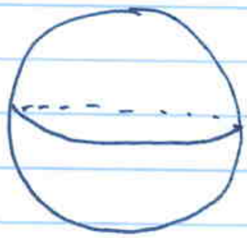
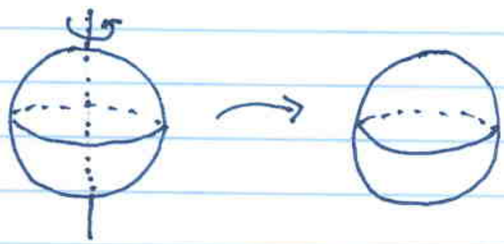
Curvature: Part Trois: Electric Boogaloo

This is a sphere:



What makes this sphere special? Well, it's name is Mike, it's working to put its life together after Tammy left, it loves romance novels, ... wait, wrong story. This sphere is special because there exist transformations that we can perform to it that leave its appearance, and any measurable quantities unchanged. For example, we can rotate the sphere about the vertical axis and it appears the same:



Measurable quantities, like distances on the sphere are clearly unchanged by such a transformation. A transformation that leaves a manifold unchanged is called a <u>symmetry</u>.

We've seen symmetries before, at the beginning of this class. We motivated special relativity by starting from requiring that the speed of light is constant in all inertial reference frames. This resulted in defining the space-time interval ds^2. In flat Minkowski space ds^2 is:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

In special relativity, we identified 6 transformations that left this interval invariant; six symmetries. These were three rotations and three boosts and are called Lorentz transformations. There are four other transformations that leave the spacetime interval invariant: translations. We can translate any coordinate by a fixed amount and the interval is unchanged. That is, transforming

$x^\mu \to x^\mu + a^\mu$ means $dx^\mu \to dx^\mu$, for a constant

vector a^μ. These 10 total symmetries are all linearly independent transformations that leave the interval unchanged. In D dimensions, a manifold that has $D(D+1)/2$ linearly independent symmetries is called a maximally symmetric space(time). This is the most symmetries possible in D dimensions, in general, a manifold has fewer than this number of symmetries.

How do we identify these symmetries? Translations are easy to find. If the metric is independent of a coordinate, then that coordinate exhibits a translation symmetry. In the case of Minkowski spacetime, the metric is independent of all coordinates, and so each of them can be translated. Back to the sphere where we began this discussion, recall that its line element is:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

That is, the metric is independent of coordinate ϕ, and so the transformation:

$\phi \to \phi + \alpha$ is a symmetry. As ϕ is the azimuthal angle this is of course nothing more that rotatation about the vertical axis.

While ~~the~~ independence of the metric on a coordinate is clearly sufficient to identify a symmetry it is not nesessary. In particular, for the sphere, we can rotate about 3 independent, perpendicular axes and the sphere is unchanged. Is there a general requirement we can identify that a symmetry must satisfy? If so, this will enable us to classify spacetimes by their symmetry, and exploit the constraints which the symmetries impose.

Our definition of a symmetry of a manifold will be any transformation of the coordinates ~~~~ that leave the interval invariant. That is, we perform a transformation of the coordinates as

$$x^\mu \to x^\mu + k^\mu, \text{ for some vector } k^\mu \text{ that itself}$$

depends on the coordinates. What is the constraint on k^μ such that this transformation is a symmetry?

Let's just plug this into the interval and see what we find. On a general manifold, the interval is:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \longrightarrow g_{\mu\nu}(x+k)\, d(x^\mu + k^\mu)\, d(x^\nu + k^\nu) \overset{?}{=} ds^2$$

Let's expand the right side in k^μ and see what we find.

First, we have that $d(x^\mu + k^\mu) = dx^\mu + dk^\mu$.

Using the chain rule, we have:

$$d(x^\mu + k^\mu) = dx^\mu + dk^\mu = dx^\mu + \frac{\partial k^\mu}{\partial x^\sigma} dx^\sigma.$$

A similar thing holds for $d(x^\nu + k^\nu)$. For the transformation in the metric, we need to Taylor expand. That is,

$$g_{\mu\nu}(x+k) = g_{\mu\nu}(x) + k^\sigma \partial_\sigma g_{\mu\nu}(x) + \dots$$

It will be suffieient to work to linear order in k in what follows. We will also drop the explicit coordinate dependence of the metric from now on:

$$g_{\mu\nu}(x) \equiv g_{\mu\nu}.$$

The derivative acting on the metric, $\partial_\sigma g_{\mu\nu}$, is a bit awkward, but we can re-write it nicely. Recall that we define the covariant derivative ∇_μ to be metric compatible:

$\nabla_\sigma g_{\mu\nu} = 0$. We can expand this in terms of partial derivatives and connections as:

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} g_{\mu\lambda} = 0$$

Then, the derivative of the metric is:

$$\partial_\sigma g_{\mu\nu} = \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} + \Gamma^\lambda_{\sigma\nu} g_{\mu\lambda}.$$

We'll replace $\partial_\sigma g_{\mu\nu}$ accordingly.

