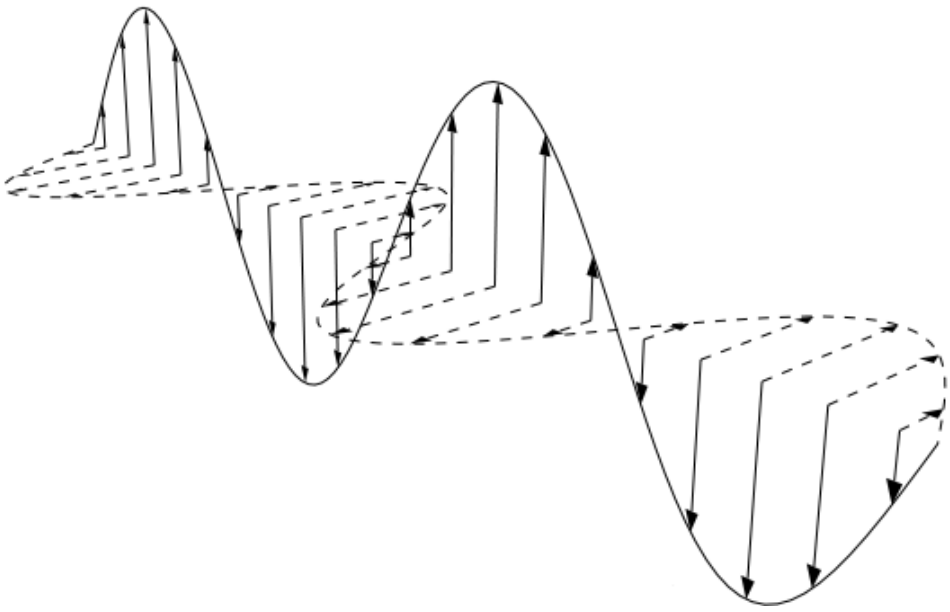
NOTES ON

PHYSICS



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A Reminder

Sometimes I hate physics for being so complicated.
I get frustrated and feel like I'll never understand.
I fail tests and break equipment,
disappointing myself and those who have taught me.
But sometimes, when the pieces snap together,
the beauty of physics appears like a mirage above the pages.
My shaky mastery of the equations feels like a great power.
And I am profoundly grateful.

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1 Mathematical Methods

1.1 Matrices

1.1.1 Properties of Matrices and Matrix Operations

- **Trace** The trace of an $n \times n$ matrix A is defined as $tr(A) = \sum_{i=0}^{n} A_i$. Alternately, this can be written in bra-ket notation as $tr(A) = \sum_k \langle k|A|k\rangle$.

  The following are some useful properties of the trace:

  - Trace is linear: $tr(A + B) = tr(A) + tr(B)$.
  - $tr(cA) = ctr(A)$.
  - $tr(A^T) = tr(A)$.
  - $tr(AB) = tr(BA)$.
  - The trace is invariant under cyclic permutations of its elements: $tr(ABC) = tr(BCA) = tr(CAB)$.

- **Conjugate Transpose (or Adjoint Matrix, Hermitian Conjugate, or Hermitian Transpose):** The complex-conjugate transpose of a matrix is given by:

$$A^{\dagger}_{ij} = A^{*}_{ji} \tag{1.1.1}$$

  The following are some properties of the conjugate transpose, for matrices A, B of the same dimensions, and a complex constant r.

  - $(A + B)^{\dagger} = A^{\dagger} + B^{\dagger}$ (Distributive)
  - $(rA)^{\dagger} = r^{*}A^{\dagger}$
  - $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$
  - $(A^{\dagger})^{\dagger} = A$
  - The eigenvalues of $A^{\dagger}$ are the complex conjugates of the eigenvalues of A.

- **Normal Matrix** A matrix is A normal if

$$A^{\dagger}A = AA^{\dagger} \tag{1.1.2}$$

  Where $A^{\dagger}$ is the conjugate-transpose of A.

- **Singular Matrix** A matrix A is said to be singular if it does not have a matrix inverse. This is the case iff the

- **Hermitian or Self-Adjoint:** A matrix A is Hermitian if it is equal to its own complex-conjugate transpose, denoted $A^{\dagger}$. Note that, in order to be Hermitian, a matrix must be square.

  The following properties hold for hermitian matrices:

  - All eigenvalues of a Hermitian matrix are real.

- All Hermitian matrices are normal
- The determinant of a Hermitian matrix is always real.

The following properties describe the Hermiticity of combinations of matrices.

- If C is a square matrix, then $C + C^\dagger$ is Hermitian.
- If C is a square matrix, then $C - C^\dagger$ is Skew Hermitian (see below).

- **Skew Hermitian or Antihermitian:** A matrix is skew hermitian if

$$A^\dagger = -A \tag{1.1.3}$$

- **Diagonalizable:** A matrix A is diagonalizable if it can be transformed into a basis in which it is a diagonal matrix. In other words, there must exist some matrix Q such that $Q^{-1}AQ$ is diagonal.

  A matrix will be diagonal when written in a basis consisting of its eigenvectors.

- **Unitary:** A matrix U is unitary if:

$$U^\dagger U = UU^\dagger = I \tag{1.1.4}$$

  Where I is the identity matrix.

  The following properties hold for any unitary matrix U:

  - **Unitary transformations preserve inner products!**. That is, for some vectors $|A\rangle$ and $|B\rangle$, $\langle UA|UB\rangle = \langle UAU^\dagger|B\rangle = \langle A|B\rangle$.
  - U is normal
  - U is diagonalizable.
  - The eigenvectors of U are orthogonal.

1.1.2 Determinants

The following are some important facts about determinants, where A and B are $n \times n$ matrices and a constant c:

- $det(AB) = det(A)det(B)$
- $det(A^{-1}) = det(A)^{-1}$
- $det(A^T) = det(A)$
- $det(cA) = c^n(A)$

1.1.3 The Eigenvalue Problem

Let A be a $n \times n$ matrix, and $\boldsymbol{x} \in R\_n$. $\boldsymbol{x}$ is an eigenvector if the following equation holds:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \tag{1.1.5}$$

Where the λ's are called the eigenvalues of A. Rearranging this equation, we obtain the condition that $A\boldsymbol{x} - \lambda\boldsymbol{x} = (A - \lambda I)\boldsymbol{x} = 0$.

We are uninterested in the trivial solution to this equation, where $\boldsymbol{x} = 0$. However, if $(A - \lambda I)^{-1}$ exists, then we can easily deduce that $\boldsymbol{x} = 0$ by applying the inverse to both sides. Therefore, for non-trivial solutions, $(A - \lambda I)$ must be singular (non-invertable). This is only the case if:

$$det(A - \lambda I) = 0 \tag{1.1.6}$$

Evaluating this determinant yields an equation in λ whose roots are the eigenvalues of the matrix A.

Once we have found the set of λ_i's, the next task is to identify the eigenvectors $\boldsymbol{x}_i$ that go with each λ_i. These can be found by the following system of n equations in $x_1, x_2, ...x_n$:

$$(A - \lambda I)\boldsymbol{x}_i = 0 \tag{1.1.7}$$

Solving this system of equations fully determines $\boldsymbol{x}_i$.

1.1.4 Change of Coordinate Matrices

Suppose we have a vector $\boldsymbol{x} \in R_n$ expressed in a basis of n vectors, $\boldsymbol{\alpha}_i$. We would like to change coordinates to another basis, $\boldsymbol{\beta}_i$. To do this, we must assemble a change of coordinates matrix.

We first express our current basis vectors, $\boldsymbol{\alpha}_i$ in terms of our new basis vectors, $\boldsymbol{\beta}_i$:

$$\boldsymbol{\alpha}_i = \sum_j c_{ij}\boldsymbol{\beta}_j \tag{1.1.8}$$

The c_{ij}'s now make up the change of coordinates matrix, where each "i" represents a column. We will write this matrix as $Q = c_{ij}$.

The vector $\boldsymbol{x}$ can now be transformed:

$$\boldsymbol{x}_\beta = Q\boldsymbol{x}_\alpha \tag{1.1.9}$$

The same change-of-coordinates matrix can be used to convert a matrix T (which could represent an operator) from the α to the β basis:

$$T_\beta = Q^{-1}T_\alpha Q \tag{1.1.10}$$

The change-of-coordinate matrix Q must be a unitary matrix: that is, we require that $QQ^\dagger = Q^\dagger Q = I$. Unitary transformations preserve the norm of a vector or matrix. This is of course necessary for a well defined coordinate transformation, which should not change the *length* of the vectors you are transforming!

1.1.5 Rotation Matrices

A particularly common type of coordinate change (and unitary transformation) is a rotation about the origin. The following matrices can be applied as change-of-coordinate matrices (to either a vector or, on either side, to a matrix) to effect a rotation.

In 2D:

$$R_{origin} = \begin{pmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{pmatrix} \tag{1.1.11}$$

This matrix rotates a clockwise rotation of the coordinate system by an angle θ.

In 3D, three rotation matrices are necessary:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{1.1.12}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{1.1.13}$$

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1.14}$$

Notice that each of the 3D rotation matrices 'contains' the 2D rotation matrix. The rows of the matrix that contain these elements are the ones that are rotating. The Line with the 1 is the axis about which the rotation is being performed. Noting these similarities will make the matrices easier to memorize.

1.1.6 Matrix Exponentiation

Occasionally, we will find ourselves with a matrix A in an equation such as e^A. We will interpret this "matrix exponential" by using the Taylor series of e^x:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{1.1.15}$$

1.2 Geometry

1.2.1 The Solid Angle

The solid angle describes the amount of surface area of a sphere spanned by some θ and ϕ, independent of the radius of the sphere. This often useful when describing a situation from the point of view of an observer at the center of the sphere. For example, if you hold up your thumb to block the Sun in the sky, your thumb and the Sun occupy the *same* solid angle.

The solid angle is defined as a differential element to be:

$$\boxed{d\Omega = \sin(\theta) d\theta d\phi} \tag{1.2.1}$$

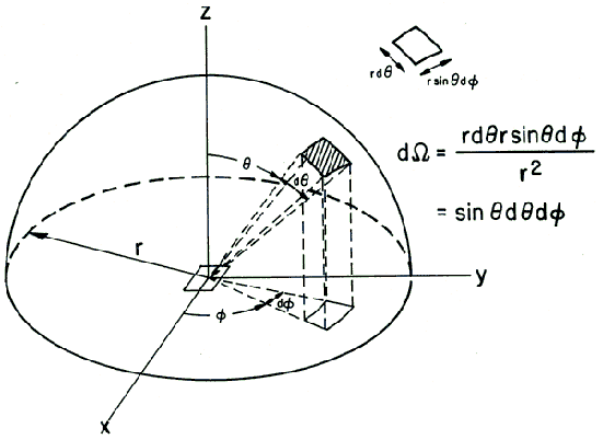


Figure 1.2.1: Solid Angle Diagram(Source:
`http://www.thermopedia.com/content/4761/2.gif`

The solid angle is often used as a nice way to simplify the spherical Jacobian term when integrating in spherical coordinates:

$$dA = r^2 \sin(\theta) d\theta d\phi = r^2 d\Omega \tag{1.2.2}$$

This solid angle is responsible for the ubiquitous factors of 4π in spherical integrals, since:

$$\int_{surface} d\Omega = 4\pi \tag{1.2.3}$$

1.3 Coordinate Systems

1.3.1 Spherical Coordinates

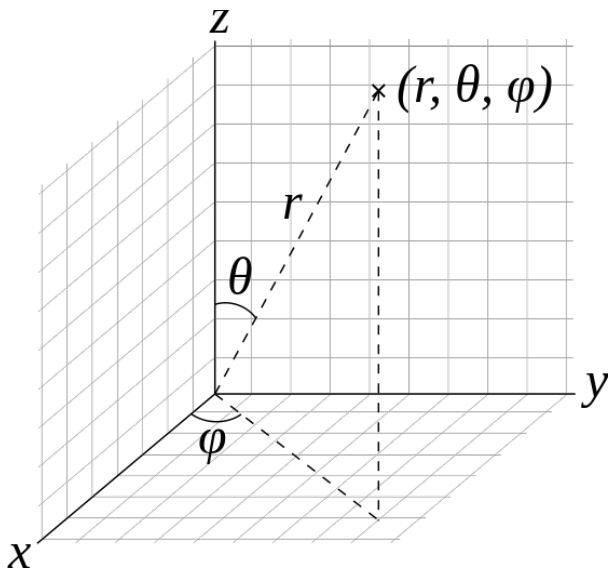


Figure 1.3.1: Spherical Coordinates. Source: Wikipedia.

Spherical coordinates are at the center of one of the most annoying convention disputes in science. In physics, the angle ϕ is chosen as the azimuthal angle and θ as the polar angle (see

fig. 1.3.1), while in mathematics the two are often reversed! There is a strong common-sense argument to be made for the mathematicians system, since polar coordinates is a projection of spherical coordinates onto a plane, and is usually denoted in r and θ. However, for the sake of consistency, I have tried to stick to the physicists convention in these notes.

Some clever geometry reveals that vectors can be transformed from Cartesian to spherical coordinates as:

$$\boxed{\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}} \tag{1.3.1}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y}\sin\theta\hat{z} \tag{1.3.2}$$

$$\hat{\phi} = -\sin\theta\hat{x} + \cos\theta\hat{y} \tag{1.3.3}$$

Really only the *hatr* expression is really necessary to memorize. If necessary, the expression for $\hat{p}hi$ is easy to derive from a diagram.

It is also often useful to know the time derivatives of some of these unit vectors:

$$\boxed{\frac{d}{dt}\hat{r} = \dot{\theta}\hat{t}heta} \tag{1.3.4}$$

$$\boxed{\frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{r}} \tag{1.3.5}$$

1.4 Calculus

1.4.1 Trig Substitution

Trig substitution makes use of trigonometric identities to simplify integrals. Starting with the identity:

$$\sin^2(x) + \cos^2(x) = 1 \tag{1.4.1}$$

If we divide both sides by $\cos^2(x)$, we get:

$$\tan^2(x) + 1 = \sec^2(x) \tag{1.4.2}$$

Whiled dividing by $\sin^2(x)$ leaves:

$$1 + \cot^2(x) = \csc^2(x) \tag{1.4.3}$$

Now, consider an integral of the form:

$$I = \int \frac{xdx}{\sqrt{a^2 \pm x^2}} \tag{1.4.4}$$

Where a is a real constant. We could simplify the integral by dividing out an a:

$$I = \frac{1}{a}\int \frac{xdx}{\sqrt{1 \pm \left(\frac{x^2}{a^2}\right)^2}} \tag{1.4.5}$$

For the moment, lets take $\pm \to +$. **We can make the argument of the square root resemble one of the trig identities above if we make the substitution**:

$$\frac{x}{a} = \tan(u) \text{ or } \frac{x}{a} = \cot(u) \tag{1.4.6}$$

In general, the first of these is going to be nicer, since the derivative of $\tan(u)$ makes it easy to write that:

$$dx = a\sec^2(u)du \tag{1.4.7}$$

Making the substitution and then applying the trig identity:

$$I = \frac{1}{a}\int \frac{a\tan(u)(a\sec^2(u)du)}{\sqrt{1+tan^2(u)}} = a\int \frac{\tan(u)\sec^2(u)du}{\sqrt{\sec^2(u)}} = a\int \tan(u)\sec(u)du = a\int \frac{\sin(u)}{\cos^2(u)}du \tag{1.4.8}$$

The resulting integral is now easy to solve with another substitution: $z = \cos(u)$.

If instead in the original integral we had $\pm \to -$, we could simply subtract 1 from both sides of the trig identities to get a new set of three easily applicable identities.

Notice that, in order for the this trick to work, we need both a term (usually $\sqrt{a^2+x^2}$) to morph into the trig identity, as well as a free x on top or bottom to help make the resulting integral tractable.

1.5 Fourier Series and Transforms

1.5.1 Fourier Series

Fourier series are a way of writing a periodic function f as a sum of (possibly infinite) Sine and cosine functions of different frequencies, weighted by appropriate constants. f is usually written:

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{1.5.1}$$

Generally a function we wish to represent with a Fourier series exists over a finite interval, but is not necessarily periodic. In order to find the Fourier series in this case, we *create* a periodic function by copying the function end-to-end, off to infinity.

The problem of *finding* the Fourier series is now reduced to finding the appropriate constants A_n and B_n.This is accomplished by way of "Fourier's Trick".

Fourier's Trick relies on the orthogonality of Sine and cosine functions over a some domain $[a, b]$ at different frequencies. For example:

$$\int_a^b \sin(nx)\sin(mx)dx = \delta_{mn} \tag{1.5.2}$$

Similarly, Sine and cosine's of all frequencies are orthogonal[1.1].

We can use this fact to find expressions for the constants in the Fourier series. For example, we will derive the expression for B_n.

Starting with the general Fourier series written above, multiply both sides of the equation by $\sin\left(\frac{m\pi x}{L}\right)$, and then integrate from $-L$ to L (where 2L is the period of $f(x)$):

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_{-L}^{L}\sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)+\sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \tag{1.5.3}$$

By the orthogonality of Sine and cosine, all but one of the terms on the right hand side is now zero!

