Manifolds Part 2

At the end of last lecture, we had introduced the differential line element, or distance measure, ds^2 for a manifold. For example, in spherical coordinates, the differential line element of a two-sphere is:

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2.$$

Compare this expression to the spacetime-interval we introduced in special relativity:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = ds \cdot ds$$

Recall that from this expression, we defined the Minkowski or flat-space metric η which determines how indices are contracted. As a matrix, η is

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

This is just the matrix of coefficients of the terms in the interval.

The flat-space metric enabled us to contract vector indices so as to construct Lorentz-invariant quantities. On a non-linear manifold, like the two-sphere, the analogy of a Lorentz transformation was a general coordinate transformation. A general coordinate transformation leaves all measurable quantities, like a distance ds^2, invariant. So, this suggests that we should be able to identify an object g that defines for us how to contract indices on a manifold in a coordinate-invariant way. g is called the <u>metric</u> and is defined from the differential line element:

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where the coordinates on the manifold are x^μ. If ds^2 is coordinate-invariant, we can determine how $g_{\mu\nu}$ must transform. Recall, by the chain rule, the coordinates dx_μ transform to new coordinates dx'_μ as

$$dx'_\mu = \frac{\partial x'_\mu}{\partial x_\nu} dx_\nu$$

Then, under a coordinate transformation, $dx^\mu dx^\nu$ transforms to:

$$dx^\mu dx^\nu \longrightarrow \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\rho} dx'^\sigma dx'^\rho$$

For ds^2 to be invariant to coordinate transformations, we then must require:

$$g_{\mu\nu} \longrightarrow \frac{\partial x'^\sigma}{\partial x^\mu} \frac{\partial x'^\rho}{\partial x^\nu} g_{\sigma\rho}.$$

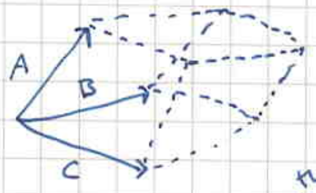
Just like the flat space metric η, g can act to raise and lower indices on vectors and tensors. For example, for a two-index tensor $T^{\mu\nu}$, we have

$T^\mu{}_\nu = T^{\mu\sigma} g_{\sigma\nu}$

The metric is symmetric in its two indices: $g_{\mu\nu} = g_{\nu\mu}$, and contracts with itself as:

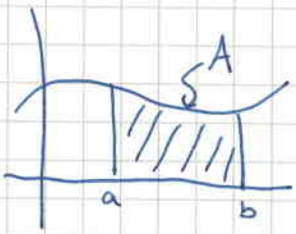
$g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu$, the Kronecker-$\delta$ object. This <u>defines</u> the metric with upper indices. Note also that the trace of the metric is just the dimension of space-time D: $g_{\mu\nu} \cdot g^{\mu\nu} = D$, In our universe, $D = 4$.

We'll return to the metric in a second, but for now we'll switch gears to differential forms. To motivate differential forms, we'll approach this in the opposite way as the textbook. The volume of a parallelepiped in three-dimensions is:



$$\text{Vol} = \vec{A} \cdot (\vec{B} \times \vec{C})$$

The magnitude of the volume the parallelepiped is the absolute value of this, while the sign means that the volume is oriented. "Oriented" means that there is a preferred direction to the volume. This might seem weird, but is actually quite familiar. For example, one-dimensional integration calculates the area under a curve:



$$A = \int_a^b dx\, f(x)$$

However, this area is also oriented: it is negative if the bounds of integration switch:

$$\int_b^a dx\, f(x) = -A.$$

So, oriented areas or volumes are actually quite familiar from Integration.

We can make the orientation of the parallelepiped explicit by introducing an anti-symmetric symbol. The three-dimensional

Volume element $d^3x = dx\,dy\,dz$ is then actually short-hand for

$d^3x = \frac{1}{3!}\varepsilon_{ijk}\,dx^i dx^j dx^k$, where $i,j,k = 1,2,3$ and represents x, y, z, for example. Under a coordinate transformation, the anti-symmetric symbol is unaffected, while each differential element transforms as: $dx^i \to \frac{\partial x^i}{\partial x'^a} dx'^a$, for new coordinates x'.

