Manifolds: Electromagnetism with Forms

We'll finish up our introduction to manifolds today with applying the differential form notation that we introduced last lecture. Let's attempt to formulate electromagnetism with differential forms. Our fundamental object in electromagnetism is the vector potential A, which is also a one-form. To construct a Lagrangian and ultimately an action, we need derivatives of the vector potential. With the exterior derivative d, we can form the field strength tensor F, which is an exact two-form:

$$F = dA.$$

Note that $dF = d^2A = 0$, because partial derivatives commute:

$$d^2 = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0.$$

In components, dF is

$$(dF)_{\mu\nu\rho} = \frac{3}{3!}\left[\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} - \partial_\mu F_{\rho\nu} - \partial_\rho F_{\nu\mu} - \partial_\nu F_{\mu\rho}\right]$$

$$= \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0,$$

which is just the Bianchi identity. For Maxwell's equations that involve the current, we need an action and the principle of least action. The action is an integral over all spacetime of the Lagrangian density:

$$S[A] = \int d^4x \, \mathcal{L}.$$

In arbitrary coordinates, or on an arbitrary spacetime manifold, we can make this coordinate invariant with two things.

First, the volume element should be multiplied by $\sqrt{|g|}$, the determinant of the metric. In flat space, with the Minkowski metric $\sqrt{|g_{\mu\nu}|}=1$, so this is trivial, but let's include it for generality. Next, the Lagrangian density should be a four-form, because spacetime is four-dimensional. Why a four-form? A four form is completely anti-symmetric, just like the Levi-Civita tensor and all of its indices can contract with the volume element. This then defines an object that is coordinate transformation invariant.

We can write the following expression to denote this:

$$S[A] = \int d^4x \sqrt{|g|}\, \mathcal{L} = \int \underline{dx^\mu dx^\nu dx^\rho dx^\sigma} \sqrt{|g|}\, \mathcal{L}_{[\mu\nu\rho\sigma]}$$
$$\equiv \int \mathcal{L}$$

We've written out the explicit indices of the Lagrangian density four-form in the second equality. On the second line, we use the compact notation which suppresses the integration measure, because we know what it must be.

So, we have a lot of constraints that we can impose to determine the E+M lagrangian:

1) It must be a four-form
2) It must be gauge invariant
3) It must contain two derivatives of A (among other things

Let's use these guiding principles to determine $\mathcal{L}$. Let's start by constructing possible four forms.

Two possible four-forms are constructed from the field strength tensor. Because it ~~its~~ is a two-form, it can be wedged with itself:

$F \wedge F$ or with its Hodge dual, $F \wedge * F$

and both are four-forms. We could also possibly wedge F with two vector potentials:

$F \wedge A \wedge A$, but this is 0 because

$(A \wedge A)_{\mu\nu} = A_\mu A_\nu - A_\nu A_\mu = 0.$

Similarly, we could wedge A with its Hodge dual $*A$, which would be a four-form. However, the term

$A \wedge * A$ is not gauge invariant.

Let's see how this works. In components, this is

$$(A \wedge * A)_{\mu\nu\rho\sigma} = \frac{4}{4!} A_{[\mu} \varepsilon^{\alpha}{}_{\nu\rho\sigma]} A_\alpha = \frac{1}{6} A_{[\mu} \varepsilon^{\alpha}{}_{\nu\rho\sigma]} A_\alpha$$

When inserted in the integral, it is contracted with $\varepsilon^{\mu\nu\rho\sigma}$ from the integration measure. Note that

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha}{}_{\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\beta\nu\rho\sigma} g^{\alpha\beta} = 6 g^{\alpha\mu}$$

That is, $\varepsilon^{\mu\nu\rho\sigma} (A \wedge * A)_{\mu\nu\rho\sigma} = A \cdot A = A_\mu A^\mu$.

Under a gauge transformation, $A_\mu \to A_\mu + \partial_\mu \lambda$ and so

$$A_\mu A^\mu \to (A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda) = A_\mu A^\mu + 2 A_\mu (\partial^\mu \lambda) + (\partial_\mu \lambda)(\partial^\mu \lambda)$$

This is not 0 for general λ. Therefore, it is not a viable term in the Lagrangian. (By the way, such a term would correspond to a photon mass.)

So, it seems like all we have is $F \wedge F$ and $F \wedge * F$. These are both gauge invariant because F is gauge invariant. Which one do we use?