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_{-L}^{L}\delta_{mn}B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \tag{1.5.4}$$

Collapsing the delta to replace m's with n's, and performing the integral, we get that

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_n L \tag{1.5.5}$$

And therefore our expression for B_n is:

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \tag{1.5.6}$$

The same trick can be used to derive an expression for A_n:

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx \tag{1.5.7}$$

With the special case of A_0 being easily written as

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{1.5.8}$$

1.5.2 Fourier Transform

A Fourier transform take a periodic function and returns a continuous function of the frequency components. The inverse Fourier transform returns from this frequency domain to completely reconstruct the original function in its original domain.

The Fourier transform is given by:

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixk}dx \tag{1.5.9}$$

while the inverse Fourier transform is given by:

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{2\pi ixk}dk \tag{1.5.10}$$

[1.1] This can be easily seen: Sine is odd, cosine is even, and the domain required to be symmetric.

These formulas are essentially a continuous version of the Fourier Series coefficient formulas derived earlier. The separate sine and cosine components are now being handled simultaneously by the complex exponential.

The following are some important properties of the Fourier Transform:

- The transform is linear: if $h(x) = af(x) + bg(x)$, then $H(k) = aF(k) + bG(k)$

There are a number of common Fourier transform relationships that are useful to have available:

- Translation in x means a phase factor in k: $f(x)$: $f(x - x_0) \leftrightarrow e^{-2\pi i x_0 k} F(k)$
- Translation in k means a phase factor in x: $F(k - k_0) \leftrightarrow e^{2\pi i k_0 x} f(x)$
- Constant multiples of x: $f(ax) \leftrightarrow \frac{1}{|a|} F\left(\frac{k}{a}\right)$
- Derivatives in x bring down multiples of k in k-space: $\frac{d^n f(x)}{dx^n} \leftrightarrow (2\pi i k)^n F(k)$
- Likewise, multiples of x in x-space correspond to derivatives in k-space:$x^n f(x) \leftrightarrow \left(\frac{i}{2\pi}\right)^n \frac{d^n F(k)}{dk^n}$

1.6 The Legendre Transform

The Legendre transform is a common variable transformation that is useful when it is more convenient to describe a function $F(x)$ in terms of the derivative $s = \frac{dF(x)}{dx}$.

The Legendre transformation can only be applied when the function $F(x)$ is convex, that is $\frac{d^2F}{dx^2} > 0$, and smooth. This is required so that there exists a $1 - 1$ relation between s and x. When this is the case, we can invert the usual relation $s(x)$ to obtain $x(s)$.

When these requirements are met, the Legendre transform is written [1.2]:

$$G(s) = sx - F(x(s)) \tag{1.6.1}$$

The following are some important properties of the Legendre transform:

- The Legendre transform is an involution, which means that it is its own inverse, such that $G(G(x)) = x$.

1.7 Vector Calculus

1.7.1 Line and Volume Elements

The line element is the differential distance in a coordinate system. Each line element in 3D space takes the form:

$$d\boldsymbol{x} = h_1 \hat{e}_1 + h_2 \hat{e}_2 + h_3 \hat{e}_3 \tag{1.7.1}$$

Where the h's are elements of the Jacobian called "scale factors" that are always the same for each coordinate system. In the three most commonly used coordinate systems:

- Cartesian: $d\boldsymbol{x} = \hat{x} + \hat{y} + \hat{z}$

[1.2] A derivation of this expression, as well as a geometric motivation, can be found in the paper **Making Sense of the Legendre Transform** by R.K.P Zia et al., Am. J. Phys. **77** (7) July 2009 pg. 614.

- Cylindrical: $d\boldsymbol{x} = \hat{\rho} + \rho\hat{\theta} + \hat{z}$
- Spherical: $d\boldsymbol{x} = \hat{r} + r\hat{\theta} + r\sin\theta\hat{\phi}$

The volume element is the differential volume in a coordinate system (i.e. what you would put at the end of an integral in order to integrate over a volume). In general, the differential volume (in a space with coordinates q_1, q_2, and q_3)is:

$$d^3V = (h_1h_2h_3)dq1dq_2dq_3 \tag{1.7.2}$$

So that:

- Cartesian: $d^3V = dxdydz$
- Cylindrical: $d^3V = \rho d\rho d\theta dz$
- Spherical: $d^3V = r^2\sin\theta dr d\theta d\phi$

1.7.2 Surface Normals

The surface normal is defined to be orthogonal to the surface, and has a magnitude of the differential area element. It is sometimes to imagine $d\boldsymbol{S} = \hat{n}dS$, where dS is the usual area element, while $\hat{n}$ is unit normal vector. Suppose you have some surface S in 3D, paramaterized by two variables to be $S(t,u)$. Then a surface normal can be constructed as $(\partial_t S \times \partial_u S)dudt$! Of course, whether this normal vector points up or down on the surface depends on the *order* of this cross product.

1.7.3 Vector Derivatives

(The best intuitive explanation of vector derivatives is given in the first chapter of Griffiths E&M book. Rather than attempt to top that, this section is focused on easy ways to remember the form of the operators in different coordinate systems).

Vector derivatives are constructed using the 'derivative vector', ∇ (formally called 'nabla', but often referred to as 'del'). In a coordinate system with coordinates q_1, q_2, and q_3:

$$\boxed{\nabla = \frac{1}{h_1}\frac{\partial}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial q_3}\hat{e}_3} \tag{1.7.3}$$

Applying ∇ to a scalar function f produces the **gradient**, which is a **vector**:

$$\boxed{\nabla = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3} \tag{1.7.4}$$

Dotting ∇ with a scalar function f produces the **divergence** which is a **scalar** (and is therefore often called the scalar product):

$$\boxed{\nabla \cdot f = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}(fh_2h_3) + \frac{\partial}{\partial q_2}(fh_1h_3) + \frac{\partial}{\partial q_3}(fh_1h_2)\right)} \tag{1.7.5}$$

Notice that the h's inside the derivative operator are all but the one cooresponding to the coordinate being differentiated against in that term.

Crossing ∇ with a vector function $\boldsymbol{A}$ produces the **curl** which is is a **vector** (and is therefore often called the vector product):

$$\boxed{\nabla \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}} \tag{1.7.6}$$

The only commonly used second-derivative operator is the **Laplacian**, ∇^2, which acts on a scalar function f and returns a **scalar**. The Laplacian is defined as $\nabla^2 f = \nabla \cdot \nabla f$, and can be calculated this way from the results above. The result is:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1} + \frac{\partial}{\partial q_2}\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2} + \frac{\partial}{\partial q_3}\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right) \tag{1.7.7}$$

Notice that all four of these vector derivatives reduce to a simple form in Cartesian coordiates. It is worth memorizing these Cartesian results, but then also memorizing these general equations (along with the appropriate line element) so that you can conjure up the appropriate derivatives in whatever coordinate system you happen to find yourself. There's no need to memorize the Laplacian: you rarely need it, and if you do, you can just take $\nabla \cdot \nabla f$ to find it.

Here are some useful vector identities concerning vector derivatives:

- Curl of a gradient is zero: $\nabla \times (\nabla \phi) = 0$
- Divergence of a curl is zero: $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$
- Divergence of a vector with a scalar coefficient: $\nabla \cdot (c\boldsymbol{A}) = c(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla c)$
- Curl of a vector with a scalar coefficient: $\nabla \times (c\boldsymbol{A}) = c(\nabla \times \boldsymbol{A}) + (\nabla c) \times \boldsymbol{A}$
- Curl of a curl: $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
- Divergence of a cross product: $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
- Curl of a cross product: $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{\,2}}, \qquad \nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{\,2}} \tag{1.7.8}$$

1.7.4 The Divergence Theorem

Consider a vector field $\boldsymbol{F}$ and and some surface, S, enclosing a volume V. The divergence theorem states that:

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{F} dV \tag{1.7.9}$$

Where $d\boldsymbol{S}$ is the (outward facing) surface normal along the surface.

This theorem makes sense in light of a couple examples. Consider first a vector field that is diverging from a source inside the surface. In this case, the divergence on the RHS would be positive, while the dot product on the LHS would also be positive. The same idea, but with opposite sign, holds if a sink is inside the surface.

Now, imagine a constant vector field. This field is divergence free, so the RHS is zero. On the LHS, the dot product is positive on half of the surface, but negative on the *other* half of the surface. These terms cancel, making the LHS also zero.

1.7.5 The Curl Theorem (Stokes Theorem)

Consider a vector field $\boldsymbol{F}$ and a curve C the encloses some surface S. The curl theorem states that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{C} = \int_S \nabla \times \boldsymbol{F} d\boldsymbol{S} \tag{1.7.10}$$

Where $d\boldsymbol{C}$ is the line element along the curve C, and $d\vec{S}$ is the area element of the surface, with direction defined normal to the surface S.

Of course, the direction around which the integral along C is carried out will affect the sign of the result, as well the choice of an inward or outward oriented $d\boldsymbol{S}$. The positive surface normal corresponds to integrating along C counter-clockwise.

A convenient visual explanation of the theorem can be seen by considering the dot product of various vector fields with the line elements of various curves. For a constant vector field, these dot products will cancel around the (closed) curve. However, for a vector field with curl, the sum of the dot products may be non-zero (as the field points different directions at different points around the loop).

1.7.6 The Helmholtz Theorem

The Helmholtz Theorem (or Helmholtz Decomposition)[1.3] states that any vector field $\boldsymbol{F}$ can be decomposed into a divergence-free and a curl-free component:

$$\boldsymbol{F} = -\nabla\phi + \nabla \times \boldsymbol{A} \tag{1.7.11}$$

Where ϕ is a scalar function called the **potential**, and $\boldsymbol{A}$ is a vector function called the **vector potential**. Notice that, since $\nabla \times \nabla\phi = 0$ by an identity, the first term is curl free, while since $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ by another identity, the second term is divergence free.

If **we further assume that** $\boldsymbol{F} \to 0$ **faster than** $\frac{1}{r}$ **as** $r \to \infty$, then the theorem states that:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi} \int_{allspace} \frac{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dV' \tag{1.7.12}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi} \int_{allspace} \frac{\nabla' \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dV' \tag{1.7.13}$$

[1.3] `http://en.wikipedia.org/wiki/Helmholtz_decomposition`

1.8 The Calculus of Variations

1.8.1 Functional Derivatives

A functional derivative is the derivative (still 'rate of change') of a functional as you vary one of its parameters. If:

$$J[f] = \int L[x, f(x), f'(x)]dx \tag{1.8.1}$$

Then, if we perturb the functional L by some variation, δf:

$$L[x, f(x), f'(x)] \to L[x, f(x) + \delta f, f'(x) + deltaf'] \tag{1.8.2}$$

Then, to first order in the variation. the functional derivative is:

$$\delta J = \int \frac{\delta J}{\delta f} \delta f dx \tag{1.8.3}$$

Where the 'differentiation' is carried out just as you would expect if it were written $\frac{\partial J}{\partial f}$.

1.9 Complex Analysis

Broadly speaking, complex analysis is the calculus of complex numbers. There are many useful results from this field that translate into physics, most of which relate to special integration tricks.

1.9.1 Some Useful Identities

A complex number C is defined as $C = a + ib$, where $Re(C) = a$ and $Im(C) = b$, while $C^\star = a - ib$. Directly from these definitions, we can derive:

$$C + C^\star = 2a \tag{1.9.1}$$

Which, along with $C - C^\star = 2ib$, yields the expressions:

$$a = \frac{C + C^\star}{2} \tag{1.9.2}$$

$$b = \frac{C - C^\star}{2i} \tag{1.9.3}$$

These expressions are very useful whenever you need to get rid of (or conjure out of nowhere) a $Re(C)$ or $Im(C)$.

1.9.2 Extension to the Complex Plane

In many cases, the functions that we want to apply the techniques of complex analysis to will actually be defined on the real numbers. In order to apply our techniques, we will need to **extend these functions to the complex plane**. This is done by simply making the substitution:

$$f(x) \longleftrightarrow f(z), \quad x \in \mathbb{R}, z \in \mathbb{C} \tag{1.9.4}$$

The previously 1D function now exists on the complex plane. Notice that when $\text{Im}(f(z)) = 0$, the function once again lies only on the real axis (the x-axis).

All of this is rigorously mathematically justifiable, but if actually want to see a proof of that, you should probably have been a mathematician.

1.9.3 Poles and Residue

A pole is a point where a complex function diverges to ∞ or $-\infty$. Loosely speaking, a pole behaves "like" the function $f(z) = \frac{1}{z^n}$. The integer n is called the **order** of the pole, and quantifies *how fast* the function diverges at that location.

Poles can be separated into two categories. A **simple pole** is of order $n = 1$. All other poles are simply referred to as **poles** or **higher-order poles**.

Contour integrals *around* a pole are proportional to a constant, known as the *residue* of the pole. This property is as fantastically useful as it is surprising! The residue of poles is written as $\text{Res}(f, c)$, where $f(z)$ is the function and c is the location of the pole on the complex plane. The value of the residue can be calculated using one of several formulas:

- **Simple poles ($n = 1$):**

$$\text{Res}(f, c) = \lim_{z \to c}(z - c)f(z) \tag{1.9.5}$$

  However, for simple poles it often turns out that we can express $f(z) = \frac{g(z)}{h(z)}$. When this is the case, we can use the much simpler formula:

$$\text{Res}(f, c) = \frac{g(c)}{h'(c)} \tag{1.9.6}$$

  Where the prime represents differentiation with respect to z.

  Higher Order Poles ($n > 1$):

  For higher order poles, you're stuck with this equation:

$$\text{Res}(f, c) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}}((z - c)^n f(z)) \tag{1.9.7}$$

  However, it's not *nearly* as bad as it looks! The factor of $(z - c)^n$ will generally cancel part of $f(z)$, making the differentiation easier rather than harder.

  This equation is technically just a more general version of the one for simple poles: notice that it reduces to the first formula when $n = 1$.

1.9.4 The Cauchy Residue Theorem

The Cauchy Residue Theorem states that the value of a contour integral around some number N of poles at locations a_N is simply:

$$\boxed{\oint_C f(z)dz = 2\pi i \sum_N \text{Res}(f, a_N)} \tag{1.9.8}$$

We normally apply this theorem in order to calculate the the integral from $-\infty$ to ∞ of some

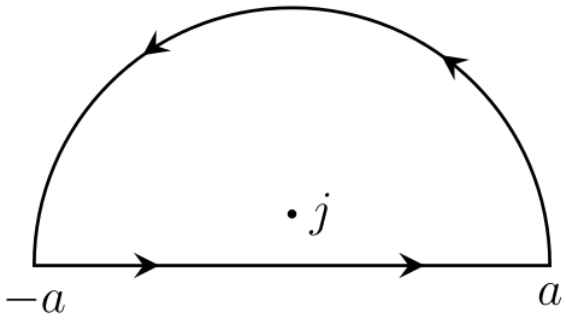


Figure 1.9.1: Contour for evaluating a real integral. Source: Wikipedia

real-valued function that we have analytically continued to the complex plane[1.4]. In this case, we want to pick a contour that *includes all of the x-axis, while not including any other substantial contributions.* **In the limit where $f(z)$ falls off faster than $\frac{1}{z}$ as $z \to \infty$**, it is legitimate to chose a contour like the one shown in figure 1.9.1. Since the entire arc exists very far from the origin, the contribution of the integral is negligible then, and:

$$\oint_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx \tag{1.9.9}$$

One potential sticky point arises when a pole exists *on* the real axis. In this case, we must

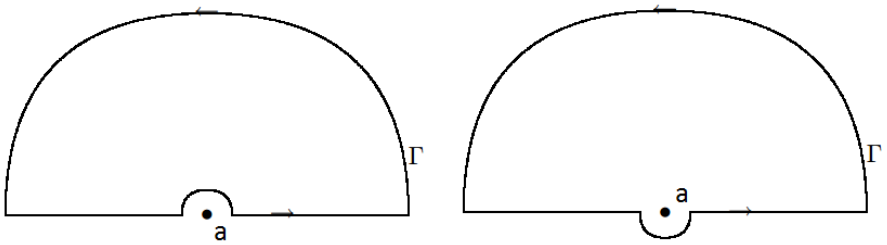


Figure 1.9.2: Two options for closing a contour around a pole on the real axis. Source: `http://www.nhn.ou.edu/~milton/p5013/chap7.pdf`.

define the **principal value** of an integral[1.5]. For a simple pole at a we define:

$$P\int_{-\infty}^{\infty} f(x)dx = \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{a-\epsilon} f(x)dx + \int_{a+\epsilon}^{\infty} f(x)dx \right) \tag{1.9.10}$$

With this definition in place, we will now add a "bump" to our contour that circumnavigates the troublesome pole (as shown in figure 1.9.2. Using this contour, it turns out that:

$$P\int_{-\infty}^{\infty} f(x)dx = \oint_C f(z)dz = 2\pi i \left(\sum_N \text{Res}(f, a_N) + \frac{1}{2}\text{Res}(f, a) \right) \tag{1.9.11}$$

Where the second pole a is the troublesome pole at the origin. In other words, **the net result is that a simple pole on the origin only contributes half of it's residue to the integral**.

[1.4]The Wikipedia page has an excellent worked out example: `http://en.wikipedia.org/wiki/Residue_theorem`.

