Lecture 4 Manifolds

Now, I don't mean to pick on Nikki, but they said something that I originally laughed at, but on deeper reflection, realized there was a profound question. Nikki is unsure if the Earth is round or flat. How do we know? If you couldn't fly, how could you ever know? Everything at human scales is flat; we say that locally the Earth is flat (up to mountains and valleys and such). I think this is something that everyone could agree on. What about the property of the entire Earth? Is the whole thing flat? [Is the only possibility for the whole Earth to be flat if it appears so at small scales?] #1

Further, is it possible to conclusively determine, from local measurements, what the entire earth looks like? You might say, go watch a ship sail over the horizon. Boom, Earth is round. That is not a local measurement, and to really see the ship, you need a telescope. [No, I mean can you conclusively determine the shape of the entire Earth exclusively from looking no more than a foot in any direction?] #2

We'll answer this second question in a bit; let's revisit the first. Is a flat Earth the only thing consistent with a locally flat Earth? Locally flat mathematically means that we are able to define a plane (a flat surface) tangent to our point on Earth.

To be able to define a tangent plane just requires that we can take a derivative of the surface of the Earth at our point of interest to set the slope of the plane. A generic object which enables differentiation on it is called a differentiable manifold. A manifold colloquially is any object in any number of dimensions. A plane is a manifold, and so is a 2-sphere, and a torus, etc. A differentiable manifold is smooth: if you zoom in far enough, all the wrinkles flatten, the noise subsides, and the manifold looks flat. Just like we see on Earth. This answers the first question: the only thing that is required of Earth if it appears locally flat is that it is a differentiable manifold.

Our first step toward the second question will be asking the following. If we sit in the grass in front of Eliot Hall, is there any measurement we could do on our one square foot patch of grass to determine the curvature of Earth. I'll let you think about it, but if you think of something in the positive, I will be shocked. Earth is locally flat, which is to say that by sitting and staring at the grass, it will just look flat. We can make this a more profound statement. Let's broaden our horizons beyond Earth to the entire universe. Is there any measurement we can do in a cubic foot, or inch, or nanometer that can determine what shape the universe is? No? If not, then the universe is also locally flat. If it's locally flat then in general it is some differentiable manifold.

This is an extremely profound conclusion. From the observation that the universe on small scales appears flat (that is, described by special relativity), and so a tangent plane can be defined, requiring that there's no way just from measurements at a point to say otherwise, leads to the conclusion that the universe is a differentiable manifold. This is a four-dimensional manifold because locally flat implies that special relativity applies locally, so space and time are mixed. Further, gravity is mixed up in this. You feel a force on you because you are sitting in a chair. We ascribe this to Earth's gravity, but would it feel any different if you were in a spaceship accelerating at g? So, is it gravity or acceleration?

At this point, you might be having an existential crises. Gravity, as you have always been told, pulls you toward Earth. "Pulls" invokes forces and Newton's second law. However, is there <u>any</u> measurement that you can do locally, on your patch of grass to distinguish "gravity" (whatever that means) and acceleration? No? If not, then "gravity" is <u>not</u> a force. Indeed, if you felt <u>no</u> forces on you (no pushing or pulling) you would be in free-fall, like jumping out of a plane or riding an ~~ex~~ exciting roller coaster. But, even when you feel no forces, gravity can still be there, like when jumping out of a plane. This leads to a general definition of free-fall:

Free fall is being exclusively under the influence of gravity.

You can't escape gravity: it is everwhere. The only thing that is everywhere, even far from stars, galaxies, light, far from any maddening crowd, is space and time. If gravity and space and time are the only things that are (or define) everywhere, then

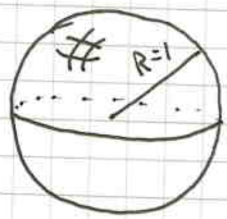
$$\boxed{\text{Gravity} = \text{Space and time}}$$

I emphasize this point and draw out the physical understanding because this is the foundation for everything that follows in this class. Gravity manifests itself as a differentiable space-time manifold, so if we are to make progress, we need to understand manifolds.

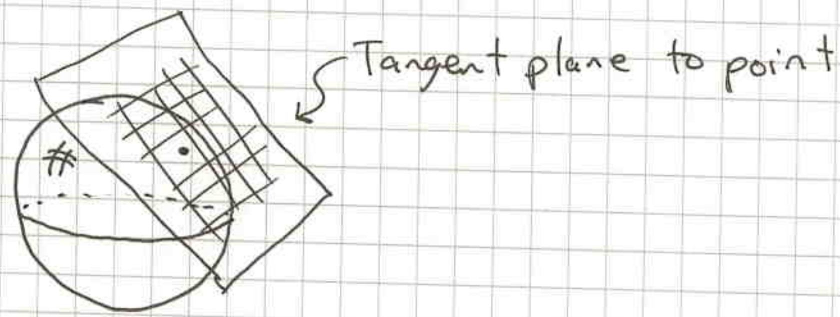
A manifold is some set of points. For example, the two-sphere is defined as the set of points (x,y,z) such that

