Black Hole Thermodynamics

Last week, we had studied the physics of what was happening *just* outside the event horizon of a black hole. To describe a black hole, we of course need general relativity which defines the strength of gravity at the event horizon. By restricting attention to extremely close to the event horizon, we need quantum mechanics via Heisenberg uncertainty to predict the relevant energy or momenta of particles created there. We had argued that particle-antiparticle pairs will fluctuate out of the vacuum and some of the time, one of them will get sucked into the black hole. When that happens, the particle with no partner is then cursed to fly off to ∞, never annihilating. QM + GR enabled us to calculate the characteristic energies of these particles to be:

$$E = \frac{\hbar c^2}{G_N M},$$

where M is the black hole mass. Many, many, many such particles would appear to be emitted from the black hole. So, like the air in this room, the cumulative effect of a statistically large sample of particles would be a <u>temperature</u>. The characteristic temperature of the black hole is thus

$$T_H \cong \frac{\hbar c^2}{k_B G_N M},$$ which is called the Hawking temperature.

Importantly, all of the radiation that it appears the black hole is emitting is quantumly <u>random</u>, and so there are no correlations whatsoever between different particles that we observe at ∞. Therefore, the black hole is a blackbody. There is only one bit of information that quantifies the radiation from a blackbody: its temperature.

In this lecture, we'll work a bit to make this more precise. In particular, if black holes have a temperature, then ~~they~~ they must obey the laws of thermodynamics. Let's recall the three laws of thermodynamics:

0) States in thermal equilibrium with each other share a quantity called temperature, T

1) Total energy is conserved.

2) The entropy of a closed system does not decrease with time.

Let's see how a black hole manifests these three laws. As we have discussed, a black hole has a well-defined temperature; therefore it is in thermal equilibrium. A requirement of this statement is that the temperature at every point on the horizon be the same, otherwise it hasn't reached equilibrium. Therefore, the temperature of a black hole is defined by a quantity that is constant over the horizon. We have already defined such a quantity: the surface gravity, κ. The surface gravity is indeed constant on the horizon of a black hole if that horizon is also a Killing Horizon; i.e., there is a time-like Killing vector that is null on the horizon. For a Schwarzschild black hole, the surface gravity is ofcourse

$\kappa = \frac{1}{4G_N M}$. In units where $k_B = c = \hbar = 1$,

the Hawking temperature is thus: $T_H = \frac{\kappa}{2\pi}$.

Now, onto the first law, energy conservation. In theramodynamics, we often express differential energy dE as:

$$dE = T\,dS - p\,dV.$$

Of course $p\,dV$ represents the work we do on the system by applying pressure p and changing volume by dV. T is the temperature which measures the internal energy of the system and dS is the change in entropy, which is a measure of the number of accessible states of the system.

For now, let's ignore the $p\,dV$ component and just focus on the internal energy of a black hole:

$$dE = T\,dS.$$

Starting on the left, what is the energy of a Schwarzschild black hole? As it is a point mass, its energy is just

$E = M$, in units with $c=1$. Therefore, to change the energy, we have to change the mass:

$$dE = dM.$$

On the right, we've already determined the temperature of our black hole: $T \cong \frac{c^2\hbar}{k_B G_N M} \cong \frac{1}{G_N M}$.

This then provides a definition of the entropy of a black hole. Re-arranging, note that the change in entropy is:

$$dS = \frac{dE}{T} \cong M\,dM, \quad \text{or that entropy } S \cong M^2.$$

The squared mass of a black hole is correspondingly proportional to the square of its Schwarzschild radius:

$$M^2 \propto R_s^2 \propto A,$$ the area of the event horizon. The Beckenstein entropy of a black hole is proportional to its event horizon surface area:

$$S = \frac{A}{4G_N}.$$

Another way to interpret a black hole is as the maximal entropy spacetime volume with surface area A.

