More Black Holes

So far in our study of exact solutions to Einstein's equations, we have identified the Schwarzschild and Reissner-Nördstrom black holes, which are static, spherically-symmetric and may include electric charge. Both of these black holes exhibited event horizons: points beyond which one was doomed never to return. In the Reissner-Nördstrom case, however, depending on the relationship between mass and charge, there may or may not exist event horizons. Such "naked singularities" violate a sense of decency, and more troublesome, seem to violate conservation of energy. So, we conjecture that all space-time singularities are equipped with an event horizon, called the cosmic ~~[illegible]~~ censorship conjecture. Nevertheless, even with cosmic censorship, your fate with a Reissner-Nördstrom black hole is not inevitable: the singularity is time-like; so you can, within the event horizon stop and stare it down.

Today, we add another complication: angular momentum. We will search for axially-symmetric, stationary solutions to Einstein's equation in vacuum. While the Schwarzschild solution had four Killing vectors (time translations and 3 rotations), this rotating black hole has only two: time translations (its rate of rotation is constant) and ~~[illegible]~~ one rotation about the angular momentum axis. The reduction of symmetries means that finding a solution to Einstein's equations is much more challenging. The general form of a metric with axial symmetry can be expressed as:

$$ds^2 = -e^{2\alpha(r,\theta)}dt^2 - e^{2\beta(r,\theta)}(dt\,d\phi + d\phi\,dt)$$
$$+ e^{2\gamma(r,\theta)}dr^2 + e^{2\delta(r,\theta)}d\theta^2 + e^{2\varepsilon(r,\theta)}d\phi^2.$$

This metric is further no longer diagonal either! So, we can't use our simple formulas to calculate Christoffel connections. Nevertheless, a solution can be found, called the Kerr metric:

$$ds^2 = -\left(1 - \frac{2G_N M r}{\rho^2}\right)dt^2 - \frac{2G_N M a r \sin^2\theta}{\rho^2}(dt\,d\phi + d\phi\,dt)$$
$$+ \frac{\rho^2}{\Delta}dt^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

where $\Delta = r^2 - 2G_N M r + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$.

Here a is the angular momentum per unit mass of the black hole. You might be ~~asked~~ asking why we use these coordinates (called Boyer-Lindquist) because we can always consider a diagonal metric. We could express such a diagonal metric for the Kerr solution in cylindrical coordinates:

$$ds^2 = -e^{2\alpha(\rho,z)}dt^2 + e^{2\beta(\rho,z)}d\rho^2 + e^{2\gamma(r,z)}dz^2 + e^{2\delta(r,z)}d\phi^2,$$

where r is now the distance from the $\hat{z}$-axis. We could use such coordinates, but Boyer-Linquist are preferred because they reduce to Schwarzschild when $a \to 0$, with a static black hole. Further, the "t" and "ϕ" that are in these cylindrical coordinates aren't the "t" and "ϕ" we expect.

Then kerr metric, just like Schwarzschild and R-N, exibits event horizons. These are found by identifying the ~~radii~~ radii at which the rr component of the metric diverges:

$$\frac{\Delta}{\rho^2} = 0$$

This is just when $\Delta = r^2 - 2G_N M r + a^2 = 0$ or for the radii

$$r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - a^2}$$

So, just like RN, we see there are different configurations of horizons. If $a > G_N M$, then this seems to violate energy conservation because angular momentum energy is greater than mass energy. Further, the singularity at $r=0$ is naked. The extremal solution, when $G_N M = a$ exhibits one horizon, but is unstable because any addition of angular momentum would elimate the horizon. So, we'll just focus on the case where $G_N M > a$, for which there are two horizons.

Unlike the case for Schwarzschild or R-N, these event horizons are not killing horizons for the time translation killing vector. Dotting this Killing vector with itself, we have

$$K^\mu \cdot K_\mu = g_{\mu\nu} K^\mu K^\nu = -\left(1 - \frac{2G_N M r}{\rho^2}\right).$$

This vanishes when $\rho^2 = 2G_N M r = r^2 + a^2 \cos^2\theta$ or when

$$r^2 - 2G_N M r + a^2 \cos^2\theta = 0$$

This is not the same equation as for the event horizon. Apparently, the time translation killing vector becomes spacelike for

$$r^2 - 2G_N M r + a^2 \cos^2\theta < 0$$

which is mostly outside of the outer event horizon, r_+, which satisfies

$$r_+^2 - 2G_N M r_+ + a^2 = 0.$$

These expressions are identical for $\theta = 0, \pi$, the poles about which the black hole is rotating. The region where the time translation killing vector is space-like is called the ergosphere: in this region, you must move in the direction of rotation of the black hole, but you are still freely allowed to leave.

The actual black hole singularity of the Kerr metric is weird too. Note that the metric has factors of $1/\rho^2$ all over. This suggests that the singularity is where $\rho^2 = 0$, which can be verified by calculating coordinate invariants. This is the requirement that

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0$$

Thus, the singularity exists in the region where $r = \cos\theta = 0$. What does this mean? We can understand it by taking the $M \to 0$ limit of the metric, with a constant. This reduces to flat space, but in funky coordinates:

$$ds^{2\,M\to 0} = -dt^2 + \frac{(r^2+a^2\cos^2\theta)}{r^2+a^2}dr^2 + (r^2+a^2\cos^2\theta)^2 d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2$$

Flat space is now expressed in ellipsoidal coordinates for which:

$$x = (r^2+a^2)^{1/2}\sin\theta\cos\phi$$
$$y = (r^2+a^2)^{1/2}\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

So, $r=0$ collapses to a disk of radius a:

$$x^{r=0} = a\sin\theta\cos\phi$$
$$y^{r=0} = a\sin\theta\sin\phi$$
$$z^{r=0} = 0$$

The radial coordinate on the disk is $\sin\theta$. When $\cos\theta=0$, $\theta=\pi/2$ and $\sin\theta=1$. Therefore the singularity of the Kerr metric is a ring, and not a point. Strange.

You can pass through the ring, as the interior of the ring is non-singular. Doing so puts you in another spacetime for which you can pass through its singular ring and enter another spacetime, etc. This ring opens up an infinity of different spacetimes to explore.

In the ring, we can do some crazy stuff. Because there is only a singularity at $\rho^2=0$, the radial coordinate can meaningfully be negative. We can consider trajectories for which t, θ, and r are constant, and r is sufficiently small and negative. Then, such a trajectory is just some loop in ϕ with the interval:

$$ds^2 = a^2\left(1+\frac{2G_N M}{r}\right)d\phi^2.$$

In writing this expression, we Taylor expanded $g_{\phi\phi}$ to lowest order in $r \to 0$. For sufficiently small negative r, this can be negative, and so such a curve would be time-like. As it is a loop ($\phi = 0$ is the same as $\phi = 2\pi$) it is closed. That is, you can move along a closed, time-like curve and get back to the time at which you started. Which is likely not a good thing.