Gravitational Fluctuations

Recall Maxwell's equations, written as a tensor equation:

$$\partial_\mu F^{\mu\nu} = -J^\nu \, , \quad \partial_{[\mu} F_{\nu\rho]} = 0$$

Expressed in terms of the vector potential A_μ, the Bianchi identity is trivial (everything explicitly cancels) while the other equation is:

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = -J^\nu$$

If we define $\partial_\mu \partial^\mu = \partial \cdot \partial = \Box$, the d'Alembertian, this is compactly represented as:

$$\Box A^\nu - \partial^\nu \partial \cdot A = -J^\nu .$$

For now, we'll just focus on the source-free Maxwell's equations / equations of motion for A_μ. This sets $J_\mu = 0$ and so

$$\Box A_\mu - \partial_\mu \partial \cdot A = 0 . \qquad (*)$$

This equation completely governs the dynamics of electromagnetic radiation in a region of space with no charges (i.e., light). Let's see if we can make sense of what it is telling us. First, it is a Lorentz invariant equation. If we have in one coordinate system $(*)$ satisfied, then rotating or boosting to another frame with matrix Λ yields:

$$\Lambda^\nu{}_\mu \left(\Box A_\nu - \partial_\nu \partial \cdot A \right) = 0 \, , \quad \text{still.}$$

Further, this is a linear equation, even when a source is included. This means that we can consider electromagnetic fluctuations on top of, or in addition to, some background charge configuration. In fact, we can just add this solution in vacuum to a solution with charge and the sum is also a solution. Of course, this is nothing more than superposition.

What about the number of degrees of freedom of A_μ? Naively, A_μ is real-valued and has four components (four values of μ), so we might expect it to have four degrees of freedom. However, let's take a closer look at this equation. Let's examine each component, starting with $\mu = 0$:

$$\Box A_0 - \partial_0 \partial \cdot A = \left(-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2\right) A_0 - \partial_0\left(-\partial_0 A_0 + \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3\right)$$

$$= 0$$

or that,

$$\nabla^2 A_0 - \partial_0 \vec{\nabla} \cdot \vec{A} = 0$$

This corresponds to the source-free Gauss's Law. Note that there is no time derivative of the A_0 component. As such, this is not an "equation of motion"; with a time derivative, A_0 is not able to propagate through time. Therefore, A_0 does not correspond to a degree of freedom. So, A_μ has at most 3 degrees of freedom.

Further, there is another component of A_μ that does not propagate. If a term explicitly cancels in the difference of terms in the equation of motion, then it also cannot propagate. That is, consider some vector ϕ_μ such that

$$\Box \phi_\mu = \partial_\mu \partial \cdot \phi$$

This is trivially satisfied by $\phi_\mu = \partial_\mu \lambda$, where λ is a scalar function of the coordinates. Note that

$$\Box \partial_\mu \lambda = \partial_\mu \partial \cdot \partial \lambda = \Box \partial_\mu \lambda.$$

So, we are allowed, with impunity, to add the gradient of a scalar function λ to the vector potential A_μ and no physics is affected. Of course, this is just a gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu \lambda$$

Because λ is an arbitrary function of space and time, its presence means that there is another component of A_μ that cannot propagate. If one Fourier transforms $\partial_\mu \lambda$, this corresponds to the direction of momentum (the Poynting vector) of A_μ. That is, there are no electromagnetic waves that oscillate in the direction of motion. Of course you knew this: electric and magnetic fields are orthogonal to the Poynting vector.

So, it turns out that A_μ has only two degrees of freedom, and these can be identified with the transverse electric and magnetic fields (which must be perpendicular to each other.).

This might seem a bit weird why a spin-1 particle like the photon has only two degrees of freedom. I thought that spin-1 should have 3: spin +1, 0, -1? For a more complete answer, you'll need to take particle physics next semester (☺), but here's a brief argument. Photons are massless, and so always travel at the speed of light. As such their direction of motion/momentum defines an unambiguous vector/direction along which we can quantize spin. The spin can either be aligned or anti-aligned with the momentum (spin "up" or spin "down"), so there are only two spin states, not 3.

Okay, that is E&M. What about gravity? Does there exist an equation of motion for "gravitational waves"? To identify such an equation, we have to be a bit careful. As we discussed, Einstein's equations are non-linear, and so do not satisfy the principle of superposition. To identify gravitational waves, we need to consider small fluctuations on some background geometry, defined by a metric $g^{(0)}_{\mu\nu}$. We will call the fluctuations $h_{\mu\nu}$, so that the complete metric is

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$$

and then we will expand Einstein's equations to linear order in $h_{\mu\nu}$. While one can, in general, consider any background metric $g^{(0)}_{\mu\nu}$, we'll just study fluctuations on flat space-time, for which $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. We'll also assume we're in vacuum for which $T_{\mu\nu}=0$ everywhere. Then, we need to work through the usual Christoffels → Riemann → Ricci to get Einstein's equations.

Recall that the Christoffels are defined as

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)$$

If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then the derivatives only act on $h_{\mu\nu}$. To linear order in h, we can just set $g^{\rho\lambda} = \eta^{\rho\lambda}$. Then, the Christoffels are:

$$\Gamma_{\mu\nu}^{\rho} \cong \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)$$

Now, we can plug this into the expression for the Riemann tensor. Recall its schematic form is

$R \sim \partial\Gamma + \Gamma\Gamma$, but Γ is already linear in the metric, so Γ^2 is quadratic in h. So, to linear order in the metric, we only need the derivatives contribution to the Riemann tensor:

$$R^{\mu}{}_{\nu\rho\sigma} = \delta^{\mu}_{\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \delta^{\mu}_{\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho}$$

$$= \frac{1}{2}\left(\partial_\rho\partial_\nu h^{\mu}{}_{\sigma} + \partial_\sigma\partial^{\mu} h_{\nu\rho} - \partial_\sigma\partial_\nu h^{\mu}{}_{\rho} - \partial_\rho\partial^{\mu} h_{\nu\sigma}\right)$$

Contracting indices μ and ρ produces the Ricci tensor:

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right)$$

Further contracting indices to get the Ricci scalar one finds:

$R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h$, where $h = \eta_{\mu\nu} h^{\mu\nu}$, the trace of the metric fluctuation. It then follows that Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = 0 = \frac{1}{2}\Big(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h\Big)$$

Note also that by contracting indices this requires that the Ricci scalar vanishes:

$$R = 0 = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h = 0.$$

This can then be re-inserted into Einstein's equations (or, just setting $R_{\mu\nu} = 0$):

$$R_{\mu\nu} = 0 = \partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}$$

Today, we'll just identify the trivial solutions to these equations which represent transformations one can perform on $h_{\mu\nu}$ with no physical consequences. In general, a metric can be transformed by a coordinate transformation and no physics changes:

$$g_{\mu\nu} \rightarrow \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}$$

Because we have expanded the metric to linear perturbations, these coordinate transformations will be simplified into gauge transformations. To identify these, let's just focus on the vanishing Ricci scalar and its trivial solutions, denoted by some symmetric, two-index tensor $\phi_{\mu\nu}$:

$$\partial_\mu \partial_\nu \phi^{\mu\nu} = \Box \phi^{\mu\nu}.$$

This is trivially satisfied for $\phi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, where ξ_μ is an arbitrary one-form. That is, we can transform $h_{\mu\nu}$ by this trivial solution:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu,$$

and no physics is changed. This will have consequences for the degrees of freedom counting of these fluctuations $h_{\mu\nu}$, but we'll get to that next time.