Quantum Mechanics and Gravity

Let's do something silly. Let's focus way in to very near the event horizon of a Schwarzschild black hole. Recall that the metric is

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

For future reference, we note the conserved energy per unit mass E is

$$E = \left(1-\frac{R_s}{r}\right)\frac{dt}{d\lambda}$$ and the trajectory of

a time-like geodesic satisfies

$$1 = \left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2$$ or that

$$\frac{dr}{d\lambda} = \sqrt{E^2 - \left(1-\frac{R_s}{r}\right)}$$

Focusing way into the Schwarzschild radius, means that we are within, say, $\Delta x \ll R_s$ of the event horizon. With this assumption, the expressions from above simplify:

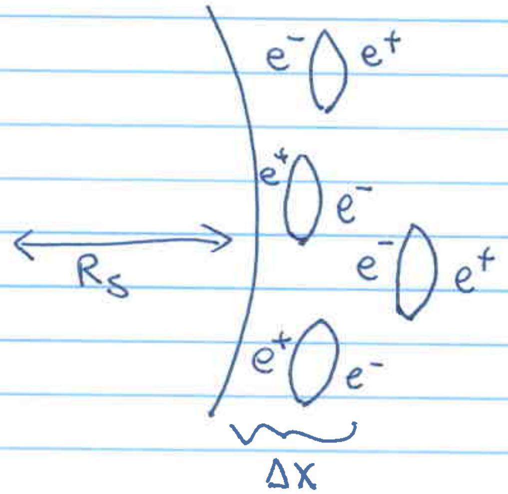
$$E = \left(\frac{r-R_s}{r}\right)\frac{dt}{d\lambda} \simeq \frac{\Delta x}{R_s}\frac{dt}{d\lambda}$$

$$\frac{dr}{d\lambda} = \sqrt{E^2 - \left(\frac{R_s - r}{r}\right)} \simeq \sqrt{E^2 - \frac{\Delta x}{R_s}}$$

We can correspondingly find the coordinate velocity for a massive particle near the event horizon:

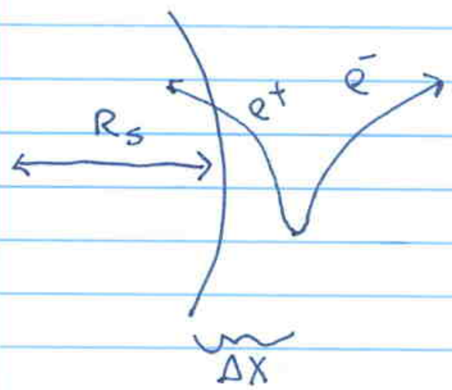
$$\frac{dr}{dt} = \frac{\frac{dr}{d\lambda}}{\frac{dt}{d\lambda}} \simeq \frac{\Delta x}{E R_s} \sqrt{E^2 - \frac{\Delta x}{R_s}}$$

Without knowing Δx or E this is all we can do. However, let's see what's happening near the event horizon:



Zooming in close enough to the event horizon, we see a roiling mess of particles and anti-particles produced. Focusing in on a distance Δx from the horizon, the momentum of these particles would be:

$\Delta p \, \Delta x \sim \frac{\hbar}{2}$, from the Heisenberg uncertainty principle. The closer we look to the horizon, the higher the momentum of particles. As drawn, all of the particle-antiparticle pairs (here, electrons and positrons) are created and then annihilate shortly after, about a distance of Δx away. If we get close enough to the horizon, the momentum $\Delta p \to \infty$ and so it becomes very likely that one of those particles gets sucked in:



If one of the particles gets sucked in, then the other has no partner to annihilate with so it will fly off. To an observer far away, it will look like the black hole emitted this particle. What energy would an observer measure?

The characteristic quantum time of the pair creation and annihilation is given by another of Heisenberg's uncertainties:

$\Delta E \, \Delta t \simeq \frac{\hbar}{2}$. That is, the energy of the emitted particle, at the point of emission, is

$E_{part} \simeq \frac{\hbar}{2\Delta t}$, where Δt is the time that the particle-anti-particle pair lives. Importantly, this is coordinate time; roughly, it is the coordinate time for the antiparticle to get sucked into the black hole.

Then, the coordinate velocity of the emitted particle is

$$\frac{\Delta x}{\Delta t} \sim \frac{dr}{dt} \sim \frac{E_{part}}{\Delta p}$$

Importantly, note that both E_{part} and Δp are proportional to the particle mass, so this is equivalent to:

$\frac{dr}{dt} \sim \frac{E}{p}$, the energy and momentum per unit mass at the time of particle pair creation. Heisenberg says that the momentum p is then

$$p = \frac{\Delta p}{m} = \frac{\hbar}{2m\Delta x}.$$

Earlier, we had calculated the coordinate velocity

$\frac{dr}{dt}$, so we have an expression for the momentum:

$$p = \frac{E}{\frac{dr}{dt}} = \frac{\hbar}{2m\Delta x} \cong \frac{E^2 R_s}{\Delta x \sqrt{E^2 - \frac{\Delta x}{R_s}}}$$

This is an oddly dimensionalized expression: we have $\hbar$ explicit, but have set $c=1$. We'll restore c at the end. In terms of the distance Δx from the event horizon, the energy of the escaping particle is then:

$$\frac{\hbar}{2m}\sqrt{E^2 - \frac{\Delta x}{R_s}} \cong E^2 R_s \Rightarrow R_s^2 E^4 - \frac{\hbar^2}{4m^2}E^2 + \frac{\hbar^2 \Delta x}{4m^2 R_s} = 0$$

Solving for E^2, we find

$$E^2 = \frac{\hbar^2}{8m^2R_s^2} \pm \frac{\sqrt{\frac{\hbar^4}{16m^4} - \frac{\hbar^2 \Delta x R_s}{m^2}}}{2R_s^2} = \frac{\hbar^2}{8m^2R_s^2}\left(1 \pm \sqrt{1 - \frac{16m^2\Delta x R_s}{\hbar^2}}\right)$$

In the limit that $\Delta x \to 0$ as we approach the event horizon, the energy approaches a sensible limit:

$E \to \frac{\hbar}{2\sqrt{2}mR_s}$. We don't necessarily trust the constants $2, \sqrt{2}$, etc., (as we've made a lot of assumptions), so let's ignore them. That is, the energy per unit mass of a quantum mechanical particle near a black hole is:

$$E = \frac{\hbar}{mR_s}.$$

Energy is conserved, so let's calculate what this would mean for an observer at $r=\infty$.

Because this energy is conserved, this is what an observer at $r=\infty$ will also observe. That is, the energy of such a particle at $r=\infty$ is

$$E_{tot} = mE = \frac{\hbar}{R_S} \propto \frac{c^2\hbar}{G_N M}.$$

This quantum mechanical process of particles produced near the event horizon will happen over and over and over and we will observe innumerable such particles at $r\to\infty$ with energy given by E_{tot}. Such a large statistical sample of particles therefore has a temperature corresponding to the energy divided by the Boltzmann constant:

$$T_H = \frac{E_{tot}}{k_B} \propto \frac{c^2\hbar}{G_N k_B M}.$$

This is one of the most important results in gravity. T_H is called the Hawking temperature, and a black hole, simply by existing in a quantum mechanical universe, necessarily emits radiation at a temperature inversely proportional to its mass.

While we won't justify it, it is true that the particles emitted from the black hole have a Boltzmann distribution and therefore black holes are blackbodies. As Stephen Hawking would say: black holes ain't so black.

We'll study further consequences of this observation next week.