Let's massage $F \wedge F$ a bit. With exterior derivatives and the vector potential, this is:

$$F \wedge F = dA \wedge dA = d(A \wedge dA) + A \wedge d^2 A$$

In writing this expression, I have used the product rule for exterior derivatives. In components, we have

$$\begin{aligned}(d(A \wedge dA))_{\mu\nu\rho\sigma} &= 4\, \partial_{[\mu} A_\nu F_{\rho\sigma]} \\ &= 6 \left(\partial_{[\mu} A_\nu\right) F_{\rho\sigma]} - 8 A_{[\nu} \left(\partial_\mu F_{\rho\sigma]}\right) \\ &= \left(dA \wedge dA - A \wedge d^2 A\right)_{\mu\nu\rho\sigma}\end{aligned}$$

Note the $-$ sign from transposing the μ and ν indices. This is a bit different from the standard product rule, but just follows from anti-symmetry.

We also know that $d^2 = 0$ so $F \wedge F$ is simple:

$F \wedge F = d(A \wedge dA)$, which is a total derivative!

This term corresponds to the Lagrangian studied last week:

$$\mathcal{L}' = \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma},$$

which can be expressed as a total derivative.

The quantity $A \wedge dA$ is called the Chern-Simons current. So, for determining Maxwell's equations, $F \wedge F$ doesn't help.

What about $F \wedge * F$? In components, this is

$$(F \wedge * F)_{\mu\nu\rho\sigma} = 6\, F_{[\mu\nu} \cdot \tfrac{1}{2} \varepsilon^{\alpha\beta}{}_{\rho\sigma]} F_{\alpha\beta}$$

$$= 3\, F_{[\mu\nu}\, \varepsilon^{\alpha\beta}{}_{\rho\sigma]}\, F_{\alpha\beta}$$

As earlier, in the action we need to contract with $\varepsilon^{\mu\nu\rho\sigma}$ from the integration measure. Note that

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta}{}_{\rho\sigma} = g^{\alpha\delta} g^{\beta\epsilon} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\delta\epsilon\rho\sigma} = 4\left(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}\right)$$

Trust me that all the combinatorics work out! ☺
This then implies that:

$$\varepsilon^{\mu\nu\rho\sigma} (F \wedge * F)_{\mu\nu\rho\sigma} = F_{\mu\nu} F^{\mu\nu},$$ which is indeed

what we need for the action of E+M, up to an overall constant. Therefore, up to ignoring charges, the Lagrangian of E+M can be written as

$$\underline{\mathcal{L} = F \wedge * F.}$$

What about currents and charges? We have the current one-form J and the vector potential A and so to construct a four-form we need to use the Hodge $*$. In particular,

$A \wedge * J$ is a four form.

Note that it isn't possible to have a Lagrangian term in which the current couples to the field strength. It's not possible to make a non-zero four form from J and F. The only thing that is possible is:

$$dJ \wedge F = d(J \wedge F) - J \wedge dF = d(J \wedge F),$$

which is just a total derivative and so doesn't affect the equations of motion.

Then, the action of E+M can be written as:

$$S[A] = \int F \wedge * F + A \wedge * J$$

(up to overall constants :)).

To end this lecture, I want to study the four-forms $F \wedge F$ and $F \wedge * F$ a bit more. These two four-forms equal one another (up to a sign) if the field strength is selfdual or ~~self dual~~. antiself-dual

$$F = i * F \quad (\text{dual}) \qquad F = -i * F \quad (\text{anti})$$

Recall that, in components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

The Hodge dual of F in components is:

$$i(*F)_{\mu\nu} = i\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & +E_z & -E_y \\ B_y & -E_z & 0 & +E_x \\ B_z & +E_y & -E_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iB_x & -iB_y & -iB_z \\ iB_x & 0 & -iE_z & iE_y \\ iB_y & iE_z & 0 & -iE_x \\ iB_z & -iE_y & iE_x & 0 \end{pmatrix}$$

So, self-dual $F = i {*F}$ means that $\vec{E} = i\vec{B}$ and anti-self dual means that $\vec{E} = -i\vec{B}$. For self-dual field strenths and no charges, the equations of motion are identical to the Bianchi identity:

$$d * F = 0 = d^2 F.$$

So, a self-dual field trivially satisfies the equations of motion. In standard differential form, a self-dual electric (or magnetic) field satisfies:

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \times \vec{E} = i \frac{\partial E}{\partial t}$$

The electric field that solves this is:

$$\vec{E} = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{y}) e^{-ipt + ipz},$$ where we assume

that it is traveling along the $\hat{z}$ direction. This electric field is referred to as "right-handed" helicity. The electric field is circularly polarized in the right-handed sense. Left-handed helicity corresponds to the anti-self dual field.