[1.5]Here's a good resource on this: `http://www.nhn.ou.edu/~milton/p5013/chap7.pdf`.

A hand-waving explanation for this fact is that, as the residue "emerges" from the point a in all directions, only half of it is "caught" by the upper half of the plane.

1.10 Solving Differential Equations

1.10.1 Separation of Variables

Separation of variables allows us to solve a differential equation (usually $\nabla^2 F(x) = 0$) by assuming that the solution can be written as a product of functions that each only depend on one of the variables:

$$\nabla^2 F(x,y,z) = \nabla^2 X(x)Y(y)Z(z) \tag{1.10.1}$$

This assumption is NOT generally true. However, it turns out that the solution space is spanned by these solutions, so we can write any other solution as a linear combination of these "separable" solutions.

When applied in 3D with Cartesian coordinates, separation of variables goes as follows:

$$\nabla^2 F(x,y,z) = 0 \to yz\frac{d^2}{dx^2}X(x) + xz\frac{d^2}{dy^2}Y(y) + xy\frac{d^2}{dz^2}Z(z) = 0 \tag{1.10.2}$$

Dividing both sides by xyz:

$$\frac{1}{x}\frac{d^2}{dx^2}X(x) + \frac{1}{y}\frac{d^2}{dy^2}Y(y) + \frac{1}{z}\frac{d^2}{dz^2}Z(z) = 0 \tag{1.10.3}$$

Now for the crucial argument. Since each of these terms depends only on one variable, but they sum to zero, *each term itself must be equal to a constant.* Otherwise, changing x while leaving the other terms fixed would violate the equation. Typically, once a problem is solved, it is realized that some strange definition of these constants (such as $l(l+1)$) is convenient. However, in the simplest case:

$$\frac{1}{x}\frac{d^2}{dx^2}X(x) = \alpha, \quad \frac{1}{y}\frac{d^2}{dy^2}Y(y) = \beta, \quad \frac{1}{z}\frac{d^2}{dz^2}Z(z) = \gamma \tag{1.10.4}$$

Notice that there is now a relationship between these constants, namely:

$$\alpha + \beta + \gamma = 0 \tag{1.10.5}$$

1.10.2 Roots of the Characteristic Polynomial

This method of solving differential equations works whenever the coefficients of the differential equation are constants. It relies on two facts about second order homogeneous differential equations:

1. The linear combination of two solutions to a differential equation is itself a solution
2. The general solution of a second order homogeneous differential equation is a linear combination of any two linearly independent solutions.

The characteristic polynomial can be found by substituting r^n for each $\frac{d^n y}{dr^2}$:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \implies ar^2 + br + c = 0 \tag{1.10.6}$$

This procedure is motivated by considering a trial solution of the form $y = e^{rx}$, where $y'' = r^2 y$, $y' = ry$, etc. The resulting equation is known as the **characteristic equation** or the **auxiliary equation**: $ar^2 + br + c = 0$.

The importance of this equation is that its roots can be used to find solutions to the differential equation and, therefore, produce a general solution when put together as a linear combination.

The roots of the equation as written above are clearly:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{1.10.7}$$

The particular solutions to this equation always take the form e^{rx}. However the form of the general solution depends on the sign of the discriminant, $b^2 - 4ac$, which determines which roots of the equation are real and/equal.

- If $b^2 - 4ac > 0$, then the solutions will be real and unequal, each producing an exponential as a solution, yielding the general solution:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{1.10.8}$$

- If $b^2 - 4ac = 0$, then the two roots will be real and equal: $r_1 = r_2 = r$. One solution remains the exponential e^{rx}, but the other must be found separately. It happens that xe^{rx} is also a solution. Thus, the general solution is:

$$y = c_1 e^{rx} + c_2 x e^{rx} \tag{1.10.9}$$

- If $b^2 - 4ac < 0$, then the roots will be unequal and complex. If we denote the solutions as $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, the solutions are again written:

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \tag{1.10.10}$$

  Which can be rearranged to the form:

$$y = e^{\alpha x}\Big(c_1 e^{(i\beta)x} + c_2 e^{(-i\beta)x}\Big) = e^{\alpha x}\Big(c_1 \cos(\beta x) + c_2 \sin(\beta x)\Big) \tag{1.10.11}$$

Useful Source: Stewart Calculus `http://www.stewartcalculus.com/data/CALCULUS%20Concepts%20and%20Contexts/upfiles/3c3-2ndOrderLinearEqns_Stu.pdf`

1.11 Einstein Summation Notation

1.11.1 The Summation Notation

The Einstein summation notation convention is that indices that appear multiple times in an expression should be interpreted as summations. These double indicies are also known as "dummy" indices, because they disappear if the implied sum is actually written out. For example:

$$a^i b_i = \sum_i a_i b_i \tag{1.11.1}$$

Indices that appear only once are "free", i.e. they are just variables that indicate that the equation written is just one of a set of similar equations. For example:

$$V_j = a^i b_i Z_j \implies (V)_j = (Z)_j \sum_i a_i b_i \tag{1.11.2}$$

This expression describes a set of j (the free variable) equations, each of which includes the sum $a_i b_i$. Free and dummy indices are analogous to free and bound variables. The free indices are placeholders for some value, while the bound indices already have a fixed value.

The position of the incides is used to denote column and row vectors (which are often multiplied together to form a sum). They should not be misread as exponents . For example:

$$a^i b_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ ... \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 & ... \end{bmatrix} \tag{1.11.3}$$

Here are some tips on using the notation:

- While doing index manipulations, return as much as possible to normal vector notation in between steps. For example, if you complete a sub-part of a problem, rewrite the indices in vector notation as much as possible. Removing dummy indices in this way makes it easier to see connections between different equations.
- Wikipedia suggests a helpful mnemonic for remembering which index position corresponds to which vector: "**Up**per indices go **up** to down; **l**ower indices go **l**eft to right."

1.11.2 The Dot Product

The dot product between two vectors can be easily expressed in Einstein notation:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^i b_i \tag{1.11.4}$$

1.11.3 The Levi-Civita Symbol

The Levi-Civita symbol is a constant whose value depends on the permutation of its indices. The symbol (in three dimensions) is written ϵ_{ijk}, and its value is defined by the following conditions:

- $\epsilon_{ijk} = 1$

- $\epsilon_{ikj} = -1$ (When the indices have been permuted)
- $\epsilon_{iij} = \epsilon_{ijj} = 0$ (Whenever two indices are equal)

Since it is only the permutation and not the precise order of the indices of the constant that matters, $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$.

1.11.4 Cross Product

This definition of the Levi-Civita constant allows the cross product to be easily expressed:

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a^j b^k \tag{1.11.5}$$

Where i is a free variable, and the sum is conducted over the repeated dummy variables j and k. The resulting vector is therefore:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \epsilon_{1jk} a^j b^k \\ \epsilon_{2jk} a^j b^k \\ \epsilon_{3jk} a^j b^k \end{bmatrix} \tag{1.11.6}$$

A useful simplification of the product of two Levi-Civita symbols can be made when they share an index:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.11.7}$$

Where the Kronecker delta of two indices is defined by:

- $\delta_{ij} = 1$ if $i = j$
- $\delta_{ij} = 1$ if $i \neq j$

So that:

$$\delta_{ij} a_i = a_i = a_j \tag{1.11.8}$$

When condensing an expression using this identity, it is useful to think about changing all the indices to a "preferred set", i.e. exchanging m for j and n for k whenever possible.

1.11.5 Example: Proving Vector Identities with Einstein Notation

Einstein notation and the Levi-Civita symbol make proving some vector identities easier. Consider the expression:$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$.

$$\begin{aligned} \big((\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}\big)_i &= \epsilon_{imn} (\boldsymbol{a} \times \boldsymbol{b})_m c_n \\ &= \epsilon_{imn} (\epsilon_{mjk} a_j b_k) c_n \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_j b_k c_n \\ &= a_j c_j b_k - b_k c_k a_j \end{aligned}$$

But this is equal to:

$$(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}=a_jc_jb_k-b_kc_ka_j \tag{1.11.9}$$

So we see that $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$. QED.

1.12 Special Functions

1.12.1 Legendre Polynomials

The Legendre Polynomials usually arise in problems in spherical coordinates with azimuthal (ϕ) symmetry. Formally, they are solutions to a differential equation known as "Legendre's Equation". They are usually found by means of the Rodrigues formula:

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{1.12.1}$$

However, they also emerge as coefficients in a Taylor series expansion:

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \tag{1.12.2}$$

This context is most useful when expanding the multipole expansion, which can be rewritten in this form:

$$\frac{1}{\mathscr{r}}=\frac{1}{\sqrt{r^2-2\vec{r}\cdot\boldsymbol{r'}+r'^2}}=\frac{1}{r}\frac{1}{\sqrt{1-2\hat{r}\cdot\hat{r}'\frac{r'}{r}+\frac{r'^2}{r^2}}}=\frac{1}{r}\sum_{n=0}^{\infty}P_n(x)r^n \tag{1.12.3}$$

The first few Legendre polynomials (and the only ones you really need to memorize) are:

- $P_0(x)=1$
- $P_1(x)=x$
- $P_2(x)=\frac{1}{2}(3x^2-1)$

As a set of special functions, the Legendre Polynomials have a variety of useful properties including an orthogonality relationship:

$$\int_{-1}^{1}P_n(x)P_m(x)=\frac{2}{2n+1}\delta nm \tag{1.12.4}$$

The Legendre Polynomials are alternatingly symmetric and antisymmetric:

$$P_n(-x)=(-1)^nP_n(x) \tag{1.12.5}$$

1.12.2 Associated Legendre Polynomials

Although the normal Legendre Polynomials only emerge as solutions when a problem has azimuthal symmetry, in the general case (and for the construction of the Spherical Harmonics), we want a form of the Legendre Polynomials that allows both l and m to vary. These functions are the "associated Legendre Polynomials".

The associated Legendre Polynomials are solutions to a modified version of Rodrigues's formula:

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{1.12.6}$$

These functions very rarely make an appearance by themselves, but are important as part of the definition of the Spherical Harmonics.

1.12.3 Spherical Harmonics (The Y_l^m's)

The Spherical Harmonics, or (or "YLMs") are the angular part of the general solution to Laplace's equation ($\nabla^2 \boldsymbol{f} = 0$) in spherical coordinates. As such, they often appear in E&M (as potentials) or in QM as solutions to the Schrodinger Equation.

The Spherical Harmonics are normally written in terms of the associated Legendre Polynomials [1.6]:

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos(\theta))e^{im\phi} \tag{1.12.7}$$

Notice that, in the azimuthally symmetric case where $m = 0$:

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos(\theta)) \tag{1.12.8}$$

The first few spherical harmonics are:

- **l= 0** $Y_0^0 = \frac{1}{\sqrt{4\pi}}$
- **l= 1**
  - $Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin(\theta)e^{-i\phi}$
  - $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos(\theta)$
  - $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi}$
- **l= 2**
  - $Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)e^{2i\phi}$
  - $Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin(\theta)\cos(\theta)e^{i\phi}$
  - $Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right)$

Different Spherical Harmonics are orthogonal:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) Y_{l'}^{*m'}(\theta',\phi') Y_l^m(\theta,\phi) = \delta_{l,l'}\delta_{m,m'} \tag{1.12.9}$$

[1.6] See Jackson pg. 107.

And also complete:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{l'}^{*m'}(\theta',\phi')Y_l^m(\theta,\phi) = \delta(\phi-\phi')\delta(\cos(\theta)-\cos(\theta')) \tag{1.12.10}$$

This means that any function may be expanded in terms of the Spherical Harmonics:

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m Y_l^m(\theta, phi) \tag{1.12.11}$$

Where the constant coefficients can be found to be:

$$A_l^m = \int Y_l^{*m}(\theta,\phi) f(\theta,\phi)\sin(\theta)d\theta d\phi \tag{1.12.12}$$

One extremely important result with Spherical Harmonics is the Addition theorem [1.7], which allows a Legendre Polynomial in terms of $\cos(\gamma)$ [1.8], where γ is the angle between the observer vector $\boldsymbol{r}$ and the source vector $\boldsymbol{r'}$, in terms of Spherical Harmonics in the angles of those two vectors:

$$P_l(\cos(\gamma)) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^{*m}(\theta',\phi')Y_l^m(\theta,\phi) \tag{1.12.13}$$

1.12.4 Bessel Functions and Hankel Functions

Bessel functions are solutions of the following differential equation (which often arises when solving Laplace's equation through separation of variables in spherical or cylindrical coordinates):

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-\alpha^2)y = 0 \tag{1.12.14}$$

When α is an integer, the solutions $y(x)$ to this equation are Bessel Functions, or Cylindrical Harmonics. When α is a half integer, the solutions are the Spherical Bessel Functions. Other values of α are possible, but do not generally appear in solutions to physical problems.

Since the equation above is a second-order differential equation, the most general solutions are linear combinations of two linearly independent solutions. However, just like exponential, sin,cos,sinh, and cosh can all be used to form solutions to Laplaces's equation in Cartesian coordinates, there are many different pairs of Bessel functions that can be used to most conveniently write the solutions.

There are a variety of ways of calculating the form of these function, none of which are generally used very often. Therefore, we will simply take the shape of the regular Bessel Functions to be given (just like Sine) and then describe the relationship of the other functions to these.

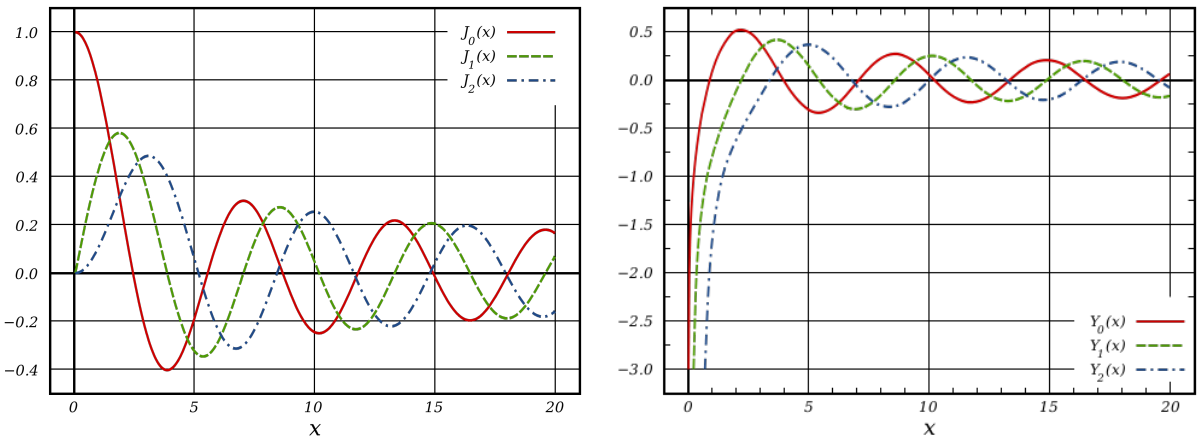
The first (and most important) solution is:

$$y(x) = aJ_\alpha(x) + bN_\alpha(x) \tag{1.12.15}$$

Where a and b are constants. These "ordinary" bessel functions are the standard solutions to

[1.7] See Jackson pg. 110 (although even HE doesn't seem to prove this).

[1.8] $\cos(\gamma)$ can also be written as $\cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi-\phi')$.



(a) Bessel Functions of the First Kind

(b) Bessel Functions of the Second Kind

Figure 1.12.1: Source: Wikipedia

Laplace's equation in cylindrical coordinates. The J_α's are **Bessel Functions of the First Kind**, pictured in Fig. 1.12.1a. It is worth noting that $J_0(0) = 1$. The functions oscillate back and forth with an irregular period. The zeros of the Bessel Functions (where it crosses the x-axis) can be calculated, and are sometimes useful.

The N_α's are called **Neumann Functions** or **Bessel Functions of the Second Kind**, and are another set of solutions to the equation that are linearly independent to the J_α's.

With these two functions, we can define the **Hankel Functions**:

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iN_\alpha(x) \tag{1.12.16}$$

$$H_\alpha^{(2)}(x) = J_\alpha(x) - iN_\alpha(x) \tag{1.12.17}$$

These rarely come up, but are apparently of some theoretical value.