Does it make sense for the entropy of a black hole to be proportional to area? At its surface, this seems very strange. The entropy is a measure of the number of microstates W that produce the same macrostate defined by energy, temperature, volume, etc:

$$S = k_B \ln W.$$

Shouldn't there be configurations within the volume of the black hole that can change without affecting the macrostate? No, not in the way we identified how the black hole had a temperature. To calculate the black hole temperature, we only considered what was happening at the surface of the black hole, and not throughout its volume. That is, the microstates that produce the temperature of the black hole are localized over an area, not a volume. Another way to say this is that, as an outsider to the black hole, we have no access to its interior, only its surface. Therefore, all of our observations must only depend on quantities, states, and configurations defined at the surface of the black hole.

Finally, the second law. With the entropy defined as the surface area of the event horizon, $S=A$, this would suggest that:

$$dS = \frac{dA}{4G_N} \geq 0.$$

However, the black hole, in isolation is not a closed system. The black hole radiates into the rest of the universe. So, the closed system consists of the black hole and its radiation. With the entropy of the radiation S_R, the second law for black holes is:

$$dS_R + \frac{dA}{4G_N} \geq 0.$$

This was proved by Jacob Beckenstein. What makes this particularly interesting is that the microstates of radiation fill a volume, while for the black hole they only fill an area.

Let's do a couple more things today. First, given

The entropy of a black hole, how many microstates does this describe? From the definition

$S = k_B \ln W$, the number of microstates W is

thus $W = e^{S/k_B}$. With the total energy $E = Mc^2$

and Hawking temperature $T_H = \frac{c^3\hbar}{8\pi k_B G_N M}$, the

entropy is: $dS = \frac{dE}{T} = \frac{Mc^2}{c^3\hbar} 8\pi k_B G_N M = 8\pi \frac{k_B G_N}{c\hbar} M^2$ so $S = 4\pi \frac{k_B G_N}{c\hbar} M^2$

That is, the total number of microstates is:

$$W = \exp\left[4\pi \frac{G_N M^2}{c\hbar}\right]$$

Plugging in numbers for a solar mass black hole, we find the number of microstates to be:

$W \sim \exp[10^{77}]$, which cannot be evaluated in Mathematica. :(

A possible explanation for where these microstates come from we'll hear about in the presentations. At any rate this number of microstates calls into question the no-hair theorem that states that black holes are defined by a few quantities (M, Q, J, etc.). What are these microstates?

Finally, if the black hole radiates then it must lose energy and so after sufficient time evaporates away. We can estimate the time for evaporation by the Stefan-Boltzmann law. Recall that the power radiated ~~b~~ per unit area for a perfect blackbody is

$$\frac{P}{A} = \sigma T^4.$$

Power is dE/dt, and for a black hole $c^2 dM/dt$, the surface area of the event horizon is

$$A = 4\pi \cdot 4 \frac{G_N^2}{c^4} M^2 = 16\pi \frac{G_N}{c^4} M^2$$

and the fourth power of the temperature is:

$$T^4 = \frac{c^{12} \hbar^4}{4096 \pi^4 k_B^4 G_N^4 M^4}.$$

The Stefan-Boltzmann constant is:

$$\sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2}$$

and so the rate of mass lost by the black hole is:

$$\frac{dM}{dt} = -\cancel{16}\cancel{\pi} \frac{G_N^2}{\cancel{c^4}} \frac{\cancel{M^2}}{\cancel{c^2}} \cdot \frac{\cancel{\pi^2} \cancel{k_B^4}}{60 \cancel{\hbar^3} \cancel{c^2}} \frac{c^{\cancel{12}\,4} \cancel{\hbar^4}}{\underset{256}{\cancel{4096}} \cancel{\pi^4} \cancel{k_B^4} G_N^{\cancel{4}\,2} M^{\cancel{4}\,2}}$$

$$= -\frac{c^4 \hbar}{\underbrace{15360 \pi G_N^2}_{=\alpha}} \frac{1}{M^2}$$

As M decreases, the rate of emission increases. This can be integrated to find:

$$M(t) = -(3\alpha)^{1/3} t^{1/3} + M_0,$$

where M_0 is the initial mass.

The total evaporation time is then when $M(t_{ev}) = 0$

$$t_{ev} = \frac{5120 \pi G_N^2}{c^4 \hbar} M^3.$$

For a solar mass black hole, this is about 10^{60} times longer than the age of the universe.