Cosmology

Let's describe the universe with general relativity. To do this, we need to identify what the symmetries of the universe are; correspondingly, we need to identify the killing vectors that generate transformations of our spacetime. Further, our goal will be to describe the universe on the largest of scales. At these scales, we average over the miniscule fluctuations of planets, stars and even galaxies. That is, to first approximation, what does the universe look like as a whole?

Our fundamental guiding principle will be the Copernican principle. Historically, this was the statement that our location in space was not special. From this perspective, it is much more natural to describe and physically understand the orbits of planets around the sun, for instance. We will use a generalization of the Copernican principle applied to the universe. Indeed, our location in the universe is not special, but this is still a geocentric perspective of thinking that "we" are not special. No location in the universe is special and as such, we must see the same thing in any direction (isotropy) from any point in the universe (homogeneity). Further, we at first may also believe that our time in the universe isn't special. Thus, we might expect all space-time coordinates to exhibit isotropy and homogeneity. Isotropy in 4D means that we can rotate any pair of coordinates into one another and nothing is changed. This corresponds to $6 = \binom{4}{2}$ Killing vectors of our spacetime. Homogeneity in 4D means that we can translate any coordinate and nothing is changed. This corresponds to an additional 4 Killing vectors. In D dimensions, we had argued that there were $\frac{D(D+1)}{2}$ independent Killing vector solutions to Killing's equation:

$$\nabla_{(\mu}K_{\nu)} = 0.$$

All of these solutions are nontrivial iff the spacetime is maximally symmetric. The Copernican principle in its most extreme limit means that the universe is a maximally-symmetric spacetime.

With this assumption, let's categorize the possible maximally-symmetric spaces. When we had first introduced the Riemann tensor and curvature, we noted a few properties. The Ricci scalar R is a constant value over the entire maximally-symmetric manifold. Correspondingly, the Riemann tensor for a maximally-symmetric spacetime is

$$R^{\rho}{}_{\sigma\mu\nu} = \frac{R}{12}\left(\delta^{\rho}_{\mu} g_{\sigma\nu} - \delta^{\rho}_{\nu} g_{\sigma\mu}\right).$$

The factor of 1/12 is unique to 4D, but has a generalization to arbitrary dimension. It then follows that the Ricci tensor is

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \frac{R}{12}\left(\delta^{\rho}_{\rho} g_{\mu\nu} - \delta^{\rho}_{\nu} g_{\mu\rho}\right)$$
$$= \frac{R}{4} g_{\mu\nu}$$

Therefore, Einstein's equation for a maximally symmetric spacetime is:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{R}{4} g_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\frac{R}{4} g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

That is, the stress-energy tensor of a maximally symmetric spacetime is proportional to the metric:

$$T_{\mu\nu} = -\frac{R}{32\pi G_N} g_{\mu\nu}.$$

As we have discussed before, such an energy density corresponds to a cosmological constant, or vacuum energy. The sign of the Ricci scalar and hence the sign of the vacuum energy defines the particular maximally symmetric spacetime:

$R = 0$, $\Lambda = 0$: Minkowski spacetime; $ds^2 = -dt^2 + d\vec{x}^2$

$R > 0$, $\Lambda > 0$: de Sitter spacetime (dS)

$R < 0$, $\Lambda < 0$: anti-de Sitter spacetime. (AdS)

One can derive the metrics for dS and AdS, which is done in the book. However, we have a bigger problem to worry about. These maximally symmetric spacetimes exclusively have cosmological constants which is the only allowed form of energy. However, our universe consists of light, matter, bosons, fermions, etc., which therefore explicitly cannot be described by a maximally symmetric spacetime. So, our Copernican principle was taken too far.

Let's step it back a bit. We observe spatially, that the universe is homogeneous and istropic on the largest scales. Time is much harder to say (as of now) because we can't "look through time" so straightforwardly. So, let's weaken the maximally symmetric spacetime idea to just maximally symmetric space. In that case, we can express the metric most generally as:

$$ds^2 = -dt^2 + a^2(t)\left[e^{2\beta(r)}dr^2 + r^2 d\Omega^2\right]$$

If the space is maximally symmetric, then it definitely exhibits spherical/rotational symmetry. So, we can and have expressed the spatial components in spherical coordinates.

We can determine the function $\beta(r)$ exactly like we have done many times before. For a three-dimensional space that is maximally symmetric, the Ricci tensor is:

$R_{ij} = \frac{R}{3} g_{ij}$, where R is the Ricci scalar,

which is just a number for a maximally symmetric space.

Continuing, we can recycle our results from the calculation of the Ricci tensor from Schwarzschild long ago, by simply setting the time scaling function $\alpha(r)=0$. The Ricci tensor is then:

$$R_{rr} = \frac{2}{r}\partial_r\beta = \frac{R}{3}e^{2\beta}$$

$$R_{\theta\theta} = e^{-2\beta}(r\partial_r\beta - 1) + 1 = \frac{R}{3}r^2$$

$$R_{\phi\phi} = \left[e^{-2\beta}(r\partial_r\beta - 1) + 1\right]\sin^2\theta = \frac{R}{3}r^2\sin^2\theta$$

Note that the first equality can be expressed as:

$$e^{-2\beta}\frac{2}{r}\partial_r\beta = -\frac{1}{r}\partial_r e^{-2\beta} = \frac{R}{3}$$

The solution of this is:

$$e^{-2\beta} = c - r^2\frac{R}{6}\text{, where } c \text{ is a constant.}$$

We can determine the constant from the $\theta\theta$ equation:

$$e^{-2\beta}(r\partial_r\beta - 1) + 1 = e^{-2\beta}\left(\frac{r^2R}{6}e^{2\beta} - 1\right) + 1$$

$$= \frac{r^2R}{6} - c + r^2\frac{R}{6} + 1 = \frac{R}{3}r^2$$

Therefore, $c = 1$. Finally, the metric for a maximally symmetric space is

$$\boxed{ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - \frac{R}{6}r^2} + r^2 d\Omega^2\right]}$$

We often denote $k = \frac{R}{6}$, which is just any real number. This metric is called the Robertson-Walker metric and the function $a(t)$ is called the <u>scale factor</u>.

There are three possible configurations that lead to different asymptotic behavior ($r \to \infty$) of the metric. If the metric is flat in space, $k=0$, then the spatial metric is just Euclidean. However, as ~~the~~ time moves forward, distances will expand or contract according to $a(t)$. If $k>0$, corresponding to a positively-curved space, then this is the metric of a 3-sphere:

$$d\sigma^2 = \frac{dr^2}{1-|k|r^2} + r^2 d\Omega^2 .$$

Note that there is a maximum radius, $r = 1/\sqrt{k}$, of this space which is a manifestation of its compactness. Finally, if $k<0$, the metric is that of a three-dimensional hyperboloid:

$$d\sigma^2 = \frac{dr^2}{1+|k|r^2} + r^2 d\Omega^2$$

A hyperboloid is non-compact because the radial coordinate r can be unbounded. It's therefore also called "open".

One more word about these metrics. A metric only defines the geometry <u>locally</u> on a manifold, not globally. I've been a bit cavalier, and only mentioned the most familiar manifolds which have the corresponding geometry. However, for $k=0$, for example, the manifold could also be a three-torus $S^1 \times S^1 \times S^1$, which is flat (zero curvature everywhere), but ~~it~~ just has identifications of the boundaries so that if ~~to~~ you travel in any direction far enough you get back to where you started.

We'll study Einstein's equations next time...