We can also define the **Modified Bessel Functions** $I_\alpha(x)$ and $K_\alpha(x)$ here, which are another pair of solutions (that can be written in terms of the ordinary Bessel Functions) that are valid for imaginary arguments. Again, we can assemble a solution using this pair of functions:

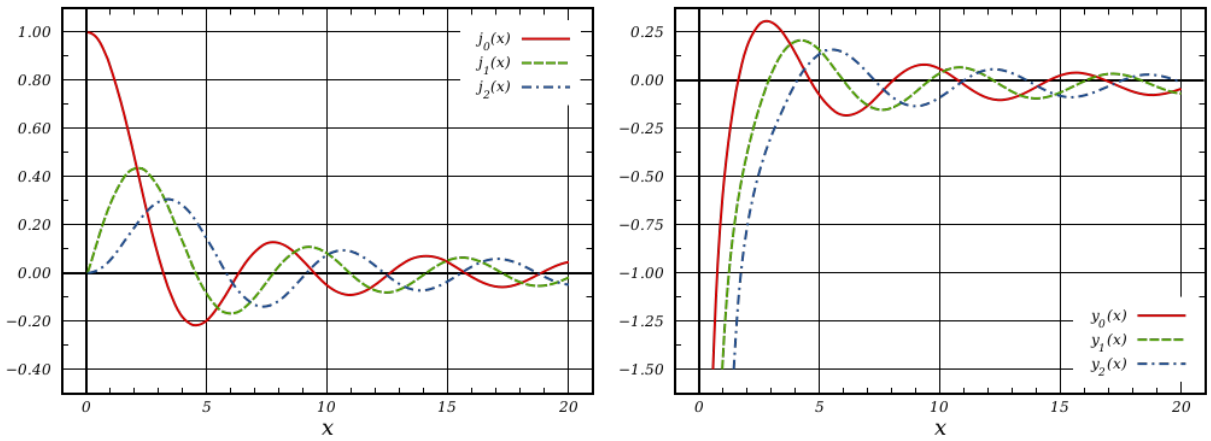
$$y(x) = aI_\alpha(x) + bK_\alpha(x) \tag{1.12.18}$$

Another form of Bessel functions arises when considering the Helmholtz equation ($\nabla^2 Y + k^2 Y = 0$) in spherical coordinates. These are known as **Spherical Bessel Functions**, and are pictured in Figure 1.12.2. The Spherical Bessel Functions can be written in terms of the ordinary Bessel Functions[1.9].

In general, the spherical $Y)\alpha$'s will be eliminated from physical solutions because they are not finite at the origin. It is, however, useful to remember the first few Spherical Bessel Functions of the first kind:

- $J_0(x) = \frac{\sin(x)}{x}$

[1.9] See Wikipedia for details.



(a) Spherical Bessel Functions of the First Kind (b) Spherical Bessel Functions of the Second Kind

Figure 1.12.2: Source: Wikipedia

- $J_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$
- $J_2(x) = \left(\frac{3}{x^2} - 1\right)\frac{\sin(x)}{x} - \frac{3\cos(x)}{x^2}$

The spherical bessel functions reduce asymptoticlly to the following functions for $x << 1$:

$$j_l \to \frac{2^l l!}{(2l+1)!}x^l, \qquad x << 1 \tag{1.12.19}$$

$$n_l \to -\frac{(2l)!}{2^l l!}\frac{1}{x^{l+1}}, \qquad x << 1 \tag{1.12.20}$$

You can also define **Spherical Hankel Functions** using the Spherical Bessel Functions in the exact same way that the ordinary Hankel Functions are defined in terms of the ordinary Bessel Functions:

$$h_l^{(1)}(x) = j_l(x) + in_l(x) \tag{1.12.21}$$

$$h_l^{(2)}(x) = j_l(x) - in_l(x) \tag{1.12.22}$$

These functions are solutions to the radial spherical equation[1.10]:

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u = -k^2u \tag{1.12.23}$$

Asymptotically, (for large x), these functions go as $h_l^{(1)} \approx \frac{(-i)^{l+1}}{x}e^{ix}$ and $h_l^{(2)} \approx \frac{(i)^{l+1}}{x}e^{-ix}$.

[1.10]This equation often arises when the raidal coordinate has been transformed by u(r) = r R(r).

1.12.5 Airy Functions

Airy functions[1.11] are solutions to the following differential equation (which often appears when considering linear potentials):

$$\frac{d^2f(x)}{dx^2} = xf(x) \tag{1.12.24}$$

There are two linearly independent solutions to this equation, which are called Ai(x) and Bi(x). The most general solution can then be written:

$$f(x) = aAi(x) + bBi(x) \tag{1.12.25}$$

for some constants a and b. These solutions can be represented as integrals:

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{s^3}{3} + sx\right)ds \tag{1.12.26}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[e^{\frac{s^3}{3}+sx} + \sin\left(\frac{s^3}{3} + sx\right)\right]ds \tag{1.12.27}$$

More commonly, however, you will encounter approximations of the Airy functions far away from the origin:

- $x >> 0$:

$$Ai(x) \approx \frac{1}{2\sqrt{\pi}x^{\frac{1}{4}}}e^{-\frac{2}{3}x^{\frac{3}{2}}}, \quad x >> 0 \tag{1.12.28}$$

$$Bi(x) \approx \frac{1}{\sqrt{\pi}x^{\frac{1}{4}}}e^{\frac{2}{3}x^{\frac{3}{2}}}, \quad x >> 0 \tag{1.12.29}$$

- $x << 0$

$$Ai(x) \approx \frac{1}{\sqrt{\pi}(-x)^{\frac{1}{4}}}\sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right), \quad x << 0 \tag{1.12.30}$$

$$Bi(x) \approx \frac{1}{\sqrt{\pi}(-x)^{\frac{1}{4}}}\cos\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right), \quad x << 0 \tag{1.12.31}$$

1.13 Probability and Combinations

1.13.1 Combinations

The combination $\binom{n}{k}$ is the number of ways to chose k objects out of n objects, disregarding order. This can be calculated as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.13.1}$$

[1.11] See Griffiths QM pg. 327

1.14 Statistics

1.14.1 The Autocorrelation function

Suppose you have a data set that consists of N pieces of data, each of which is a time series $x(t)$ of some process taking place: $\{x_1(t), x_2(t), ...x_N(t)\}$. You suspect that the same process is taking place in each data set, but that the process takes place at a different point (i.e. different time) in each data set. The autocorrelation (or "serial correlated errors). of the data set allows this to be determined.

The auto correlation function is defined to be:

$$R_{xx}(t_1, t_1 + \tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t_1) x_k(t_1 + \tau) \tag{1.14.1}$$

1.15 Miscellaneous

1.15.1 The Feynman Integration Trick

The Feynman integration trick is a way of taking one integral you *know* and using it to generate a whole family of other integrals merely by differentiating. The general idea will be to differentiate both sides of the known integral (possibly with respect to a constant!) to generate the desired expression. For example, consider the integral:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \tag{1.15.1}$$

If we differentiate *both sides* of this expression with respect to x, we generate:

$$-2a \int_{-\infty}^{\infty} x e^{-ax^2} dx = 0 \tag{1.15.2}$$

An even more useful trick allows us bring down even powers of x. Pretend for a moment that $f(x) = e^{-ax^2}$ is really $f(x, a)$. Then we can take the derivative of both sides with respect to a:

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}} \tag{1.15.3}$$

This new expression must of course still hold if we now again set $a = \text{const}$. Repeating the process can generate any integral of the form $\int_{-\infty}^{\infty} x^{2N} e^{-ax^2} dx$!

1.15.2 Raleigh's Formula

Raleigh's Formula: expression of a plane wave in spherical coordinates:

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l + 1) j_l(kr) P_l(\cos\theta) \tag{1.15.4}$$

Where $j_l(kr)$ is a spherical bessel function of the first kind, and P_l is a Legendre polynomial.

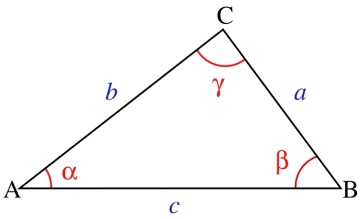


Figure 1.15.1: Law of Cosines (Source: Wikipedia)

1.15.3 The Law of Cosines

The Law of Cosines is a more general version of the Pythagorean theorem, and states (for the triangle illustrated above) that:

$$c^2 = a^2 + b^2 - 2ac\cos\gamma \tag{1.15.5}$$

The same relationship holds permuting a, b, c, and the associated angles.

1.15.4 Taylor Series

The Taylor series allows us to approximate a function near a point by a series of terms, each of which depends on a derivative of the function. In principle, well behaved functions can be perfectly represented in this way by a Taylor series with infinite terms. However, we usually truncate the series after a few terms; making an approximation that makes the problem tractable.

The Taylor series for a function $f(x)$ about a point a is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...\frac{f^n(a)}{n!}(x-a)^n \tag{1.15.6}$$

A Taylor series can similarly be written to approximate a vector function $\boldsymbol{A}(\boldsymbol{r})$:

$$\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{\epsilon}) = \boldsymbol{A}(\boldsymbol{r}) + (\boldsymbol{\epsilon}\cdot\nabla)\boldsymbol{A}(\boldsymbol{r}) + \frac{1}{2}(\boldsymbol{\epsilon}\cdot\nabla)^2\boldsymbol{A}(\boldsymbol{r}) + ... \tag{1.15.7}$$

Almost always, this series is truncated to the first two terms.

1.15.5 The Binomial Theorem

The binomial theorem is used for expanding repeated products of a binomial, i.e. for calculating $(x+y)^n$. The theorem states that:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \tag{1.15.8}$$

For example, this theorem generates the basic result that:

$$(x+y)^2 = x^2 + 2xy + y^2 \tag{1.15.9}$$

Since $\binom{2}{0} = \binom{2}{2} = 1$ and $\binom{2}{1} = 2$.

If n is a positive integer, then the sum above will converge, yielding the correct expanded polynomial. However, if n is negative, a fraction, or both, the sum will be infinite!

In this latter case we can compute the coefficients of each term manually:

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n \tag{1.15.10}$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2!} \tag{1.15.11}$$

and etc.[1.12]

In particular it is useful to remember that, for small x:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \tag{1.15.12}$$

And that in therefore in the special case where $n = \frac{1}{2}$:

$$(1+x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + ... \tag{1.15.13}$$

1.15.6 The Schwartz Inequality

Begin by noting that for some complex number λ

$$(\langle\alpha| + \lambda^\star\langle\beta|)(|\alpha\rangle + \lambda|\beta\rangle) \geq 0 \tag{1.15.14}$$

This must be true because the LHS is just an inner product of a vector with its complex conjugate, which is positive definite.

Now, make the divinely inspired choice of $\lambda = \frac{-\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$. We can now write the product above as:

$$(\langle\alpha| + \frac{-\langle\alpha|\beta\rangle\langle\beta|}{\langle\beta|\beta\rangle})(|\alpha\rangle + \frac{-\langle\beta|\alpha\rangle|\beta\rangle}{\langle\beta|\beta\rangle}) \geq 0 \tag{1.15.15}$$

Now, expanding this expression and rearranging the resulting terms yields the Schwartz inequality:

$$\boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2} \tag{1.15.16}$$

1.16 Useful Facts

You'll be saving yourself a lot of pain if you memorize these facts early and use them often!

1.16.1 Integrals

$$\int_0^\infty x^n e^{-\frac{x}{a}} dx = n! a^{n+1} \tag{1.16.1}$$

$$\int_0^\infty x^{2n} e^{-\frac{x^2}{a^2}} dx = \sqrt{\pi}\frac{(2n)!}{n!}a^{2n+1} \tag{1.16.2}$$

[1.12] Useful link: https://www.physicsforums.com/threads/binomial-theorem-for-fractional-exponents.240990/

$$\int_0^\infty x^{2n+1} e^{-\frac{x^2}{a^2}} dx = \frac{n!}{2} a^{2(n+1)} \tag{1.16.3}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \tag{1.16.4}$$

1.16.2 Series

Note: All of these identities are series, but on some I have placed the sum on the left of the equals sign, while for others they are on the right. This merely reflects which way you are more likely to use the identity.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 1 + x + \frac{x^2}{2!} + ... \tag{1.16.5}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \tag{1.16.6}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... \tag{1.16.7}$$

Notice from here that $e^{ix} = \cos(x) + i \sin(x)$.

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + ... \qquad \text{for } x << 1 \tag{1.16.8}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ for } |x| < 1 \text{ (The Geometric Series)} \tag{1.16.9}$$

$$\sum_{n=0}^{\infty} e^{-an} = \frac{1}{1-e^{-a}} \tag{1.16.10}$$

1.16.3 Vectors

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 \tag{1.16.11}$$

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \text{ (any permutation)} \tag{1.16.12}$$

1.16.4 Equations

$$A + iB \to e^{i\phi} \text{ with } \phi = \tan^{-1}\left(\frac{B}{A}\right) \tag{1.16.13}$$

The following procedure (completing the square) is useful when you want to rearrange a polynomial to *look* like a perfect square, shifted by some constant.

$$ax^2 + b^2 + c = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a} \text{ (completing the square)} \tag{1.16.14}$$

1.16.5 Theorems/Results

The Baker-Campbell-Hausdorff formula:

$$e^{AB} = e^A e^B e^{-\frac{[A,B]}{2}} \tag{1.16.15}$$

$$\nabla \frac{1}{\boldsymbol{r}} = -\frac{-\boldsymbol{r}}{\boldsymbol{r}^3} \tag{1.16.16}$$

$$\nabla^2 \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{1.16.17}$$

Proof:

Taking the gradient of $\frac{1}{\boldsymbol{r}}$ gives $-\frac{-\boldsymbol{r}}{r^3}$. Now, taking the divergence (spherical coordinates make this easy) shows that this is always $= 0$. However, applying Gauss's law over a sphere and the result above: $\int_V \nabla \cdot (\nabla \boldsymbol{r}) dV = int_S \frac{-\boldsymbol{r}}{r^3} \cdot dS = \int_S \frac{1}{r^2} dS = \int_S \frac{1}{r^2} r^2 \sin(\theta) dr d\theta d\phi = 4\pi$. Since we have shown that $\nabla^2 \frac{\hat{r}}{r} = 0$, and also that this integral around the origin is NON-zero, it must be that $\nabla^2 \frac{\hat{r}}{r} = 4\pi\delta(\boldsymbol{r})$.

When you have a delta function in the z direction, you can rewrite it as:

$$\delta(z) = \frac{\delta(\cos(\theta))}{r} \tag{1.16.18}$$

2 Classical Mechanics

2.1 Poisson Brackets

The Poisson bracket of two quantities, A and B, with respect to two variables q and p, is defined to be:

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{2.1.1}$$

2.1.1 Identities and Useful Relations

- Clearly

$$\{A, B\} = -\{B, A\} \tag{2.1.2}$$

- Poisson brackets distribute over addition. This follows directly from the fact that derivatives distribute over addition.

$$\{A, B + C\} = \{A, B\} + \{A, C\} \tag{2.1.3}$$

- Poisson brackets obey the product and quotient rules, which also follow directly from the

corresponding properties of derivatives.

$$\{A, BC\} = \{A, B\}\, C + \{A, C\}\, B \tag{2.1.4}$$

$$\left\{A, \frac{B}{C}\right\} = \frac{C\,\{A, B\} - B\,\{A, C\}}{C^2} \tag{2.1.5}$$

Notice that this relation is identical to the corresponding relationship for commutators, except that, since all terms in a Poisson bracket commute, it does not matter which side of the brackets B and C are pulled out to.

- The Poisson bracket $\{A, q_i\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial q_i}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial q_i}{\partial q_i} = -\frac{\partial A}{\partial p_i}$, since $\frac{\partial q_i}{\partial p_i} = 0$ and $\frac{\partial q_i}{\partial q_i} = 1$. Therefore we have:

$$\{A, q_i\} = -\frac{\partial A}{\partial p_i} \tag{2.1.6}$$

And similarly:

$$\{B, p_i\} = \frac{\partial B}{\partial q_i} \tag{2.1.7}$$

2.1.2 Poisson Bracket form of Hamilton's Equations of Motion

The total time derivative of a function $A(q, p, t)$ is often of interest in the Hamiltonian formalism of classical mechanics. This time derivative can be written:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i\right) \tag{2.1.8}$$

Hamilton's equations of motion tell us that $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$. Therefore we can rewrite the above expression as:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{2.1.9}$$

Or, more compactly:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\} \tag{2.1.10}$$

2.2 Derivation of Virial Theorem

The goal of the viral theorem is to find a convenient expression for the time average value of kinetic energy, $\langle T \rangle$. Consider the following expression for a particle moving (for simplicity) in the x direction:

$$\sum_i \boldsymbol{p_i} \cdot \boldsymbol{r_i} = \sum_i m_i \dot{\boldsymbol{x}}_{\boldsymbol{i}} \cdot \boldsymbol{x_i} \tag{2.2.1}$$

Now consider the time derivative of this expression:

$$\frac{d}{dt}\Big(\sum_i m_i \dot{\boldsymbol{x}}_{\boldsymbol{i}} \cdot \boldsymbol{x}_{\boldsymbol{i}}\Big) = \sum_i m_i \ddot{\boldsymbol{x}}_{\boldsymbol{i}} \cdot \boldsymbol{x}_{\boldsymbol{i}} + \sum_i m_i \dot{\boldsymbol{x}}_{\boldsymbol{i}}{}^2$$

Notice that $\sum_i m_i \dot{\boldsymbol{x}}_{\boldsymbol{i}}{}^2 = 2T$ and that $\sum_i m_i \ddot{\boldsymbol{x}}_{\boldsymbol{i}} \cdot \boldsymbol{x}_{\boldsymbol{i}} = \sum_i \boldsymbol{F} \cdot \boldsymbol{x}_{\boldsymbol{i}} = \sum_i \boldsymbol{\nabla}_{\boldsymbol{i}} V \cdot \boldsymbol{x}_{\boldsymbol{i}}$. Now, making these substitutions, consider the time average of the equation:

$$\Big\langle \frac{d}{dt}\Big(\sum_i m_i \dot{\boldsymbol{x}}_{\boldsymbol{i}} \cdot \boldsymbol{x}_{\boldsymbol{i}}\Big)\Big\rangle = \langle \sum_i \boldsymbol{\nabla}_{\boldsymbol{i}} V \cdot \boldsymbol{x}_{\boldsymbol{i}} \rangle + 2\langle T \rangle \tag{2.2.2}$$

If we suppose that that path of the particle being considered is either cyclic or bound, than the time average of the time derivative (the LHS) will be zero. This is true for any bounded function. The time average of an arbitrary function is defined as:

$$\langle f(t) \rangle = \frac{1}{T}\int_0^T f(t)dt \tag{2.2.3}$$

If $f(t)$ is a total time derivative of some function F, so that $f(t) = \frac{dF}{dt}$, and we consider the limit as $t \to \infty$, then we have:

$$\langle \frac{dF}{dt} \rangle = \lim_{t\to\infty} \frac{1}{T}\int_0^T \frac{dF}{dt}dt = \lim_{t\to\infty} \frac{1}{T}(f(T) - f(0)) \tag{2.2.4}$$

If f(t) is bound, then $lim_{t\to\infty} f(t)$ is finite while t goes to ∞, and therefore $\langle f(t) \rangle = 0$. Applying this reasoning to our derivation, the LHS vanishes, and the remaining terms can be rearranged to produce the statement of the virial theorem:

$$2\langle T \rangle = -\langle \sum_i \boldsymbol{\nabla}_{\boldsymbol{i}} V \cdot \boldsymbol{x}_{\boldsymbol{i}} \rangle \tag{2.2.5}$$

2.3 Derivation of the Euler-Lagrange Equation (1D)

The Euler-Lagrange equation is one of those derivations in which we assume a function exists with some properties, and then use those properties to create an equation that, in the end, solve for the function we supposed at the beginning.