~~$x^2+y^2=1$, where the radius is 1.~~ $x^2+y^2+z^2=1$

This of course looks like

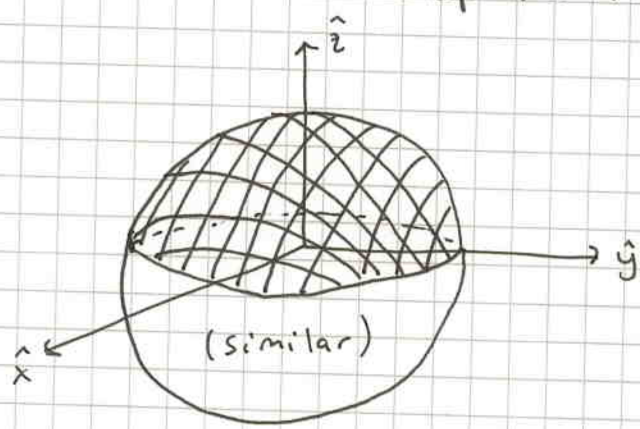


This will be our manifold poster-child for lecture, while I refer to the book for more general features and definitions. This is indeed a differentiable and smooth manifold. That is, at any point on the sphere there is a well-defined tangent plane:



That is, locally, the sphere is flat.

The sphere and points on it can be represented in many different ways that do not affect its essence. We've already introduced one coordinate system for the points on the sphere, Cartesian coordinates. Drawn on the sphere these look like (kind-of)



These coodinates are subject to the constraint that

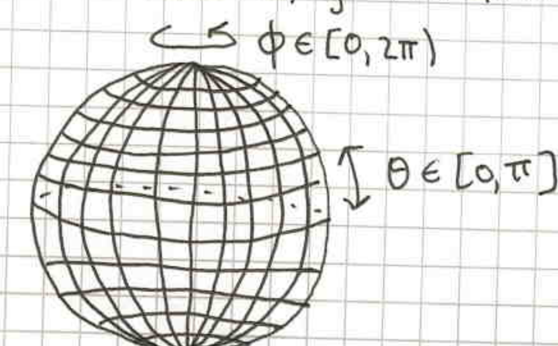
$$x^2+y^2+z^2=1.$$

We can perform a <u>coordinate</u> <u>transformation</u> by introducing a new set of coordinates. For example, spherical coordinates are:

$$1=r=\sqrt{x^2+y^2+z^2}\,,\quad x=r\cos\phi\sin\theta,\ y=r\sin\phi\sin\theta,\ z=r\cos\theta$$

which look like:

Note that the shape of the sphere is the same; the way we reference a point has changed.



The "shape" of a manifold is called the topology and a coordinate transformation does not affect topology.

A distance on the manifold ds^2 is determined by the appropriate Pythagorean theorem. For example, in Cartesian coordinates, the distance between two points would be

$$ds^2 = dx^2 + dy^2 + dz^2, \quad \text{subjected to } x^2+y^2+z^2=1.$$

Using this constraint, we can reexpress ds^2 exclusively in terms of distances dx and dy. I'll leave it for you to show that

$$\text{Cartesian: } ds^2 = \frac{1-y^2}{1-x^2-y^2}dx^2 + \frac{1-x^2}{1-x^2-y^2}dy^2 + \frac{2xy}{1-x^2-y^2}dx\,dy. \quad (\ast)$$

A coordinate transformation maintains distances; it just re-expresses them. Let's transform this to spherical coordinates, where $x = \cos\phi\sin\theta$, $y = \sin\phi\sin\theta$.

Then

$$dx = -\sin\phi\sin\theta\,d\phi + \cos\phi\cos\theta\,d\theta$$

$$dy = \cos\phi\sin\theta\,d\phi + \sin\phi\cos\theta\,d\theta$$

I'll also leave it to you to show that inserting these expressions in $(\ast)$ yields:

$$ds^2 = d\theta^2 + \sin^2\theta\,d\phi^2.$$

Again, distances are measurable quantities, and so cannot depend on how we represent the manifold.

Finally for today, I just want to mention the representation of a general coordinate transformation. Given some initial coordinates x_μ and other coordinates x'_μ, the coordinate transformation is:

$$dx'_\mu = \frac{\partial x'_\mu}{\partial x_\nu} dx_\nu .$$

This fancy looking expression has a fancy name: the chain rule. By the way, Lorentz transformations are coordinate transformations. In that case, what ~~[illegible]~~ is the derivative object $\frac{\partial x'_\mu}{\partial x_\nu}$? It's just a Lorentz matrix!

$$\Lambda_\mu{}^\nu = \frac{\partial x'_\mu}{\partial x_\nu} \quad \text{for Lorentz transformations.}$$