Putting these pieces together, the transformed interval is:

$$g_{\mu\nu}(x+k)\, d(x^\mu + k^\mu)\, d(x^\nu + k^\nu) = g_{\mu\nu}\, dx^\mu dx^\nu + g_{\mu\nu} \frac{\partial k^\mu}{\partial x^\alpha} dx^\alpha dx^\nu$$

$$+ g_{\mu\nu} \frac{\partial k^\nu}{\partial x^\alpha} dx^\alpha dx^\mu + g_{\lambda\nu} \Gamma^\lambda_{\sigma\mu} k^\sigma dx^\mu dx^\nu + g_{\mu\lambda} \Gamma^\lambda_{\sigma\nu} k^\sigma dx^\mu dx^\nu$$

$$+ \cdots = g_{\mu\nu}\, dx^\mu dx^\nu = ds^2.$$

Therefore, if k represents a symmetry transformation, we must enforce that

$$g_{\mu\nu}\, dx^\alpha dx^\beta \left(\partial_\alpha k^\mu \delta^\nu_\beta + \partial_\beta k^\nu \delta^\mu_\alpha + \Gamma^\mu_{\sigma\alpha} k^\sigma \delta^\nu_\beta + \Gamma^\nu_{\sigma\beta} k^\sigma \delta^\mu_\alpha \right) = 0.$$

Here, we've just rearranged terms and relabeled indices. Further rearranging terms, we find:

$$g_{\mu\nu}\, dx^\alpha dx^\beta \left[\delta^\nu_\beta \left(\partial_\alpha k^\mu + \Gamma^\mu_{\sigma\alpha} k^\sigma \right) + \delta^\mu_\alpha \left(\partial_\beta k^\nu + \Gamma^\nu_{\sigma\beta} k^\sigma \right) \right] = 0$$

$$= g_{\mu\nu}\, dx^\alpha dx^\beta \left[\delta^\nu_\beta \nabla_\alpha k^\mu + \delta^\mu_\alpha \nabla_\beta k^\nu \right] = 0$$

Let's ~~ka~~ now lower the index on the vector k:

$$\nabla_\alpha k^\mu = \nabla_\alpha g^{\mu\sigma} k_\sigma = g^{\mu\sigma} \nabla_\alpha k_\sigma, \text{ by metric}$$

compatibility. Making this replacement, we have

$$g_{\mu\nu}\, dx^\alpha dx^\beta \left[\delta^\nu_\beta g^{\mu\sigma} \nabla_\alpha k_\sigma + \delta^\mu_\alpha g^{\nu\sigma} \nabla_\beta k_\sigma \right]$$

$$= dx^\alpha dx^\beta \left[\nabla_\alpha k_\beta + \nabla_\beta k_\alpha \right] = 0.$$

Finally, we find that if a transformation induced by a vector k_μ is a symmetry, then, necessarily:

$$\boxed{\nabla_{(\mu} k_{\nu)} = 0 = \nabla_\mu k_\nu + \nabla_\nu k_\mu}$$

This is called <u>Killing's equation</u> and a vector K_μ that satisfies it is called a <u>killing vector</u>.

How many possible linearly independent Killing vectors/symmetries are there in D <u>dimensional</u> space? The indices μ and ν can each take D values. If μ and ν are the same, there are D ~~too~~ such possibilities. If μ and ν are different, μ can take D values while ν can only take D-1 values. The Killing equation is symmetric in $\mu \Leftrightarrow \nu$, so we need to divide by 2. Finally, the killing equation actually represents

$$D + \frac{D(D-1)}{2} = \frac{D(D+1)}{2} \quad \text{independent equations.}$$

Thus, there are at most $\frac{D(D+1)}{2}$ solutions/killing vectors/symmetries, as promised earlier.

Let's work through the Killing equation for the sphere. Again, its metric is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}_{\mu\nu}$$

You will show in problem 3 that its only non-zero connection terms are:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \,, \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

The sphere is two dimensional, so there are $\frac{2\cdot 3}{2} = 3$ Killing equations. They are:

$$\nabla_\phi K_\phi = 0 = \partial_\phi K_\phi - \Gamma^{\alpha}_{\phi\phi} K_\alpha = \underline{\partial_\phi K_\phi + \sin\theta\cos\theta\, K_\theta}$$

$$\nabla_\theta K_\theta = 0 = \partial_\theta K_\theta - \Gamma^{\alpha}_{\theta\theta} K_\alpha = \underline{\partial_\theta K_\theta}$$

$$\nabla_\theta K_\phi + \nabla_\phi K_\theta = \partial_\theta K_\phi + \partial_\phi K_\theta - \Gamma^\alpha_{\theta\phi} K_\alpha - \Gamma^\alpha_{\phi\theta} K_\alpha$$

$$= \partial_\theta K_\phi + \partial_\phi K_\theta - 2\cot\theta\, K_\phi = 0$$

Let's see if we can solve these equations for a Killing vector K. Let's assume that we want a Killing vector with components:

$$K_\mu = (0, f(\theta))_\mu, \quad \text{for } \mu = 1 = \theta,\ \mu = 2 = \phi,$$

for some function of ~~the~~ θ, f. The first two equations are trivially satisfied because $K_\theta = 0$ and $\partial_\phi K_\phi = 0$. The third equation becomes:

$$\partial_\theta K_\phi + \partial_\phi K_\theta - 2\cot\theta\, K_\phi = 0 = \frac{\partial}{\partial\theta} f(\theta) - 2\cot\theta\, f(\theta).$$

That is, $\frac{\partial f}{\partial \theta} = 2\cot\theta\, f$ or $\frac{\partial f}{f} = 2\cot\theta\, d\theta$.

Integrating both sides, we have

$$\ln f = A + 2\ln\sin\theta \quad \text{or that} \quad f(\theta) = \frac{A}{\sin^2\theta}$$

for some constant A. That is, one Killing vector is

$K_\mu = \left(0, \frac{1}{\sin^2\theta}\right)_\mu$, which just represents rotation about the vertical axis of the sphere.