Consider the following integral, called the action:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \tag{2.3.1}$$

The fundamental physics behind the Lagrangian formulation of classical mechanics is that motion occurs so as to minimize the action. Therefore, let us suppose that there exists a function $f(t)$ such that

$$S = \int_{t_1}^{t_2} L(f(t), \dot{f}(t), t)dt \tag{2.3.2}$$

is minimized. Since $f(x)$ is a minimum, any perturbation to $f(x)$ will necessarily increase the action integral above. Let us then introduce such a perturbation:

$$g(t) = f(t) + \epsilon\eta(t) \tag{2.3.3}$$

where ϵ is a scaling constant. We will require that the perturbation goes to zero at both endpoints, that is to say $\eta(t_1) = \eta(t_2) = 0$.

We can now insert this new function into the action integral to obtain

$$S_\epsilon = \int_{t_1}^{t_2} L(g(t), \dot{g}(t), t)dt \tag{2.3.4}$$

Now we will maximize S_ϵ with respect to ϵ by setting

$$\frac{dS_\epsilon}{d\epsilon} = 0 \tag{2.3.5}$$

We can easily calculate this derivative by bringing the ϵ derivative inside the action integral to act directly upon the integrand:

$$\frac{dL}{d\epsilon} = \frac{dg}{d\epsilon}\frac{dL}{dg} + \frac{d\dot{g}}{d\epsilon}\frac{dL}{d\dot{g}} + \frac{dt}{d\epsilon}\frac{dL}{dt} \tag{2.3.6}$$

The last term of this expression vanishes, because t is not dependent on ϵ, and therefore $\frac{dt}{d\epsilon}$ is zero. Two of the other derivatives can be explicitly calculated from the definition of g:

$$\frac{dg}{d\epsilon} = \eta(t) \tag{2.3.7}$$

$$\frac{d\dot{g}}{d\epsilon} = \dot{\eta}(t) \tag{2.3.8}$$

Leaving us with the final expression:

$$\frac{dL}{d\epsilon} = \eta(t)\frac{dL}{dg} + \dot{\eta}(t)\frac{dL}{d\dot{g}} \tag{2.3.9}$$

We know from calculus that the integral of this expression ($\frac{dS_\epsilon}{d\epsilon}$) is equal to zero when $\epsilon = 0$, since ϵ was a perturbation from the minimum function $f(t)$. Evaluating $\epsilon = 0$ changes our variables back, so that $g \to f$. Plugging the expression back into the integral:

$$\int_{t_1}^{t_2} \left[\eta(t)\frac{dL}{df} + \dot{\eta}(t)\frac{dL}{d\dot{f}}\right]dt \tag{2.3.10}$$

This integral can be further simplified by an integration by parts on the second term, where $u = \frac{dL}{d\dot{f}}$ and $v = \eta(t)$. This gives us:

$$\int_{t_1}^{t_2} \left[\frac{dL}{df} - \frac{d}{dt}\frac{dL}{d\dot{f}}\right]dt\eta(t)dt + \left[\eta(t)\frac{dL}{d\dot{f}}\right]\Bigg|_{t_1}^{t_2} \tag{2.3.11}$$

The last term goes to zero based on the boundary conditions we required for η at the beginning: $\eta(t_1) = \eta(t_2) = 0$.

The remaining integral is:

$$\int_{t_1}^{t_2} \left[\frac{dL}{df} - \frac{d}{dt}\frac{dL}{d\dot{f}}\right]dt\eta(t)dt = 0 \tag{2.3.12}$$

The last step is made by the Fundamental Lemma of the Calculus of Variations, which states that, for this case, the integrand must be equal to zero. This leaves us with the completed

Euler-Lagrange equation:

$$\frac{dL}{df} - \frac{d}{dt}\frac{dL}{d\dot{f}} = 0 \tag{2.3.13}$$

2.4 Constraints

The beauty of the Lagrangian formulation of classical mechanics lies in how easily it handles constraining forces. There are two types of constraints, and two methods of representing them.

2.4.1 Types of Constraints

- **Holonomic constraints**, depend only on the q_i's and t, but not on the derivatives, $\dot{q}_i$. Holonomic constraints also only involve equalities, not inequalities. Therefore, a general holonomic constraint can be written:

$$\phi(q_i, t) = 0 \tag{2.4.1}$$

- **Non-holonomic constraints** depend on the derivatives of coordinates, and may involve inequalities. Our general methods will not work with non-holonomic constraints.

  One exception to this statement must be more for so-called integrable constraints. These are non-holonomic constraints for which

$$\phi(q_i, \dot{q}_i, t) = \frac{d}{dt}\phi(q_i, \dot{q}_i, t) = 0 \tag{2.4.2}$$

  Such that $\phi(q_i, \dot{q}_i, t)$ is constant. In this case, the non-holonomic constraint can be easily reduced to a holonomic constraint, and the usual methods applied.

In any case, the "standard form" of a constraint is taken to be $\phi = 0$. Constraints should be rearranged into this form before being used in any of the equations described in this section.

2.4.2 Lagrange Multipliers

Holonomic constraints can be included in Lagrangian by introducing an extra degree of freedom, λ, for each constraint. The action then takes the form

$$S = \int_t \left(L(q_i, \dot{q}_i, t) + \sum_\alpha \lambda_\alpha \phi_\alpha(q_i, t) \right) dt \tag{2.4.3}$$

Where there are α constraints. The corresponding Euler-Lagrange equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial q_i} \tag{2.4.4}$$

The resulting system of α equations will fully determine the values of the λ_α constants. Once these have been found, the force exerted *by* the constraint(s) in the i direction can be found to

be:

$$F_i = \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial q_i} \tag{2.4.5}$$

The great advantage of the Lagrange multiplier method is that it allows these forces to be determined, while the generalized coordinate method does not.

2.4.3 Generalized Coordinates

The goal of the generalized coordinates method is to choose a coordinate system which naturally includes the constraints. For example, if a ball is constrained to roll along the inside of a spherical shell, good generalized coordinates would be θ and ϕ. In this example, the fact that r is implicitly fixed embodies the constraint.

Generalized coordinates can also be thought of as simply reducing the number of degrees of freedom in the Lagrangian using the constraint relationships. Each of the constraints can be solved for a variable in the Lagrangian, then substituted in. The result is a simpler Lagrangian (fewer degrees of freedom) that now contains the constraint information implicitly.

2.5 Conserved Quantities and Noether's Theorem

Conserved quantities (or "integrals of motion" or "constants of motion") are quantities that do not change during motion, i.e. quantities with a zero time derivative. In other words, a quantity Q is conserved if and only if $\frac{dQ}{dt} = 0$.

2.5.1 Equivalent Lagrangians

During the derivation of the Lagrangian, it becomes clear [2.1] that two Lagrangians may differ by a total time derivative but still yield the same equations of motion (the total time derivative disappearing during the variation of the action). Two Lagrangians L and L' are thus equivalent if:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}\Lambda(q, t) \tag{2.5.1}$$

2.5.2 Noether's Theorem

Noether's theorem [2.2] states that a conserved quantity exists for every symmetry of the system.

A "symmetry of the system" exists whenever a variable in the Lagrangian can be perturbed by a constant to yield a different but *equivalent* Lagrangian. In this case, the two Lagrangians must differ by a total time derivative $\frac{d\Lambda}{dt}$ (see above section).

Suppose the coordinate q_i has been perturbed, so that $q_i'(t, \epsilon) = q_i(t) + \epsilon\delta q_i + O(\epsilon^2)$. Λ can then be calculated via the equation:

$$\frac{\partial}{\partial \epsilon} L \bigg|_{\epsilon=0} = \frac{\partial}{\partial t}\Lambda \tag{2.5.2}$$

[2.1] See Landau and Lifshitz pg. 4

[2.2] Proven by Emmy Noether, who is for some reason often passed over in classical mechanics textbooks.

We can write the perturbation δq_i as:

$$\delta q_i = \frac{\partial q_i'(t,\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} \tag{2.5.3}$$

There are two commonly occurring special cases that are worth mentioning.

- If the perturbation in question is simply a first order translation such as $q_i' = q_i + \epsilon$, then we see that $\delta q_i = 1$.
- If the perturbation is in time itself, i.e. $q_i'(t,\epsilon) = q_i(t+\epsilon)$, then we see that $\delta q_i = \dot{q}_i$. Note that $\frac{\partial}{\partial \epsilon} q_i'(t,\epsilon) = \frac{\partial}{\partial \epsilon} q_i(t+\epsilon) = \frac{\partial}{\partial t} q_i(t) = \dot{q}_i$, which allows us to easily integrate the equation above for Λ, finding that $\Lambda = L$.

Given some perturbation, Noether's theorem then states that the conserved quantity Q is then given by

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - \Lambda \tag{2.5.4}$$

Q is sometimes referred to as the Nother Charge.

2.5.3 Conserved Quantities

Translations in space and time and rotations can be shown (via Noether's theorem) to generate the common constants of motion.

- Translations in time yield conservation of energy. In the case where L is not explicitly time dependent, $\frac{\partial L}{\partial t} = 0$, we see that $\Lambda = L$ [2.3], and therefore:

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - L \tag{2.5.5}$$

  Where H is now the Hamiltonian (or total energy).
- Linear momentum is generated by translations.
- Angular momentum is generated by rotations.

2.6 Derivation of Hamilton's Equations of Motion

We start by writing the differential of the Lagrangian L:

$$dL = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) + \frac{\partial L}{\partial t} dt \tag{2.6.1}$$

Our goal is to eliminate $d\dot{q}$ in favor of dp. By definition, momentum is $p_i = \frac{\partial L}{\partial \dot{q}_i}$

So we can rewrite the above equation as

$$dL = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + p_i d\dot{q}_i\right) + \frac{\partial L}{\partial t} dt \tag{2.6.2}$$

[2.3] See remarks above about the perturbation for this case.

Now since $d(p_i\dot{q}_i) = dp_i\dot{q}_i + p_i d\dot{q}_i$, we can eliminate the $p_i d\dot{q}_i$ term in the above equation. We also note that, from the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \implies \dot{p}_i = \frac{\partial L}{\partial q_i} \tag{2.6.3}$$

Making these two substitutions and rearranging, our expression for dL is now:

$$dL - d(p_i\dot{q}_i) = \sum_i (\dot{p}_i dq_i - \dot{q}_i dp_i) + \frac{\partial L}{\partial t}dt \tag{2.6.4}$$

But, since the Hamiltonian is defined as:

$$H = \sum_i p_i\dot{q}_i - L \tag{2.6.5}$$

We see that the right hand side of our equation is just $-dH$!

$$dH = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t}dt \tag{2.6.6}$$

More directly, we also see that

$$dH = \sum_i (\frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial t}dt \tag{2.6.7}$$

These two equations look very similar: in fact, they have matching terms, each multiplied by a matching differential element! Equating the coefficients of these differentials gives us Hamilton's equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2.6.8}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.6.9}$$

And, less helpfully,

$$\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t} \tag{2.6.10}$$

2.7 Canonical Coordinates and the Hamilton-Jacobi Equation

2.7.1 Canonical Coordinates and Transformations

The Euler-Lagrange equations are invariant with respect to coordinate changes in position of the form $Q_i = Q_i(q,t)$. One of the major advantages of the Hamiltonian formulation is that Hamilton's equations are *also* invariant with respect to coordinate changes in momentum.

Let us consider two very general transformations, which we will allow to be time dependent: $Q_i = Q_i(p,q,t)$ and $P_i = P_i(p,q,t)$. While all transformations of this form are *allowed*, only

certain transformations will maintain the form of Hamilton's equations, that is:

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \dot{P}_i = -\frac{\partial H'}{\partial Q_i} \tag{2.7.1}$$

Where H' is the Hamiltonian in the new coordinates: $H(Q_i, P_i)$. Coordinates that obey these relations are called **canonical coordinates**, and the transformations that produce them **canonical transformations**.

Poisson brackets are also invariant under canonical transformations, i.e:

$$\{A(Q,P), B(Q,P)\}_{Q,P} = \{A(q,p), B(q,p)\}_{q,p} \tag{2.7.2}$$

It can be proved [2.4] that transformations Q_i and P_i are canonical iff:

- $\{Q_i, Q_j\}_{p,q} = 0$
- $\{P_i, P_j\}_{p,q} = 0$
- $\{Q_j, P_i\}_{p,q} = \delta_{ij}$

2.7.2 The Method of Generating Functions

The method of generating functions allows us to easily find canonical coordinate transformations.

Suppose we wish to transform from one set of given coordinates, q and p, to new (as-yet unknown) set of coordinates Q and P. For this transformation to be canonical, the form of Hamilton's equations must be preserved. Fundamentally, this is the same as requiring that $S(q,p) = S(Q,P)$:

$$\int (p\dot{q} - H(p,q))dt = \int (P\dot{Q} - K(P,Q))dt \tag{2.7.3}$$

Where we have rewritten L in the action using the definition of the Hamiltonian. The "new" Hamiltonian, $K(P,Q)$, is (half jokingly) referred to as the Kamiltonian.

In order for two actions to be equivalent, they must be related by a total time derivative [2.5]. Therefore, if we introduce a new function F:

$$p\dot{q} - H(p,q) = P\dot{Q} - K(P,Q) + \frac{d}{dt}F \tag{2.7.4}$$

Now we can chose among several possible F's in terms of combinations of our new and old coordinates, namely $F_1(q,Q)$, $F_2(q,P)$, $F_3(p,Q)$, $F_4(p,P)$. For now, we will proceed with $F = F_1(q,Q)$. The above equation then reduces to:

$$p\dot{q} - H(p,q) = P\dot{Q} - K(P,Q) + \frac{\partial F_1(q,Q)}{\partial t} + \frac{\partial F_1(q,Q)}{\partial q}\dot{q} + \frac{\partial F_1(q,Q)}{\partial Q}\dot{Q} \tag{2.7.5}$$

Grouping terms with like coefficients, we can then derive that:

$$p = \frac{\partial F_1(q,Q)}{\partial q} \tag{2.7.6}$$

[2.4] See Landau pg. 144
[2.5] See Landau pg. 4.

$$P = -\frac{\partial F_1(q,Q)}{\partial Q} \tag{2.7.7}$$

And, somewhat less usefully,

$$K = H + \frac{\partial F_1(q,Q)}{\partial t} \tag{2.7.8}$$

Similar relations can be be derived for the other "generating functions" by the same method. In total, therefore,

- $F_1(q,Q)$: $p = \frac{\partial F_1}{\partial q}$, $P = -\frac{\partial F_1}{\partial Q}$
- $F_2(q,P)$: $p = \frac{\partial F_2}{\partial q}$, $Q = \frac{\partial F_2}{\partial p}$. Among these, F_2 seems to be the one most regularly used.
- $F_3(p,Q)$: $q = -\frac{\partial F_3}{\partial p}$, $P = -\frac{\partial F_3}{\partial Q}$
- $F_4(p,P)$: $q = -\frac{\partial F_4}{\partial p}$, $Q = \frac{\partial F_3}{\partial P}$

Once these relationships have been established, a single generating function can then be used to generate an associated canonical transformation!

2.7.3 The Hamilton-Jacobi Equation

In the Hamiltonian formulation, the principal of least action becomes the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H(q,p,t) = 0 \tag{2.7.9}$$

Where S is the action.