Therefore, the integration measure transforms as:

$$d^3x = \frac{1}{3!}\varepsilon_{ijk}\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^c}dx'^a dx'^b dx'^c$$

$$= \frac{1}{3!}\left|\frac{\partial x^i}{\partial x'^a}\right| \varepsilon_{abc}\, dx'^a dx'^b dx'^c$$

$\left|\frac{\partial x^i}{\partial x'^a}\right|$ is the Jacobian of the change of variables which is just the determinant of the derivative matrix:

$$\left|\frac{\partial x^i}{\partial x'^a}\right| = J = \det\frac{\partial x^i}{\partial x'^a} = \begin{vmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\ \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3} \end{vmatrix} \cancel{= \frac{1}{6!}\varepsilon_{ijk}\varepsilon^{abc}} \frac{\partial x^i}{\partial x'^a}$$

$$= \varepsilon_{ijk}\frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^b}\frac{\partial x^k}{\partial x'^c}$$

You are likely very familiar with this change of variables formula. Just like we were able to construct a coordinate-invariant line element, we want to construct a coordinate invariant volume element. d^3x isn't quite what we want because it picks up a Jacobian. However, recall the coordinate transformation law of the metric:

$$g_{\mu\nu} \to \frac{\partial x'^{\sigma}}{\partial x^{\mu}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} g_{\sigma\rho}$$

The determinant of the metric, which we denote by $|g|$, therefore transforms as

$$|g| \to \left| \frac{\partial x'^{\sigma}}{\partial x^{\mu}} \right|^2 |g| = J^{-2} |g|,$$ where J is the Jacobian.

So, to exactly cancel the Jacobian from the volume element d^3x, we multiply by $\sqrt{|g|}$. This defines a coordinate-invariant volume element: $\boxed{\mathrm{Vol} = \sqrt{|g|}\, d^n x}$, which we generalize to n dimensions. Let's see what this is in an example.

The line element of flat 3D space expressed in cartesian and spherical coordinates is:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

The cartesian metric g_c is then

$$g_c = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$ while the spherical metric is $$g_s = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

The corresponding determinants are:

$$g_c = 1, \quad g_s = 1 \cdot r^2 \cdot r^2 \sin^2\theta = r^4 \sin^2\theta.$$

Therefore, the volume elements are related by:

$$dx\,dy\,dz = \sqrt{r^4 \sin^2\theta}\, dr\, d\theta\, d\phi = r^2 dr\, d(\cos\theta)\, d\phi,$$

which is of course correct.

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This invariant volume element will be extremely useful in formulating a Lagrangian for gravity.

To end today, I want to introduce differential forms. A p-form is simply an antisymmetric tensor will p indices downstairs:

$$A_{\mu_1 \mu_2 \cdots \mu_p}$$

In D dimensions, p is at most D. If p > D, a p-form is automatically 0 by antisymmetry. Two forms can be combined with a wedge product. A p-form and a q-form A and B combine with a wedge product as

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$$

An exterior derivative is just a partial derivative that acts through the wedge product on a form. On our p-form, the exterior derivative d is

$$(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}$$

The exterior derivative is special because its action produces a tensor. For example, when acting on a one-form A_μ, the exterior derivative is $(dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Coordinate transforming, we have:

$$\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial}{\partial x'^\rho} \left(\frac{\partial x'^\sigma}{\partial x^\nu} A_\sigma \right) - \frac{\partial x'^\rho}{\partial x^\nu} \frac{\partial}{\partial x'^\rho} \left(\frac{\partial x'^\sigma}{\partial x^\mu} A_\sigma \right) = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right)$$

Second derivatives, which would spoil nice transformation properties, explicitly cancel. With the exterior derivative, electromagnetism is simply expressed. The field-strength tensor (which recall, was antisymmetric) is just

$F = dA$. The Bianchi identity is now trivial. The field strength tensor is an <u>exact</u> form; it is the exterior derivative of another form. Therefore it is also <u>closed</u>:

$dF = d(dA) = 0$. Note that $d(dA) = d^2 A = 0$ because partial derivatives commute: $d^2 = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0$.

To get the rest of Maxwell's equations, we need the Hodge star operator $*$. On a p-form this operator acts as

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \sqrt{|g|}\, \varepsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p},$$

which maps a p-form to an n-p form in n-dimensional space. Maxwell's equations with charges are then expressed in differential form notation as:

$$d(*F) = *J.$$

You'll study this in homework.