables us to dramatically simplify the Einstein's equations: Their trace, therefore, is:

$$g^{\mu\nu}\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) = g^{\mu\nu} 8\pi G_N T_{\mu\nu} = 0$$

$\Rightarrow$ $R - \frac{4}{2}R = 0$ or that $R = 0$. ~~Further, the stress tensor~~ So, Einstein's equations are:

$R_{\mu\nu} = 8\pi G_N T_{\mu\nu}$. We calculated the Ricci tensor for the spherical symmetry of the metric we are studying. We had found:

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\alpha\right]$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\beta$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

Now, we just set these equal to the corresponding stress-energy tensor components. Let's first focus on the tt and rr components:

$$e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] = e^{-2\beta}\frac{Q^2}{32\pi^2 r^4}\cdot 8\pi G_N$$

$$-\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\beta = -e^{-2\alpha}\frac{Q^2}{32\pi^2 r^4}\cdot 8\pi G_N$$

Just like we've seen over and over, we can combine these equations in a dramatically simple form:

$$e^{-2(\alpha-\beta)}\left(R_{tt}-8\pi G_N T_{tt}\right)+\left(R_{rr}-8\pi G_N T_{rr}\right)$$
$$=\frac{2}{r}\left(\partial_r\alpha+\partial_r\beta\right)=0$$

So, we find the familiar result that $\beta'=-\alpha'$, which we can set the time component appropriately so that $\beta=-\alpha$. Now, using this, the $\theta\theta$ component equation yields:

$$-e^{+2\alpha}\left[2r\,\partial_r\alpha+1\right]+1=\frac{Q^2}{32\pi^2 r^2}\cdot 8\pi G_N$$

or that

$$-\partial_r\left(r\,e^{2\alpha}\right)+1=\frac{Q^2}{32\pi^2 r^2}\cdot 8\pi G_N=\frac{G_N Q^2}{4\pi r^2}$$

The solution of this equation is:

$$e^{2\alpha}=1-\frac{2G_N M}{r}+\frac{Q^2 G_N}{4\pi r^2}$$

That is, the charged black hole metric is:

$$ds^2=-\left(1-\frac{2G_N M}{r}+\frac{Q^2 G_N}{4\pi r^2}\right)dt^2+\left(1-\frac{2G_N M}{r}+\frac{Q^2 G_N}{4\pi r^2}\right)^{-1}dr^2+r^2 d\Omega^2.$$

This is called the Reissnor-Nördstrom metric.

It is weird. Just like the Schwarzschild solution in these coordinates, this metric has singularities in general. However, just like Schwarzschild, these are fake, just coordinate singularities. The only real singularity on the manifold is located at $r=0$.

Also just like schwarzschild, it has horizons. Those horizons are located at radii satisfying

$$1-\frac{2G_NM}{r}+\frac{G_NQ^2}{4\pi r^2}=0,$$ which is where the

rr component of the metric diverges. This is just a quadratic equation which we can solve:

$$r_\pm = G_NM \pm \sqrt{G_N^2M^2 - G_NQ^2}$$

(For brevity, we've removed factors of 4π.) Because this is a quadratic equation, there are <u>two</u> potential solutions. Two solutions exist if

$$G_NM^2 > Q^2$$

Having two horizons is strange. If you fell in on a time like geodesic, starting at $r>r_+$, the horizon is null and for $r_-<r<r_+$ is a region in which you will be inevitably pulled to smaller radius. In this region, you might feel doomed, but once you reach $r\leq r_-$, your geodesic becomes timelike again, and you can stop! The honest singularity at $r=0$ is a point where the time component of the metric is negative; thus we call it a time-like singularity. The singularity of Schwarzschild is space-like: the time component is positive:

$$g_{tt}=-\left(1-\frac{2G_NM}{r}\right)\to+\infty \text{ as } r\to 0.$$

So, unlike Schwarzschild, the singularity in Reissner-Nordstrom is avoidable.

Okay, what about other configurations? If $G_NM^2<Q^2$, there are <u>no</u> solutions to the horizon equation.

so there is no horizon around the singularity. Such a configuration is called a naked singularity. There is a conjecture called "cosmic censorship" that states that no physical spacetime singularity exists without an event horizon. This, while not proven, is a good thing. It means that you will never just happen on a singularity and get sucked away. You can identify a possible singularity by first identifying the event horizon (by observing extreme redshifting, for example).

Finally, there's the configuration where $G_N M^2 = Q^2$. If the naked singularity is unphysical, this is the largest possible electric charge for a given mass that could exist; it is therefore called an "extremal" black hole. The metric of the extremal black hole depends on the function:

$$1 - \frac{2GM}{r} + \frac{G_N Q^2}{r^2} = 1 - \frac{2G_N M}{r} + \frac{G_N^2 M^2}{r^2} = \left(1 - \frac{G_N M}{r}\right)^2.$$

That is, the extremal Reissner-Nordstrom metric is

$$ds^2 = -\left(1 - \frac{G_N M}{r}\right)^2 dt^2 + \left(1 - \frac{G_N M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2.$$

There is one horizon at $r = G_N M$, but it hides a time-like singularity! This is different than Schwarzschild, and again it can be avoided.

The moral: If you see a black hole out there, first toss in some electric charge to ensure that the thing is Reissner-Nordstrom: you won't die. (Well, you will, but not from tidal forces of a singularity!)