Now, the Euler-Lagrange equation tells us that $\dot{p}_i = \frac{\partial L}{\partial q_i}$. Integrating both sides of this equation with respect to t, and pulling the integral through the q_i derivative, we can write $p_i = \frac{\partial S}{\partial q_i}$. Therefore, the Hamilton-Jacobi equation can be written:

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0 \tag{2.7.10}$$

Now, when writing out the full Hamiltonian where $p_i \to \frac{\partial S}{\partial q_i}$:

$$\left(\frac{1}{2m}|\nabla S|^2 + V\right) + \frac{\partial S}{\partial t} = 0 \tag{2.7.11}$$

Where V is the potential.

2.8 The Harmonic Oscillator

2.8.1 Solutions of the 1D Harmonic Osscilator

Useful Sources: http://scipp.ucsc.edu/ haber/ph5B/sho09.pdf

The simple harmonic oscillator is characterize by the equation:

$$m\ddot{x} + kx = 0 \tag{2.8.1}$$

Since we expect an oscillatory solution, we will use the trial solution:

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \tag{2.8.2}$$

We will apply the boundary conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ to eliminate the required two degrees of freedom to obtain a unique solution. Taking the first derivative and solving for A and B in terms of these conditions, we can rewrite the trial solution as:

$$x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t) \tag{2.8.3}$$

Taking the second derivative:

$$\ddot{x}(t) = \omega^2\left(\frac{-v_0}{\omega}\sin(\omega t) - x_0\cos(\omega t)\right) = -\omega^2 x(t) \tag{2.8.4}$$

Plugging this function into the original differential equation then gives us the relationship:

$$(-m\omega^2 + k)x(t) = 0 \tag{2.8.5}$$

Or, since $x(t)$ is not everywhere zero:

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2.8.6}$$

Which is the frequency of the harmonic oscillator.

A linear combination of sines and cosines can be combined together into a single phase shifted sine function using the following identity :

$$\begin{aligned} A\cos(\alpha+\beta) = A\cos(\alpha)\cos(\beta) - A\sin(\alpha)\sin(\beta) \implies \\ A\cos(\omega t+\phi) = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) \end{aligned}$$

So, identifying the arbitrary constants $A\cos(\phi) = x_0$ and $ASin(\phi) = \frac{-v_0}{\omega}$, we have

$$x(t) = A\cos(\omega t + \phi) \tag{2.8.7}$$

A here is interpreted as the amplitude of the oscillations, while ϕ is the phase shift of the oscillator from a cosine dependence on position.

2.8.2 Solutions of the Damped 1D Harmonic Oscillator

The damping force on a harmonic oscillator is modeled by introducing a $-b\dot{x}$ $(b > 0)$ term into the force equation:

$$m\ddot{x} = -b\dot{x} - kx \tag{2.8.8}$$

Rearranging this equation, and letting $\beta = \frac{b}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

$$\ddot{x} + 2\beta\dot{x} + {\omega_0}^2 x = 0 \tag{2.8.9}$$

By analyzing the characteristic equation of this differential equation (as described in the mathematical methods section), we see that it has two roots:

$$r_1 = -\beta + \sqrt{\beta^2 - {\omega_0}^2}, r_2 = -\beta - \sqrt{\beta^2 - {\omega_0}^2} \tag{2.8.10}$$

The exact form of the general solution is determined by the sign of the discriminant of these roots, $\beta^2 - {\omega_0}^2$. The general solution will be:

$$x(t) = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2 - {\omega_0}^2}t} + A_2 e^{-\sqrt{\beta^2 - {\omega_0}^2}t}\right) \tag{2.8.11}$$

- If ${\omega_0}^2 > \beta^2$, so that $\beta^2 - {\omega_0}^2 < 0$, the roots will be complex. This creates oscillatory behavior that is known as **under-damped motion**.

  In this case, the general solution can be simplified by introducing a frequency for the oscillatory motion, ${\omega_1}^2 = {\omega_0}^2 - \beta^2$. With this substitution, the equation for under-damped motion can be rewritten:

  $$x(t) = e^{-\beta t}\left(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right) = Ae^{-\beta t}\cos(\omega_1 t - \delta) \tag{2.8.12}$$

  This solution clearly describes an initial oscillation that dies away with time due to the $e^{-\beta t}$ term. Note that the frequency ω_1 is not a true frequency, because the motion is not truly periodic. However, it does accurately reflect the spacing between zero crossings, and is therefore meaningful in this sense.

- If $\beta^2 = {\omega_0}^2 < 0$, then there will only be one, real root to the characteristic equation. This is known as the **critically damped** case. A second solution can be produced by multiplying the first by t, and so a general solution can be written:

  $$x(t) = (A + Bt)e^{-\beta t} \tag{2.8.13}$$

  This equation describes the shortest path for the oscillator to its equilibrium value.

- If ${\omega_0}^2 < \beta^2$, so that $\beta^2 - {\omega_0}^2 > 0$, the roots will be real. This is known as the **over-damped case**. Again assigning a new frequency variable, $\omega_2 = \sqrt{\beta^2 - {\omega_0}^2}$, the general solution can be written:

  $$x(t) = e^{-\beta t}\left(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right) \tag{2.8.14}$$

  Noting that, since the exponents are now real, no oscillatory behavior emerges. Instead, the over-damped solutions swing wildly back and forth in a seemingly unpredictable fashion.

Useful sources: Marion & Thorton pg. 108

2.8.3 Finding the Frequency of Small Osscillations: Approximations of a SHO

Many systems behave like harmonic oscillators in the region immediately around an equilibrium point. In many cases, these oscillations can be modeled as SHO, allowing us to calculate their approximate frequency and amplitude. This is done by Taylor expanding the potential of the system in question about its equilibrium point until the result *looks* like a simple harmonic oscillator potential. At this point, the frequency of oscillations can simply be read off.

The potential of a true harmonic oscillator is:

$$U_{SHO}(x) = \frac{1}{2}kx^2 \tag{2.8.15}$$

Consider some arbitrary potential $U(x)$ acting on a particle with mass m. Suppose U has an equilibrium point at $x = x_0$. We are free to shift the potential such that $U(x_0) = 0$. At x_0, the potential is at a minimum. This means by definition that:

$$\frac{dU}{dx}(x_0) = 0 \tag{2.8.16}$$

Now, consider the first several terms of the Taylor expansion of this potential about x_0:

$$U(x) \approx U(x_0) + \frac{dU}{dx}\bigg|_{x_0}(x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}\bigg|_{x_0}(x - x_0)^2 + ... \tag{2.8.17}$$

As stated above, the first two terms of this expansion go to zero, leaving

$$U(x) \approx \frac{1}{2}\frac{d^2U}{dx^2}\bigg|_{x_0}(x - x_0)^2 \tag{2.8.18}$$

But, since $x - x_0$ is the distance of the particle from the equilibrium point, this is exactly the potential of as SHO! Once derivatives have been taken and evaluated at x_0, some constant k_U will remain, allowing us to write the equation in the form:

$$U(x) \approx \frac{1}{2}k_U(x - x_0)^2 \tag{2.8.19}$$

Therefore, the approximate frequency of small oscillations of the particle about the equilibrium point can be written:

$$\omega_0 = \sqrt{\frac{k_U}{m}} \tag{2.8.20}$$

2.9 Orbital Motion

2.9.1 The Reduced Mass

The reduced mass allows us to simplify the motion of two bodies orbiting one another by transforming the problem so that it resembles a single body orbiting around the center of mass of the system. Consider the Lagrangian for two orbiting bodies in a potential $U(r)$:

$$L = \frac{1}{2}m_1\dot{r_1}^2 + \frac{1}{2}m_2\dot{r_2}^2 - U(r) \tag{2.9.1}$$

Now suppose that we care only about motion that occurs relative to the center of mass of the system, neglecting motion of the center of mass, such as translation of the entire system. We can then set the center of mass as our origin, which implies:

$$m_2\boldsymbol{r_2} + m_1\boldsymbol{r_1} = 0 \tag{2.9.2}$$

Let us also introduce a new vector $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$. Together these two equations produce the following relations:

$$\boldsymbol{r_1} = \frac{m_2}{m_1 + m_2}\boldsymbol{r}, \boldsymbol{r_2} = -\frac{m_1}{m_1 + m_2}\boldsymbol{r} \tag{2.9.3}$$

Now substituting these expressions back into the Lagrangian, we obtain:

$$L = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{r}^2 - U(r) = \frac{1}{2}\mu\dot{r}^2 - U(r) \tag{2.9.4}$$

Where we have now defined the reduced mass:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.9.5}$$

The system is now represented as one body, with mass μ, orbiting another body at the center of mass with mass $m_{cm} = m_1 + m_2$, connected by the vector $\boldsymbol{r}$.

2.9.2 Orbital Conserved Quantities

The Lagrangian of an orbiting body can be written:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{2.9.6}$$

It is clear that the Lagrangian is cyclic in θ, i.e. $\frac{\partial L}{\partial \theta} = 0$. Therefore, there angular momentum in this direction, p_θ, is conserved. By Nother's theorem, this angular momentum is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = l \tag{2.9.7}$$

2.9.3 Kepler's Second Law

If we ask how much area the radius vector "sweeps out" in for a given $d\theta$, the $\frac{1}{2}BH$ formula for the area of a triangle gives us:

$$dA = \frac{1}{2}r(rd\theta) \tag{2.9.8}$$

Dividing by the time, dt:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{l}{2\mu} \tag{2.9.9}$$

Since $\frac{dA}{dt}$ is proportional to the angular momentum, which is conserved, we know that the radius vector must "sweep out" the same amount of area per unit time throughout the orbit.

2.9.4 Deriving Orbits with Newtonian Mechanics: The Integral Equation

Steps:

1. Write the energy of the orbiting body
2. Rearrange for $\dot{r}$
3. Rewrite as an integral for θ using $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr$ and the angular momentum

We will start by writing the energy of an orbiting body with reduced mass μ:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r) \tag{2.9.10}$$

This can be rearranged for $\dot{r}$:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}} \tag{2.9.11}$$

We would prefer, however, an equation for $\theta(r)$. We can combine the fact that $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$ with the equation for the angular momentum, $\dot{theta} = \frac{l}{\mu r^2}$, to get:

$$d\theta = \frac{l}{\mu r^2 \dot{r}}dr \tag{2.9.12}$$

We can combine this with our equation for $\dot{r}$ to write the function we desire as an integral:

$$\theta(r) = \int \frac{\pm\frac{l}{r^2}}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}dr \tag{2.9.13}$$

This equation can now technically be used to calculate orbits, although it is awfully cumbersome.

It is interesting to examine the third term under the radical. The quantity in the parentheses has units of Energy, and $U - \frac{l^2}{2\mu r^2}$ can be treated as an effective potential. In addition to the regular potential, there is a third term that is due to the fictitious "centrifugal" force:

$$U_{eff} = U(r) + \frac{l^2}{2\mu r^2} \tag{2.9.14}$$

From this potential we can calculate the value of the centrifugal force:

$$F_c = -\frac{d}{dr} - \frac{l^2}{2\mu r^2} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 \tag{2.9.15}$$

This effective potential due to the centrifugal force is the reason that there are bound states of this system! While neither term in the effective potential has a minumum by itself, their *sum* does.

Source: Marion & Thorton: pg. 291.

2.9.5 Deriving the Orbit Equation

Lim 1055 features a nicer way of computing the derivatives of r here that flows better conceptually. I plan on replacing this with that eventually. In the mean time, check pg. 85 in Lim Classical Mechanics.

Steps:

1. Write the Lagrangian for an orbiting body, then apply the Euler-Lagrange equation
2. Make the substitution $u = \frac{1}{r}$
3. From the definition of u and the angular momentum equation, derive (using lots of chain rule) expressions for the r terms in the equation
4. Substitute for the r terms and rearrange into the final form.

The Lagrangian of an orbiting body is:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{2.9.16}$$

Applying the Lagrange-Euler equation gives us:

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \tag{2.9.17}$$

This equation can be solved using the substitution $u = \frac{1}{r}$. First we will a relation to substitute to eliminate $\ddot{r}$ and $r\dot{\theta}^2$. First:

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{l}\dot{r} \tag{2.9.18}$$

Where the last step used the equation for angular momentum, $\dot{\theta} = \frac{l}{\mu r^2}$.

Now:

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\Big(-\frac{\mu}{l}\dot{r}\Big) = \frac{dt}{d\theta}\frac{d}{dt}\Big(-\frac{\mu}{l}\dot{r}\Big) = -\frac{\mu}{l\dot{\theta}}\ddot{r} = -\frac{\mu^2}{l^2}r^2\ddot{r} \tag{2.9.19}$$

Where again the last step uses the angular momentum equation. From these two expressions, we can write:

$$\ddot{r} = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2} \tag{2.9.20}$$

$$r\dot{\theta}^2 = \frac{l^2}{\mu^2}u^3 \tag{2.9.21}$$

We can now substitute these results back into our original result from the Lagrange-Euler equation to obtain:

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u}) \tag{2.9.22}$$

This is the orbit equation. Substituting back in for F and r:

$$\frac{d^2}{d\theta^2}\frac{1}{r} + \frac{1}{r} = \frac{\mu}{l^2}r^2\frac{\partial U}{\partial r} \tag{2.9.23}$$

Source: Marion & Thorton: pg. 292.

2.9.6 The Kepler Problem (Explicitly Equations for Orbits)

We will start with the integral orbit equation, substituting the central potential $U(r) = \frac{-k}{r}$:

$$\theta(r) = \int \frac{\pm \frac{l}{r^2}}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} dr \tag{2.9.24}$$

If we assign the minimum value of r to be $\theta = 0$, we get (elaborate? How?):

$$\cos(\theta) = \frac{\frac{l^2}{\mu k}\frac{1}{r} - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}} \tag{2.9.25}$$

If we assign the following constants :

$$\alpha = \frac{l^2}{\mu k} \tag{2.9.26}$$

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \tag{2.9.27}$$

Which allows us to rewrite the orbit equation as:

$$\frac{\alpha}{r} = 1 + \epsilon\cos(\theta) \tag{2.9.28}$$

This equation describes conic sections, which represent all posible orbits. The shape of these orbits can be grouped by the value of the eccentricity, ϵ:

- $\epsilon > 1$, $E > 0$: Hyperbola. These orbits are not bounded.
- $\epsilon = 1$, $E = 0$: Parabola. These orbits are not bounded, but are just on the edge of becoming so.
- $0 < \epsilon < 1$, $V_{min} < E < 0$: Ellipse. These orbits are bound.
- $\epsilon = 0$, $E = V_{min}$: Circle. This is the lowest energy bound orbit.

Source: Marion & Thorton: pg. 300.

2.10 Rigid Body Motion (Rotation in Euler Angles)

The Lagrangian and Hamiltonian formulations of classical mechanics discussed so far give us many ways of describing the center of mass motion of a system. However, physical objects are also free to rotate *around* their center of mass. In order to analyze this complicated motion, we will conceptually separate motion into motion of the center of mass (described by regular classical mechanics in the lab frame) and motion around the center of mass (described by rigid body motion in a frame that *rotates* along with the body, called the body frame).

This section directly follows Thorton and Marion, Chapter 11.

2.10.1 Moment of Inertia

Just as mass of an object determines how hard it is to move, the mass distribution of an object determines how difficult it is to rotate. This mass distribution is characterized by the inertia tensor.

The inertia tensor appears naturally when considering the energy of a rotating particle. Consider a rigid body made up of α particles that are both rotating and translating. The particle's velocities in the lab frame can be written

$$\boldsymbol{v}_\alpha = V_{trans} + \boldsymbol{\omega} \times \boldsymbol{r}_\alpha \tag{2.10.1}$$

Where V_{trans} is the translational velocity of the CM in the lab frame, and $\boldsymbol{\omega}$ and $\boldsymbol{r}_\alpha$ are the angular velocity and position vector of the particles in the body frame.

The kinetic energy of this system is

$$T = \sum_\alpha \frac{1}{2} m_\alpha (V_{trans} + \boldsymbol{\omega} \times \boldsymbol{r}_\alpha)^2 \tag{2.10.2}$$

Noting that $\sum_\alpha m_\alpha (\boldsymbol{V}_{trans} \cdot \boldsymbol{\omega} \times \boldsymbol{r}_\alpha) = \boldsymbol{V}_{trans} \cdot \boldsymbol{\omega} \times \left(\sum_\alpha m_\alpha \boldsymbol{r}_\alpha\right) = 0$ because, in the center of mass frame in which the r_α vectors are defined, the center of mass is at the origin, and thus $\sum_\alpha m_\alpha \boldsymbol{r}_\alpha = 0$. The kinetic energy can thus be separated into distinct translational and rotational terms:

$$T = \sum_\alpha \frac{1}{2} m_\alpha V_{trans}^2 + \sum_\alpha \frac{1}{2} m_\alpha (\boldsymbol{\omega} \times \boldsymbol{r}_\alpha)^2 \tag{2.10.3}$$

Consider now only the rotational term. Using the identity that $(\boldsymbol{A} \times \boldsymbol{B})^2 = A^2 B^2 - (\boldsymbol{A} \cdot \boldsymbol{B})^2$, we can simplify this expression to:

$$T_{rot} = \frac{1}{2} \sum_\alpha m_\alpha (\omega^2 r_\alpha^2 - (\boldsymbol{\omega} \cdot \boldsymbol{r}_\alpha)^2) \tag{2.10.4}$$

Plugging in components and making the substitution $\omega_i = \sum_j \omega_j \delta_{ij}$, we can pull out the ω's:

$$T_{rot} = \frac{1}{2} \sum_{i,j} \omega_i \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \tag{2.10.5}$$

This is now the final expression for the rotational kinetic energy. However, it makes sense to collect the position and mass information in the body frame into a single quantity, which we will define as the inertia tensor:

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)} \tag{2.10.6}$$

Now the rotational kinetic energy takes the form taught in high school physics classes,

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j \tag{2.10.7}$$

2.10.2 Calculating Inertia Tensors

When calculating the inertia tensor, it is usually easiest to use the above definition of the tensor elements. Notice that diagonal elements will end up having a form similar to

$$I_{ii} = \sum_{\alpha} m_{\alpha} \Big((x_{1,\alpha}^2 + x_{2,\alpha}^2 + x_{3,\alpha}^2) - x_{i,\alpha}^2 \Big) \tag{2.10.8}$$

While off-diagonal elements will have the form:

$$I_{ij} = \sum_{\alpha} m_{\alpha} (-x_{i,\alpha} x_{j,\alpha}) \tag{2.10.9}$$

It is clear from the definition of the tensor that it is symmetric: i.e. $I_{ij} = I_{ji}$.

