The Friedmann Equations

At the end of last lecture, we had derived the maximally-symmetric spatial metric, the Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

The real number k quantifies the curvature of space at a fixed time: $k=0$ is flat space, $k>0$ is "closed" (same local geometry of a three-sphere), and $k<0$ is "open" (same local geometry of a hyperboloid). The function $a(t)$ is the scale factor whose temporal evolution determines the size of the universe. The Einstein equations ~~gives~~ determine the differential equations that a satisfies.

From this metric, we can calculate the Christoffel connections and the Ricci tensor. I won't provide the details of this calculation as there are 11 non-zero Christoffels. When the dust settles, the components of the Ricci tensor are:

$$R_{tt} = -3\frac{\ddot{a}}{a}$$

$$R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}$$

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

The Ricci scalar is then

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

The Einstein equations are therefore

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

The 00 component of this equation is:

$$-3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] = 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = 8\pi G_N T_{00}$$

Recall that T_{00} is the energy density, $T_{00} = \rho$, so we can express this equation as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} \qquad (*)$$

Next, the rr component of Einstein's equations is

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} - \frac{6}{2}\frac{a^2}{1-kr^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$$

$$= \frac{-2a\ddot{a} - \dot{a}^2 - k}{1-kr^2} = 8\pi G_N T_{rr}$$

Rearranging, we can write this as

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G_N \frac{1-kr^2}{a^2} T_{rr} = -8\pi G_N g^{rr} T_{rr}$$

We can nicely eliminate the metric dependence by demanding that T_{rr} is proportional to the metric:

$T_{rr} = p g_{rr}$. p is called the pressure.

Then, another equation is

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G_N p. \qquad (\square)$$

We can continue, but the other equations provide no new information. We're familiar with this for the $R_{\phi\phi}$ equation from Schwarzschild and other exact solutions. It's also true for the $\theta\theta$ component because R_{rr} and $R_{\theta\theta}$ differ only by the metric.

Combining the equations $(*)$ and $(\square)$ we find the Friedmann equations:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}\,, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p).}$$

These are two equations for three unknowns: a, ρ, and p. Properly, there is in general no relationship between energy density and pressure. However, as we discussed some weeks ago, gravity only acts like we think it should for particular relationships between energy density and pressure. For example, we had discussed the weak energy condition:

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \text{ for all time-like vectors } t^\mu.$$

This is the requirement that $\rho \geq 0$ (when $t^\mu = (1,0,0,0)^\mu$). Additionally, we could consider the timelike vector $t^\mu = (\alpha, \beta, 0, 0)^\mu$, for some α and β. Then, the weak energy condition is

$$\alpha^2 T_{00} + T_{rr}\beta^2 = \alpha^2\rho + \beta^2 \frac{a^2}{1-kr^2}\, p \geq 0$$

Note that this vector is timelike as long as

$$g_{\mu\nu} t^\mu t^\nu = -\alpha^2 + \beta^2 \frac{a^2}{1-kr^2} < 0.$$

This can be arbitrarily close to zero, so a timelike vector is arbitrarily close to case when

$\frac{\alpha}{\beta} = \frac{a}{\sqrt{1-kr^2}}$. In this limit, then, the WEC

also implies: $\rho + p \geq 0$. The other energy conditions

(Dominant, Null, etc.) imply similar relationships between energy density and pressure. These relationships are all linear motivating the equation-of-state parameter:

$$w = \frac{p}{\rho}.$$

At this stage, we are just trading one expression for another, but let's visit another differential equation of the stress energy tensor we know that it satisfies: the conservation law

$$\nabla_\mu T^{\mu\nu} = 0.$$

The $\nu=0$ component of this equation can be derived from the Christoffel's and using the energy density ρ and pressure $p = w\rho$. One finds

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

Assuming that w is approximately constant, we can directly solve this equation to find:

$$\rho \propto a^{-3(1+w)}.$$

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If w were less than -1, this would lead to an arbitrary growth of the energy density if $a > 1$. This seems very undesireable, and the WEC, DEC, etc., explicitly forbid them. All forms of energy that we know about satisfy $w \geq -1$.

Let's study what these forms of energy might be. Again, we can express the stress-energy tensor as

$$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & \\ \hline & p g_{ij} \end{array}\right)$$

Isotropy of space requires that there is a unique pressure for all spatial components. We'll study four types of energy density today and then next week discuss implications for the evolution of the universe.

One type of energy density is <u>matter</u>. Matter is like what it sounds like: massive stuff like atoms, us, galaxies, etc. Because matter is massive and non-relativistic, its energy density will be dominated by mass. Further, matter (on cosmological scales) interacts extremely weakly and so is also collisionless. As such, it has no pressure: $p = 0$. (cf. to air in a room.) Thus, $w = 0$ and the matter energy density falls off like the third power of the scale factor:

$$\rho_M \propto a^{-3}.$$

<u>Radiation</u> is relativistic particles whose energy density is dominated by its momentum. Massless

particles like photons are radiation, and ~~radiation~~ photon's stress energy tensor is just that of electromagnetism:

$$T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{g_{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}.$$

We know that the trace of this tensor is 0:

$$T_{\mu\nu} g^{\mu\nu} = 0 = -\rho + 3p.$$

Therefore, the equation of state for radiation is

$w = \frac{1}{3} = \frac{p}{\rho}$. It then follows that the energy density for radiation falls off like a^{-4}:

$$\rho_R \propto a^{-4}.$$

There could be a vacuum energy density or cosmological constant. A cosmological constant's stress-energy tensor is:

$$\Lambda g_{\mu\nu} \propto T_{\mu\nu} \quad \text{or that } \rho = -\Lambda,\ p = \Lambda.$$

Thus the equation of state for the cosmological constant is

$w = -1$, which saturates the bound

at which we think the energy density makes sense. Not surprisingly, the cosmological constant is a well, cosmological constant:

$$\rho_\Lambda \propto 1 \propto a^0.$$

There is one more type of energy density, which isn't really, but has the effects as such. It's called the curvature density. In the Friedmann equations, energy density and curvature appeared together as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

so the object $-k/a^2$ acts like an energy density. Clearly, it scales with scale factor a as:

$\rho_c \propto a^{-2}$, which then implies that its

"equation of state" is: $-2 = -3(1+w)$ or $w = -\frac{1}{3}$. This is well within our requirement of $w \geq -1$.

To summarize, let's imagine that $a(t)$ grows with time:

$$\frac{\dot{a}}{a} > 0.$$

Then, at early times radiation is most important, at slightly later times matter is more important, at slightly later times curvature is more important, and then, as $t \to \infty$, all that remains, ad infinitum, is the vacuum.

We'll make this more precise next week.