awl

Curvature #2

Last lecture, we introduced the idea of a connection, $\Gamma^{\rho}_{\mu\nu}$, which enables a well-defined way to move around the manifold. We had discoverd the connection through studying geodesics, or shortest paths on a manifold, but saw that the same object appeared in the definition of the covariant derivative ∇_μ:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda$$

A covariant derivative, unlike the regular partial derivative, is a tensor; i.e., under coordinate transformations, it just transforms with Jacobian matrices. The covariant derivative is the object that enables parallel transport on the manifold. That is, to move a vector around a manifold, we need to use the covariant derivative as it maintains orientation.

Let's make this more precise. First just consider a function $f(x)$. Now, let's translate this function by a distance a:

$$f(x) \to f(x-a) = f(x) - a f'(x) + \frac{a^2}{2} f''(x) + \dots$$
$$= e^{-a\frac{d}{dx}} f(x)$$

That is, the operator that performs translation of a function is the derivative d/dx. Composing the derivative over and over produces the Taylor expansion of the function. Let's use a similar idea to understand moving a vector V^ν around a manifold.

On a non-linear manifold, the precise representation of a vector V^ν depends on the coordinates: $V^\nu = V^\nu(x)$.

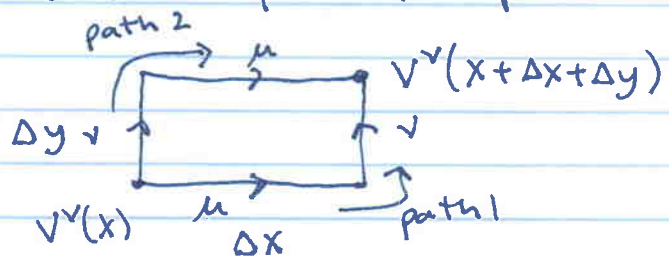
To move the vector to a new point, $x+\varepsilon$, we need to use the covariant derivative:

$$V^\nu(x+\varepsilon) = V^\nu(x) + \varepsilon^\mu \nabla_\mu V^\nu(x) + \frac{\varepsilon^\alpha \varepsilon^\beta}{2} \nabla_\alpha \nabla_\beta V^\nu(x) + \dots$$

Again, the covariant derivative ensures that the vector is still a vector (transforms appropriately) and that its orientation is maintained, though the manifold may be curved.

We can use these observations to measure the curvature of the manifold in a well-defined way, i.e., through a tensor. Last lecture we discussed how parallel transporting a vector along two different paths and comparing at the final point is a measure of the Earth's radius, for example. Let's make this more precise and see what the covariant derivative does for us.

Let's take our vector V^ν and move it along two paths from point 1 to point 2 on our manifold:



We'll move the vector a distance Δx in the μ direction and then Δy in the ν direction on path 1 and vice-versa on path 2. Once the vectors have reached $x+\Delta x+\Delta y$, we compare them, just by taking their difference. On a linear manifold, the vectors translated on the two paths will be identical, but, as we saw with the Earth, this is not true for a non-linear manifold.

First on path 1, we work backwards from $x+\Delta x+\Delta y$. That is, we taylor expand $V^\nu(x+\Delta x+\Delta y)$ first in Δy then in Δx:

$$V_1^\nu(x+\Delta x+\Delta y) = V_1^\nu(x+\Delta x) + \Delta y^\mu \nabla_\mu V_1(x+\Delta x) + \dots$$

$$= V_1^\nu(x) + \Delta x^\mu \nabla_\mu V_1^\nu(x) + \Delta y^\mu \nabla_\mu V_1^\nu(x)$$

$$+ \Delta x^\alpha \Delta y^\beta \nabla_\alpha \nabla_\beta V_1^\nu(x) + \dots$$

The "1" subscript just denotes this is for path 1. Path 2 is similar, just taking the expansion in the opposite order:

$$V_2^\nu(x+\Delta x+\Delta y) = V^\nu(x) + (\Delta x^\mu + \Delta y^\mu)\nabla_\mu V^\nu(x)$$

$$+ \Delta x^\beta \Delta y^\alpha \nabla_\alpha \nabla_\beta V^\nu(x) + \dots$$

Then, to lowest, non-trivial order in $\Delta x, \Delta y$, we find the difference:

$$V_2^\nu - V_1^\nu = \Delta x^\beta \Delta y^\alpha (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\nu + \dots$$

$$= \Delta x^\beta \Delta y^\alpha [\nabla_\alpha, \nabla_\beta] V^\nu,$$ where $[\,,]$ is the commutator.

It's a lot of algebra to evaluate this commutator, so I leave that to the book where it is done on page 122. The final result is:

$$[\nabla_\alpha, \nabla_\beta] V^\nu = \left(\partial_\alpha \Gamma^\nu_{\beta\sigma} - \partial_\beta \Gamma^\nu_{\alpha\sigma} + \Gamma^\nu_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma} - \Gamma^\nu_{\beta\lambda}\Gamma^\lambda_{\alpha\sigma}\right) V^\sigma$$

The object in parentheses is a <u>tensor</u> (i.e., transforms appropriately) and the derivatives <u>only</u> act on the connections. The terms with derivatives acting on the vector explicitly cancel when the commutator is taken. This tensor is called the Riemann tensor:

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}}$$

The Riemann tensor is analogous to the field strength tensor of electromagnetism, $F_{\mu\nu}$. $F_{\mu\nu}$ can be expressed as a commutator of appropriate covariant derivatives with a connection A_μ, the vector potential.