It is also clear that, generalizing the summation over α to an integral over all mass particles, the tensor elements for a continuous mass distribution are given by:

$$I_{ii} = \int_V \rho(\boldsymbol{r}) \Big(\delta_{ij} \sum_k x_k^2 - x_i x_j \Big) dV \tag{2.10.10}$$

It is, however, less clear that once one moment of inertia has been calculated, it is possible to easily find the moment of inertia around any other parallel axis. This result [2.6] is known as Steiner's **parallel-axis theorem**. It states that, if we have an original moment of inertia tensor J_{ij} defined about some origin, the moment of inertia about a new origin defined by the vector $\boldsymbol{a}$ from the old origin, the new inertia tensor can be written:

$$I_{ij} = J_{ij} - M(a^2 \delta_{ij} - a_i a_j) \tag{2.10.11}$$

Where M is the total mass of the object. Keep in mind that this theorem only applies to parallel axes!

A second general theorem, called the **perpendicular axis theorem**, concerns **only** objects that exist in a 2D plane (or can be approximated as such), for example a piece of paper or thin rectangular slab. If the object exists in the $x - y$ plane, then the perpendicular axis theorem states that the moments of inertia along the axes are related by:

$$I_z = I_x + I_y \tag{2.10.12}$$

2.10.3 Principal Axes of Rotation: Diagonalizing the Inertia Tensor

In general, an inertia tensor will have some non-zero off diagonal elements. However, once diagonalized (through the normal methods of linear algebra), we can give a physical interpretation to the diagonal elements.

Diagonalizing a matrix involves multiplying it by a change of coordinates matrix of eigenvectors, which amounts to a rotation of the inertia tensor about the origin. The resulting orientation of the axes are known as the **principal axes** of the inertia tensor, and are the preferred basis for describing the rotations of the body.

[2.6] For a proof, see Thorton and Marion, pg. 429)

2.10.4 Euler Angles

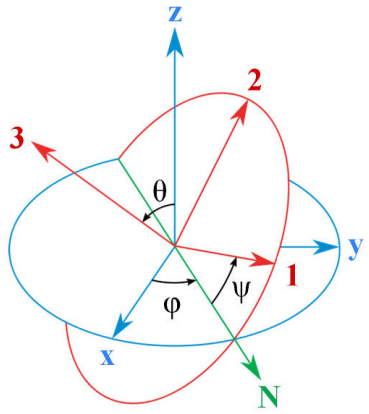


Figure 2.10.1: Euler Angles graphic (Source: Wikipedia)

Rigid body motion is essentially the study of tops: rigid objects that are fixed to rotate about one point. The Euler angles are the natural coordinate system for describing this motion, with angles corresponding to nutation (θ), precession (ϕ) and the rotation of the top around its own axis (ψ). Conveniently, many other rotating bodies can also be easily described using these angles.

In a normal Cartesian coordinate system, one is free to "give directions" in whatever order one pleases: it does not matter if you move over by x and up by y, or up by y then over by x. With Euler angles, however, this gets very confusing! The best way to think about the Euler angles is as a series of directions, specifying how to get from the "rest" position of the top (upright in the lab frame) to any position during its motion.

To get to some position with Euler angles, you follow the following steps. Let the body axes be labeled 1,2,3, and the fixed lab-frame axes x,y,z. Imagine the "body" to be a top, spinning about it's 3 axis.

1. Rotate around the lab's x or y axis (which is a convention that is different in classical and quantum mechanics) through the angle θ. The body's 3 axis is now θ away from the lab z axis. This covers all possible tilts of the top.

2. Rotate the body around the lab z axis, through an angle ϕ. This covers all possible orientations of the top around the z axis as it precesses (remember that its point is fixed at the origin).

3. Finally, rotate the top about its 3 axis. This sets the orientation of the top.

With the Euler angles so defined, we can represent each rotation as a rotation matrix, allowing (very messy) calculations of the bodies' rotational angular velocities in terms of the Euler angles, which are measured from the lab frame! [2.7]

These glorious yet disgusting equations are as follows:

$$\omega_1 = \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \tag{2.10.13}$$

[2.7] For the gory details, see Thorton and Marion pg. 442

$$\omega_2 = \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi) \tag{2.10.14}$$

$$\omega_3 = \dot{\phi}\cos(\theta) + \dot{\psi} \tag{2.10.15}$$

It is handy to note that

$$\omega_1^2 + \omega_2^2 = \dot{\phi}^2\sin^2(\theta) + \dot{\theta}^2 \tag{2.10.16}$$

2.10.5 Euler's Equations

Solving for dynamics in Euler angles generally involves solving the associated Euler-Lagrange equations. Luckily, in Euler angles these easily[2.8] reduce to a system of simple equations, known as Euler's Equations. For a body with principal moments of inertia I_1, I_2, and I_3 and an angular velocity $\boldsymbol{\omega}$ and a torque $\boldsymbol{N}$, split into components along the principal axes, we have:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1 \tag{2.10.17}$$

By permutation of the axes, we must also have the corresponding relations:

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2 \tag{2.10.18}$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3 \tag{2.10.19}$$

2.11 Classical Scattering Theory

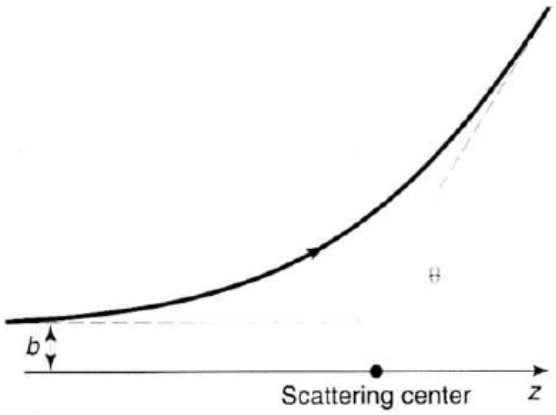


Figure 2.11.1: Classical scattering diagram (Source: Griffiths QM, pg. 394)

Classical scattering occurs when a small particle interacts with a relatively massive (and therefore effectively immovable) object. This interaction will alter the trajectory of the smaller object, causing it to "bounce" or "scatter" off of the larger object. This classical model is important in its own right (for example, it explains scattering of particles off of nuclei). However,

[2.8] See Thorton and Marion pg. 444 for a derivation

it is *particularly* important to understand as a basis for electromagnetic scattering (diffraction, etc.) and quantum mechanical scattering (scattering between quantum particles).

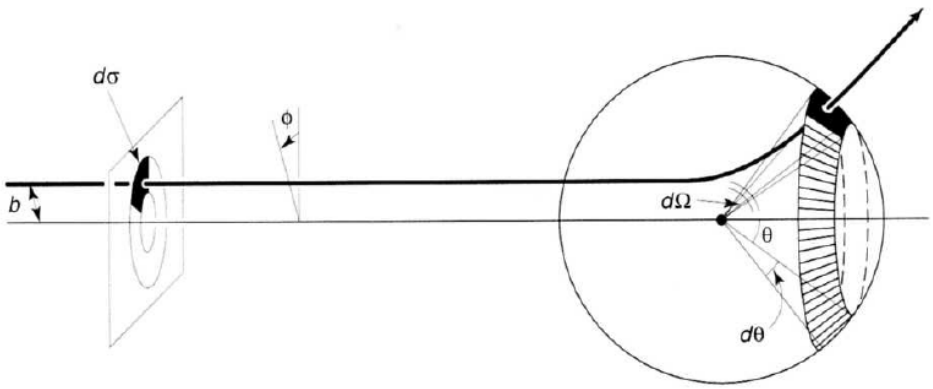


Figure 2.11.2: Classical scattering diagram (Source: Griffiths QM, pg. 396)

A particle approaches an object (or **scattering center**) with an **impact parameter** b (see fig. 2.11.1). Our goal is to determine the angular distribution of scattered particles, as well as the total cross section of the object; i.e. what area of the stream of particles it interacts with.

The *way* the particle scatters can be characterized by the ratio between the area $d\sigma$ that the particle passes through, and the angle $d\Omega$ that it scatters into (Fig. 2.11.2). We define the ratio between these two quantities as a function of θ as the **differential cross section**:

$$D(\theta) = \frac{d\sigma}{d\Omega} \tag{2.11.1}$$

$D(\theta)$ is solely a property of the object (or potential) that the particle scatters off of, and is independent of the particle itself. "Solving" the scattering problem therefore amounts to determining $D(\theta)$ for the given potential or object.

In terms of the variables of Fig. 2.11.2, $d\sigma = bdbd\phi$, while it is always true that $d\Omega = \sin\theta d\theta d\phi$. Therefore, we can also express $D(\theta)$ as[2.9]:

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{2.11.2}$$

Once $D(\theta)$ has been found, the total cross section can be found by integrating over the entire solid angle:

$$\sigma = \int D(\theta)d\Omega \tag{2.11.3}$$

2.11.1 Example: Hard Sphere Scattering

The simplest classical example of scattering is the problem of scattering off of a hard sphere of radius R. In this case, if the particle strikes the surface at an angle α, it will reflect off of the surface an angle α from the surface normal. Based on Fig. 2.11.3 we can then express $\theta = \pi - 2\alpha$. Then:

$$b = R\sin\alpha = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right) \tag{2.11.4}$$

[2.9] $\frac{db}{d\theta}$ is usually negative, so we take the absolute value by convention.

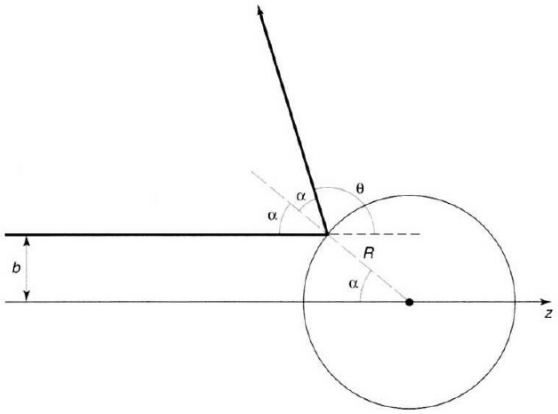


Figure 2.11.3: Hard sphere scattering (Source: Griffiths QM, pg. 395)

So:

$$\theta = \begin{cases} 2\cos^{-1}\left(\frac{b}{R}\right) & b \leq R \\ 0 & b \leq R \end{cases} \tag{2.11.5}$$

Now we can explicitly calculate $D(\theta)$:

$$\frac{db}{d\theta} = -\frac{1}{2}R\sin\left(\frac{\theta}{2}\right) \rightarrow D(\theta) = \frac{R^2}{4} \tag{2.11.6}$$

3 Relativity

3.1 The Lorentz Transformation and its Consequences

3.1.1 The Lorentz Transformation

The basic principle of special relativity insists that the speed of light be the same in all reference frames. From this assumption, we can derive the form of the Lorentz Transformation between inertial reference frames. Defining γ to be:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.1.1}$$

It is worth noting here that $\gamma \geq 1$, since $v \leq c$.

The transform into a frame moving in the x direction with velocity v can be written:

Lorentz Transformation

- $t' = \gamma(t - \frac{v}{c^2}x)$
- $x' = \gamma(x - vt)$
- $y' = y$
- $z' = z$

Of course, to obtain the inverse transformation we simply substitute $v \rightarrow -v$.

Inverse Lorentz Transformation

- $t = \gamma(t' + \frac{v}{c^2}x')$
- $x = \gamma(x' + vt')$
- $y = y'$
- $z = z'$

3.1.2 Time Dilation

One immediate consequence of accepting the Lorentz transformation is that the classical concept of simultaneity is no longer applicable.

Suppose that a clock starts at $x = 0$ at $t = 0$. The clock then moves away from the origin along the x-axis at velocity v. The clock will stop when $t' = T'$. We wish to find T: the time it takes for the clock to stop in the *rest* frame.

When the clock stops, $t' = T'$, $t = T$, and $x = v * T$. Therefore, from the time component of the Lorentz transform, we can write:

$$T' = \gamma(T - \frac{v^2}{c^2}T) = \frac{T}{\gamma} \tag{3.1.2}$$

Or, finally,

$$\boxed{T = \gamma T'} \tag{3.1.3}$$

This is the fundamental statement of time dilation. Since $\gamma \geq 1$, $T \geq T'$. MORE time will pass in the reference frame than in the moving frame. This is often rephrased that "moving clocks run slow".

3.1.3 Lorentz Contraction

Another fundamental consequence of the Lorentz transformation is that observers in different frames of reference will not agree about the lengths of objects.

Consider a thought experiment where a rod of length L' (in its own moving frame) is moving at a velocity v in the x direction. Set the origin of each coordinates system so that at $t = t' = 0$, one end of the rod is at $x = x' = 0$. In the moving (primed) frame, then, the other end of the rod will be at $x' = L'$. In the rest frame, however, the x term of the Lorentz Transformation tells us that:

$$x' = \gamma x \tag{3.1.4}$$

Therefore, calculating $L \equiv \Delta x = x - 0 =$ and $L' \equiv \Delta x' = x' - 0$:

$$\boxed{L' = \gamma L} \tag{3.1.5}$$

Or

$$L = \frac{L'}{\gamma} \tag{3.1.6}$$

Since $\gamma \geq 1$, this means that $L \leq L'$: the rod appears to shrink (lengthwise) when viewed from the rest frame. It is important to note that this phenomenon only occurs along the direction of motion: the dimensions of the rod in the y and z directions are not distorted.

3.1.4 The Einstein Velocity Addition Rule

Suppose a particle is moving *in the moving primed frame* with a velocity u', while the primed frame itself is moving with a velocity v. We wish to find the velocity of the particle in the rest frame, which we will call u.

If we set the origins of both frames so that $x = x' = 0$ at $t = t' = 0$, then the inverse Lorentz transformation gives us that:

$$\frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + v\Delta t')}{\gamma(\Delta t' + \frac{v}{c^2}\Delta x')} \tag{3.1.7}$$

Rearranging this expression, and noting that $\frac{\Delta x'}{\Delta t'} = u'$ and $\frac{\Delta x}{\Delta t} = u$, we have

$$\boxed{u = \frac{u' + v}{1 + \frac{u'v}{c^2}}} \tag{3.1.8}$$

This is the Einstein velocity addition rule. Notice that, in the case where $u', v << c$, it reduces to the classical velocity addition rule:

$$u = u' + v \tag{3.1.9}$$

Also notice that, even if $u' = c$, $u = c$. Therefore this rule reflects the fundamental rule of relativity: that light moves at the same velocity in every reference frame.

3.2 Four-Vectors

3.2.1 Four-Vectors

A four-vector is a generalized form of a normal vector in 3D space (or "three-vector"). The extra component has been added to include time as another pseudo-dimension. However, this analogy between time and the spatial dimensions should not be stretched too far, as there are some important asymmetries between the two cases.

The position four-vector is[3.1]:

$$x =< ct, x, y, z > \tag{3.2.1}$$

There are an unfortunate number of conflicting conventions for the time component of this vector. Some authors include it as the first (0th) component, while others place it last (4th). Sometimes that term is made imaginary (ict) for reasons that will be apparent later. **In these notes, all four vectors will be written according to the convention above.**

The position vector is naturally contravariant. In Einstein notation, **all contravariant**

[3.1] Notice that the c in the first term is necessary for that component to have units of length!

vectors are symbolized by upper indices:

$$x^{\mu} \tag{3.2.2}$$

The cooresponding **covariant vectors (dual vectors) are symbolizedf by lower indices**:

$$x_{\mu} \tag{3.2.3}$$

Another convention avoids a common piece of notational clutter. Sometimes while summing over four-vectors you wish to index all four components of a vector, while for others you wish to only index the three spatial components, treating the time component separately. Instead of explicitly stating which is meant, a convention is defined:

- **Greek indices** (μ, ν, etc.) are assumed to run over **all four components** of the four-vector so that, for example, $\mu = 0, 1, 2, 3$.
- **Latin indices** (i, j, etc.) are assumed to run over **only the spatial components** of the tensor, i.e. $i = 1, 2, 3$.

