More Gravitational Radiation

Last lecture, we had expanded Einstein's equations in vacuum in fluctuations about flat space. We had identified the Ricci scalar and tensor via appropriate contractions:

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right) = 0,$$

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h = 0,$$

as demanded by $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$. $h_{\mu\nu}$ is a symmetric, two-index tensor, and as such has (naïvely) 10 independent components. However, the vast majority of these components are unphysical or not allowed to propagate through time. The Ricci scalar, as a coordinate scalar, is invariant under coordinate transformations and as such there are corresponding transformations on $h_{\mu\nu}$ that can be performed that have no effect on the (physical) Ricci scalar. We had found that we could transform $h_{\mu\nu}$ as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \text{and nothing changes.}$$

This is a gravitational gauge transformation and it is responsible for eliminating those fluctuations in the direction of momentum (longitudinal modes). Because the gauge parameter ξ_μ is a four-component object, this reduces the numbers of degrees of freedom of $h_{\mu\nu}$ from 10 to $10-4=6$. But wait, there's more.

Let's consider individual components of $R_{\mu\nu}=0$, in particular when $\mu = 0$. ($R_{\mu\nu}$ is symmetric in $\mu \leftrightarrow \nu$, so this is equivalent to $\nu = 0$, too.) $R_{\mu\nu} = 0$ is then

$$R_{0\nu} = 0 = \frac{1}{2}\left(\partial_\sigma \partial_\nu h^\sigma{}_0 + \partial_\sigma \partial_0 h^\sigma{}_\nu - \partial_0 \partial_\nu h - \Box h_{0\nu}\right) = 0.$$

We are particularly interested in the $h_{0\nu}$ components of $h_{\mu\nu}$, so we focus on them. First, for the h_{00} component, its equation is:

$$-\partial_0 \partial_0 h_{00} - \partial_0^2 h_{00} - \partial_0^2 (-h_{00} + h_{ii}) - (-\partial_0^2 + \partial_i^2) h_{00} = 0$$

Note that all temporal derivatives explicitly cancel. That is, the "equation of motion" for h_{00} is actually just a constraint equation and does not propagate.

Let's now evaluate the equations of motion for h_{0i}, where i is a spatial index. We find

$$R_{0i} = 0 = \frac{1}{2}\left(\partial_j \partial_i h^j{}_0 + \partial_0^2 h^0{}_i + \partial_0^2 h_{0i} + \ldots\right) = 0$$

The ellipses denote terms with purely spatial derivatives or terms that do not involve the h_{0i} components. Note that

$$\partial_0^2 h^0{}_i = -\partial_0^2 h_{0i}\,,$$ and so all temporal

derivatives again cancel! So all components of $h_{\mu\nu}$ of the form $h_{0\nu}$ or $h_{\mu 0}$ are not allowed to propagate through time. Therefore, these four additional components do not propagate and so have no physical consequences. So, out of our original 10 degrees of freedom of $h_{\mu\nu}$, 4 are eliminated by gauge transformations and four are eliminated because they do not propagate. Apparently $h_{\mu\nu}$ has only <u>two</u> relevant, physical components. Intriguingly, this is the same number as we found for the photon of electromagnetism...

Let's now solve for these two components of $h_{\mu\nu}$ and interpret them accordingly. First, we need to identify what these two terms are. We know that $h_{\mu 0} = h_{0\mu} = 0$, and we can choose nice gauge parameters ξ_μ to isolate the two interesting components. As the book does, we work in transverse traceless gauge in which

$$h = \eta_{\mu\nu} h^{\mu\nu} = 0, \quad \partial_\mu h^{\mu\nu} = 0.$$

Under a gauge transformation, these become:

$$h \to h + 2\,\partial\cdot\xi \quad \text{and} \quad \partial_\mu h^{\mu\nu} \to \partial_\mu h^{\mu\nu} + \Box \xi^\nu + \partial^\nu \partial\cdot\xi.$$

So, for consistency of the gauge conditions, we must choose ξ_μ such that:

$$\partial\cdot\xi = \Box \xi_\mu = 0.$$

This can be done, which we assume in the following.

As we do in E+M, let's look for plane-wave solutions of the equations of motion. Assuming transeverse traceless gauge, $R_{\mu\nu} = 0$ becomes:

$$R_{\mu\nu} = 0 = \frac{1}{2}\left(\partial_\sigma \cancelto{0}{\partial_\nu h^\sigma{}_\mu} + \partial_\sigma \cancelto{0}{\partial_\mu h^\sigma{}_\nu} - \partial_\mu \cancelto{0}{\partial_\nu h} - \Box h_{\mu\nu}\right),$$

or just that $\Box h_{\mu\nu} = 0.$ $\quad (*)$

This is the wave equation or the <u>Klein-Gordon equation</u>. Let's choose our plane wave to propagate along the $+\hat{z}$ axis. Therefore, for some frequency ω and wave-number k, we can write

$$h_{\mu\nu} = C_{\mu\nu} e^{-i\omega t + ikz}$$, where $C_{\mu\nu}$ is a two-index ~~object~~ tensor with <u>constant</u> values. Our wave equation is a linear equation, so we can consider general linear combinations of solutions weighted by a function with dependence on ω and k; that is, we are just considering one Fourier mode of the solution, and one can use superposition to get the general result.