Just like the electromagnetic field strength tensor, the Riemann tensor satisfies a Bianchi identity. The Bianchi identity for E+M is

$$(dF)_{\mu\nu\rho} = \partial_{[\mu} F_{\nu\rho]} = 0$$

For the Riemann tensor, it is

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0, \quad \text{where } R_{\rho\sigma\mu\nu} = g_{\rho a} R^a{}_{\sigma\mu\nu}.$$

Note the necessity of the covariant derivative here. This just expresses the commutator identity:

$$[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0.$$

You can prove this by simply expanding out the commutators. Terms cancel pairwise.

We can have the Bianchi identity do work for us. As you study this week in homework, the stress energy tensor $T^{\mu\nu}$ which encodes energy and momentum of a system has a relationship to geometry. More relevant right now, is that it's conserved. In flat-space conserved means what you think:

$$\partial_\mu T^{\mu\nu} = 0.$$

This will be proved by you on the homework. In curved space, this conservation law is modified with the covariant derivative to:

$$\nabla_\mu T^{\mu\nu} = 0.$$

We'll discuss this next week, but for now I want to just dangle it there as a carrot. Also, the stress-energy tensor is symmetric $T^{\mu\nu} = T^{(\mu\nu)}$, and obviously a two-index tensor. So, can we use the Bianchi identity for the Riemann tensor to identify a conserved, symmetric, rank 2 tensor? Let's see.

In components, the Riemann tensor Bianchi identity is:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = \frac{1}{6}\Big[\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}$$

$$- \nabla_\lambda R_{\sigma\rho\mu\nu} - \nabla_\rho R_{\lambda\sigma\mu\nu} - \nabla_\sigma R_{\rho\lambda\mu\nu}\Big] = 0$$

The Riemann tensor is antisymmetric in its first two indices:

$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$, which follows from the

definition in terms of connections.

$$R_{\rho\sigma\mu\nu} = g_{\rho\alpha}\left(\partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right)$$

Recall that $\Gamma^a_{\beta\gamma} = \Gamma^a_{\gamma\beta}$ and $g_{a\beta} = g_{\beta\alpha}$.

Therefore the Bianchi identity reduces to

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

To make a two-index tensor from Riemann, we need to contract indices twice. We can't contract the first two indices, because they are antisymmetric. Similarly, contracting the final two indices also produces 0 because they are anti-symmetrized. So, we can only contract one of the first two with one of the second two. The contraction of the first and third indices is called the Ricci tensor:

$$R^\lambda{}_{\mu\lambda\nu} = g^{\lambda\alpha} R_{\alpha\mu\lambda\nu} \equiv R_{\mu\nu}$$

The Ricci tensor is symmetric: $R_{\mu\nu} = R_{\nu\mu}$. This follows from the definition of the Riemann tensor. The only funny term that isn't obiviously symmetric is:

$$R^\lambda{}_{\sigma\lambda\nu} \supset -\partial_\nu \Gamma^\lambda_{\lambda\sigma} = -\partial_\nu \left[\frac{1}{2} g^{\lambda\rho}\left(\cancel{\partial_\lambda g_{\sigma\rho}} + \partial_\sigma g_{\rho\lambda} - \cancel{\partial_\rho g_{\lambda\sigma}}\right)\right]$$

$$= -\frac{1}{2}\partial_\nu g^{\lambda\rho}\partial_\sigma g_{\lambda\rho}$$

$$= -\frac{1}{2}\left(\partial_\nu g^{\lambda\rho}\right)\left(\partial_\sigma g_{\lambda\rho}\right) - \frac{1}{2} g^{\lambda\rho}\partial_\nu\partial_\sigma g_{\lambda\rho},$$

which is symmetric, in $\sigma \leftrightarrow \nu$.

Further, we can contract the indices on the Ricci tensor to get the Ricci scalar:

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}.$$

Now, let's do two contractions of the Bianchi identity:

$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} \right)$$

$$= \nabla_\lambda \left(g^{\nu\sigma} g^{\mu\lambda} R_{\rho\sigma\mu\nu} \right) + \nabla_\rho \left(g^{\nu\sigma} g^{\mu\lambda} R_{\sigma\lambda\mu\nu} \right)$$

$$+ \nabla_\sigma \left(g^{\nu\sigma} g^{\mu\lambda} R_{\lambda\rho\mu\nu} \right)$$

$$= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu} = 0$$

That is, $\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0$, by contracting twice the Bianchi identity. We call

$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$ the Einstein tensor.

It has two indices, is symmetric $G_{\mu\nu} = G_{\nu\mu}$, and is conserved:

$$\nabla^\mu G_{\mu\nu} = 0.$$

Therefore, it is going to be important.