Let's first determine the relationship between ω and k. The wave equation produces

$$\Box h_{\mu\nu} = 0 = C_{\mu\nu}\left(-\partial_t^2 + \partial_z^2\right) e^{-i\omega t + ikz}$$

$$= C_{\mu\nu}\left(\omega^2 - k^2\right).$$

That is, $k = \pm\omega$. As we assume that the propagation is along the $+\hat{z}$ axis, we choose the "+" root. Therefore, the momentum vector of this gravitational disturbance is

$$k = (\omega, 0, 0, \omega)$$, which is null. We say therefore that gravitational radiation travels at the speed of light.

Next, let's see what the non-zero components of

$C_{\mu\nu}$ are. Of course, because $h = h_{\mu 0} = h_{0\mu} = 0$, those components of $C_{\mu\nu}$ are also 0. Additionally, by the transverse gauge condition

$$\partial_\mu h^{\mu\nu} = 0 = C^{\mu\nu} \partial_\mu e^{-i\omega t + i\omega z}$$

$$= C^{0\nu}(-i\omega) + C^{3\nu}(i\omega),$$

which implies that $C_{3\mu} = 0 = C_{\mu 3}$, as $C_{0\mu}$ is already 0. Therefore, $C_{\mu\nu}$ can be written as:

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}$$

We'll see why this notation is chosen shortly. This is also indeed a traceless tensor, because $h_+ - h_+ = 0$. h_+ and $h_\times$ are the remaining two degrees of freedom. These two degrees of freedom are eerily similar to what we found for the photon, A_μ.

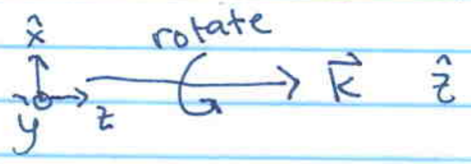
The last thing for today is to understand these two degrees of freedom a bit more. To set the stage, let's do all of this same exercise but for E+M. We'll be quick because the steps are essentially the same. We set the direction of momentum along the $+\hat{z}$ axis, work in Lorenz gauge, $\partial \cdot A = 0$, and the solution we find is

$$A_\mu = \varepsilon_\mu e^{-i\omega t + i\omega z}.$$

ε_μ is called the polarization vector, and it only has two non-zero components that are transverse to the momentum:

$$\varepsilon_\mu = (0, \varepsilon_1, \varepsilon_2, 0)_\mu .$$

We would like to identify the spin of this object. I'll sketch out how to do this, but a complete analysis requires quantum mechanics, so take particle physics next semester! Consider the momentum vector: $k = (\omega, 0, 0, \omega)$. This is invariant under rotations about the $\hat{z}$ axis:



So, this rotation cannot change any physical quantities; it must be a symmetry. However, this rotation mixes the $\hat{x}$ and $\hat{y}$ components, and so the polarization vector changes. How does it rotate?

The relevant parts of the polarization vector are $(\varepsilon_1, \varepsilon_2)$, a two-component vector. Under a rotation of the xy plane by an angle ϕ, this transforms to

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1\cos\phi - \varepsilon_2\sin\phi \\ \varepsilon_1\sin\phi + \varepsilon_2\cos\phi \end{pmatrix}$$

After a rotation by 2π, this vector goes back to itself. 2π is the smallest rotation of which the vector returns to itself. Because 2π

is the total angle of a circle, the polarization vector rotates "1" time in going around a circle. Thus we say it carries spin-1.

Now, let's do the same exercise for $C_{\mu\nu}$. The action of rotating in the xy plane requires two rotation matrices $\mathbb{M}$, one for each index of C. If we call

$$C = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix},$$ then these matrices act to

rotate C as ~~[illegible]~~ $C \to \mathbb{M} C \mathbb{M}^T$. Doing the explicit multiplication, we find

$$C \to \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} h_+ \cos 2\phi - h_\times \sin 2\phi & h_+ \sin 2\phi + h_\times \cos 2\phi \\ h_+ \sin 2\phi + h_\times \cos 2\phi & -h_+ \cos 2\phi + h_\times \sin 2\phi \end{pmatrix}$$

This clearly returns to itself after only a rotation of π. That is, rotating a gravitational wave around a full circle means that it has actually rotated twice. Thus, we call it a spin-2 object. Indeed, unlike electromagnetism, if you attempt to define a full interacting theory of a spin-2 particle, the unique result is the theory of general relativity.

So, how do we catch these waves? More next time...