With normal three dimensional Euclidian vectors, the dot product is defined as the product of a contravariant ("normal") vector with its cooresponding covariant ("dual") vector. The generalized dot product of two vectors x^{μ} and y^{ν} is:

$$g(x^{\mu}, y^{\nu}) = x^{\mu} g_{\mu,\nu} y^{\nu} \tag{3.2.4}$$

Where $g_{\mu,\nu}$ is called the **metric**, and depends on the geometry of spacetime in which you are working. In general relativity, the metric is calculated as a way to represent curved space time. In special relativity, however, we work in Minkowski space, ("flat space"), which is a pseudo-Euclidian[3.2] space in which:

$$g_{\mu,\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.5}$$

Another convention arises here: we could just as easily define the Minkowski metric above with $-1 \leftrightarrow 1$:

$$g_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.2.6}$$

This choice defines a property called the **signature** of the metric. The signature of a metric can formally be written (p, q, r), where p is the number of positive eigenvalues, q the number of negative eigenvalues, and r the number of zero eigenvalues. However, as a shorthand, the two

[3.2] Minkowski space is not *really* a Euclidian space, because of the minus sign in the metric defined below.

separate signature conventions for the Minkowski metric are usually notated[3.3]:

$$(3,1) \to (-,+,+,+), \qquad (1,3) \to (+,-,-,-) \tag{3.2.7}$$

There is no reason to prefer one over the other. However, for consistency, **in these notes we will use the** $(3,1) = (+,-,-,-)$ **metric** written above (eq. 3.2.6).

In Euclidian space, performing a dot product between two contravariant vectors involves transforming one of them into a covariant "dual" vector:

$$\begin{pmatrix} ... \\ ... \\ ... \end{pmatrix} \cdot \begin{pmatrix} ... \\ ... \\ ... \end{pmatrix} \to \begin{pmatrix} ... & ... & ... \end{pmatrix} \begin{pmatrix} ... \\ ... \\ ... \end{pmatrix} \tag{3.2.8}$$

The metric still performs a similar operation with four-vectors. The Minkowski metric defined above is a **covariant metric tensor** (hence why it was written with lower indices). For every such tensor there exists a **contravariant metric tensor** such that:

$$g^{\mu,\nu} g_{\nu,\sigma} = g_{\sigma,\nu} g^{\nu,\mu} = \delta^{\mu}_{\sigma} \tag{3.2.9}$$

Notice that indices must match up in pairs in order for multiplication of the matrices to be valid. In the special case where the indices on both metrics are the same:

$$g^{\mu,\nu} g_{\nu,\mu} = \mathbb{I} \tag{3.2.10}$$

Where I is the identity matrix. For the Minkowski metric in particular, $g_{\mu,\nu} = g^{\mu,\nu}$: the Minkowski metric is its own inverse.

The covariant and contravariant metrics have the property of transforming covariant four-vectors into contravariant four-vectors (and vice versa):

$$\begin{aligned} g^{\mu,\nu} x_{\nu} &= x^{\mu} \\ g_{\mu,\nu} x^{\nu} &= x_{\mu} \end{aligned} \tag{3.2.11}$$

As a mnumonic (although not mathematically cogent), this can be thought of as a process of canceling indices:

$$g^{\mu,\not{\nu}} x_{\not{\nu}} = x^{\mu} \tag{3.2.12}$$

3.2.2 Lorentz Transformations in Four-Vector Notation

The four-vector notation makes the Lorentz transformations look much nicer. If we define $\beta = \frac{v}{c}$, the Lorentz transformation (for motion in the x direction) is:

- $x'^0 = \gamma(x^0 - \beta x^1)$
- $x'^1 = \gamma(x^1 - \beta x^0)$
- $x'^2 = x^2$

[3.3] r=0 is understood by omiting r.

- $x'^3 = x^3$

This system of equations can be expressed using a tensor:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{3.2.13}$$

Where Λ is:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.14}$$

Of course, this holds for any arbitrary four-vector a:

$$a'^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu} \tag{3.2.15}$$

3.3 Relativistic Mass

Mass (as defined by $\boldsymbol{F} = m\boldsymbol{a}$) is not constant as speed varies in relativity. However, the **relativistic** mass, m_0, is. The two are related by:

$$m = \gamma m_0 \tag{3.3.1}$$

Note that the terms 'relativistic mass', 'invariant mass', and 'rest mass' all mean the same thing: the relativistic mass *is* invariant, which is why it is *also* equal to the rest mass.

3.4 Relativistic Energy

The total relativistic energy includes both the rest energy of a particle and any kinetic energy it may have. The relativistic energy of a particle with rest mass m moving at velocity v is:

$$E = \gamma m c^2 \tag{3.4.1}$$

Sometimes you may want to split the energy into kinetic and rest parts. The rest component is always:

$$E_{rest} = mc^2 \tag{3.4.2}$$

Which means that the kinetic energy must then be:

$$KE = E - E_{rest} = (\gamma - 1)mc^2 \tag{3.4.3}$$

Another extremely useful equation relates the total relativistic energy to the rest mass (m) and the magnitude of the three momentum (p^2):

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{3.4.4}$$

From this follows an important result: **for zero-mass particles:**

$$E = \frac{|\boldsymbol{p}|}{c} \tag{3.4.5}$$

3.5 Relativistic Three Momentum

The relativistic three momentum is simply the same kinematic momentum we are used to dealing with, but with the mass replaced by the relativistic mass:

$$\boldsymbol{p} = \gamma m v \tag{3.5.1}$$

Where m is the invariant mass.

3.6 Four Momentum

The four momentum is a generalized version of the momentum vector that now includes the energy of the particle:

$$\boldsymbol{p_\mu} = (\frac{E}{c}, p_1, p_2, p_3) \tag{3.6.1}$$

Notice that E is the *total* energy of the particle, not just its rest energy! In other words, the same E as in:

$$E = \sqrt{|p|^2c^2 + m^2c^4} \tag{3.6.2}$$

Where p is the *three* momentum.

Just like three momenta, four momenta can be added:

$$\boldsymbol{p_{1,\mu}} + \boldsymbol{p_{2,\mu}} = \boldsymbol{p_{12,\mu}} \tag{3.6.3}$$

Four momenta is also conserved *during elastic collisions*:

$$\boldsymbol{p_{\mu,i}} = \boldsymbol{p_{\mu,f}} \tag{3.6.4}$$

The magnitude of $\boldsymbol{p_\mu}$ can be found via a dot product on the Minkowski metric:

$$\boldsymbol{p_\mu p^\mu} = \frac{E^2}{c^2} - p_1^2 - p_2^2 - p_3^2 \tag{3.6.5}$$

The most important feature of the four momentum is that this product, $\boldsymbol{p_\mu p^\mu}$ is frame invariant: the same in all reference frames. If we can write the four momenta in two separate reference frames, we can be sure that their magnitudes will be the same.

One consequence of this applies to the four momenta of a single particle. If we shift into the rest frame of the particle, then all of its three-momenta components are zero, so that:

$$\boldsymbol{p_\mu p^\mu} = \frac{E^2}{c^2} = m^2c^2 \tag{3.6.6}$$

Notice that this *will not* work for multiple particles, since you cannot shift into a reference frame where they are both at rest! (unless they have the same 3 momentum vector).

Before you get too excited, this is a good point to sit back and remember that $(a+b)^2 \neq a^2+b^2$. In general, before transforming frames, you will need to combine *all* the four momenta in a system to get a total four momentum.

4 Electromagnetism

4.1 A Note on Griffith's "script r"

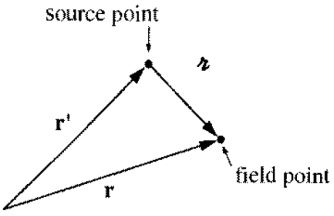


Figure 4.1.1: Diagram of Script r (Source: Griffiths "Introduction to Electrodynamics", pg. 9)

In the first chapter of *Introduction to Electromagnetism*[4.1], Griffiths defines a convenient "separation" vector, $\boldsymbol{\mathscr{r}}$, as:

$$\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}' \tag{4.1.1}$$

where $\boldsymbol{r}$ is the vector from the origin to the **field point**, and $\boldsymbol{r}'$ the vector from the origin to the **source point** (See figure 4.1.1). The important thing to remember is that $\boldsymbol{\mathscr{r}}$ points from the source point *to* the field point.

This vector is denoted by $\boldsymbol{\mathscr{r}}$, while its magnitude is denoted by $\mathscr{r}$.

In these notes, I generally use arrows rather than boldface to denote vectors, since some LaTeXcharacters don't have bold versions. However, to be consistent with Griffiths, I WILL try to use $\boldsymbol{\mathscr{r}}$ instead of $\mathscr{r}$.

Finally, the reader should be aware that I spent a fair amount of time getting the Griffiths $\mathscr{r}$ symbol set up in LaTeX. I hope you're grateful. [4.2]

Here are some generally useful facts about $\mathscr{r}$:

- $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$, $\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ (Proved in Griffiths (prob. 1.13) by direct differentiation)

4.2 The Units of Electromagnetism

Good habits of dimensional analysis have a tendency to disappear in E&M, since the units are no longer ones we are familiar with in everyday life.

Just like chemistry and thermodynamics, E&M deals with very large numbers of electrons. In chemistry and thermodynamics, we make these numbers manageable by defining Avogadro's number and the Boltzmann constant respectively. For the same reason, in E&M, we will define

[4.1] Page 9

[4.2] If you ever need to do this, you can download PDFs of the symbols and a sample file demonstrating their use from Griffith's website: http://academic.reed.edu/physics/faculty/griffiths.html

the **Columb** to be 6.241×10^{18}. It is important to remember that the columb by itself is not really a unit: it's just a number!

Charge is measured in "electrons" [4.3]; the intrinsic charge of the electron taken to be a universal constant. In practice, we will talk about columbs as a unit of charge. However, what we are really referring to is the charge of one columbs worth *of electrons*!

The most basic true unit in E&M is the **Ampere**, which is an SI base unit, defined to be the flow of one columb of electrons past a point in one second:

$$\boxed{1A = 1\frac{C}{s}} \tag{4.2.1}$$

The first *derived* unit of E&M that we should define is the **Volt**, which is defined to be the potential difference between two points that will produce 1 Joule of energy per Columb that flows between them:

$$\boxed{1V = 1\frac{J}{C}} \tag{4.2.2}$$

Directly from the volt, we can now define a unit of electric fields. Since the electric field is defined to be the gradient of the potential, it will have units of **Volts per Meter**. To my knowledge, there isn't a special name for this unit.

We can also now define a unit for capacitance. One **Farad** is defined to be the capacitance of a capacitor that holds one columb of charge and has a potential difference of one volt, leading us to write:

$$\boxed{1F = 1\frac{C}{V}} \tag{4.2.3}$$

The unit for electrical resistance come straight from Ohms law, and is the **Ohm**:

$$\boxed{1\Omega = 1\frac{V}{A}} \tag{4.2.4}$$

The units of magnetic field flux density are unfortunately rather cluttered. The current (derived) SI unit for magnetic field flux density is the **Tesla**, which is defined such that a 1 Columb test charge passing through a 1 Tesla B-field at a speed of 1 Meter per Second experiences a force of 1 Newton. In other words, by the Lorentz force law:

$$\boxed{1T = 1\frac{N \cdot s}{m \cdot C}} \tag{4.2.5}$$

There are a number of nicer ways to write this unit, but this one makes the most sense from the definition.

[4.3] Actually, the fundamental unit of charge is one-third of an electron, to account for the "fractional" charges of quarks.

4.3 Static Electric Fields

4.3.1 Electric Field Discontinuities

Electric field lines are discontinuous wherever they cross a charged surface. The exact amount of this discontinuity is highly useful, and can often be used as a boundary condition in potential theory.

The discontinuity in an electric field $\boldsymbol{E}$ crossing a surface defined by the surface normal $\hat{n}_2$ (going from side 2 to side 1, so the surface normal of section *2*) with charge σ is:

$$\hat{n}_2 \cdot (\boldsymbol{E}_1 - \boldsymbol{E}_2) = \boldsymbol{E}_{1,\perp} - \boldsymbol{E}_{2,\perp} = \frac{\sigma}{\epsilon_0} \tag{4.3.1}$$

$$\hat{n}_2 \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = \boldsymbol{E}_{1,\parallel} - \boldsymbol{E}_{2,\parallel} = 0 \tag{4.3.2}$$

The quintessential example of these discontinuities (and the easiest way to re-derive them in a pinch)is a surface charge σ on a flat surface with no external electric field. Then:

$$\boldsymbol{E}_{1,\perp} - \boldsymbol{E}_{2,\perp} = \frac{\sigma}{2\epsilon_0} - \frac{-\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} \tag{4.3.3}$$

4.3.2 Green's Reciprocity Relation

Green's reciprocity relation relation concerns two seperate charge distributions. It states that the potential energy of charge distribution A due to charge distribution B is equal to the energy of B in the field due to A. In other words:

$$U_{2,1} = \frac{1}{4\pi\epsilon_0} \int d^3r \int d^3r' \frac{\rho_2(\boldsymbol{r})\rho_1(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{4\pi\epsilon_0} \int d^3r' \int d^3r \frac{\rho_1(\boldsymbol{r}')\rho_2(\boldsymbol{r})}{|\boldsymbol{r}' - \boldsymbol{r}|} = U_{1,2} \tag{4.3.4}$$

Notice that this works because the integrals can be interchanged, and the vector difference in the denominator is a magnitude. The definition of the electrical potential allows this equation to be written in a more compact form:

$$\int d^3r \rho_2(\boldsymbol{r})\phi_1(\boldsymbol{r}) = \int d^3r' \rho_1(\boldsymbol{r}')\phi_2(\boldsymbol{r}') \tag{4.3.5}$$

This equation can sometimes be used as a trick to calculate the force between two different charge distributions.

4.3.3 Example: Force Between Spherically Symmetric Charge Distributions Using Green's Reciprocity Relation

Suppose that we have two spherically symmetric charge distributions, ρ_1 and ρ_2 (total charges Q_1 and Q_2) with 1 centered at the origin and 2 at some displacement $\boldsymbol{R}$. We want to calculate the force between these two distributions, which we can do if we can calculate the potential energy of their interaction. We know that the potential energy of ρ_1 in the field of ρ_2 is:

$$U_{1,2} = \int d^3r \rho_1(\boldsymbol{r})\phi_2(\boldsymbol{r}) \tag{4.3.6}$$

Since ρ_2 is spherically symmetric, outside of its boundarys its electric field must be identical to that of a point charge centered at $\boldsymbol{R}$. We can then rewrite ϕ_2 as $\phi_{p,2}$: the potential from a point charge.

$$U_{1,2} = \int d^3r \rho_1(\boldsymbol{r})\phi_{p,2}(\boldsymbol{r}) \tag{4.3.7}$$

Green's Reciprocity Relation allows us to rewrite this equation as:

$$U_{1,2} = \int d^3r \rho_{p,2}(\boldsymbol{r})\phi_1(\boldsymbol{r}) \tag{4.3.8}$$

The charge density of the point charge at 2 can be written as $\rho_{p,2} = \delta(\boldsymbol{r} - \boldsymbol{R})Q_2$. However, since we are now in the perspective of 2, *1* now looks like a point charge too! so $\phi_1 \to \phi_{p,1}$.

$$U_{1,2} = \int d^3r \delta(\boldsymbol{r} - \boldsymbol{R})Q_2\left(\frac{1}{4\pi\epsilon_0}\frac{Q_1}{r}\right) \tag{4.3.9}$$

Which now reduces easily to:

$$U_{1,2} = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{R} \tag{4.3.10}$$

And, finally:

$$F_{1,2} = -\frac{\partial U_{1,2}}{\partial R} = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{R^2} \tag{4.3.11}$$

4.3.4 Energy of Electrostatic Charge Distributions

Distributions of stationary charges have inherent potential energy due to their proximity to one another. Since the electrical potential is taken to be zero at infinity, charge particles have zero potential energy at infinity. Therefore, the potential energy of a charge distribution is the energy it takes to assemble this distribution by bringing each particle in one at a time from infinity. The first particle can be added for free, while the following particles must fight against the potential of all of the particles that have come before. This can be written neatly as a sum:

$$U_E = W = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N}\sum_{i<j}^{N}\frac{q_iq_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\sum_{j=1}^{N}\sum_{i\neq j}^{N}\frac{q_iq_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{4.3.12}$$

Where, in the last expression, the summations have been simplified by double counting each interaction, which is corrected by a factor of $\frac{1}{2}$.

Recognizing part of this equation as the potential of the FINISHED charge distribution, ϕ, we can write the sum as:

$$U_E = \frac{1}{2}\sum_{i=1}^{N} q_i\phi(\boldsymbol{r}_i) \tag{4.3.13}$$

For continuous distributions, this becomes:

$$U_E = \frac{1}{2}\int d^3r \rho(\boldsymbol{r})\phi(\boldsymbol{r}) \tag{4.3.14}$$

It is also worth considering the total energy of two charge distributions that have been brought

together such that $\rho(\boldsymbol{r}) = \rho_1(\boldsymbol{r}) + \rho_2(\boldsymbol{r})$:

$$U_E(\rho) = U_E(\rho_1) + U_E(\rho_2) + \frac{1}{4\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho_1(\boldsymbol{r})\rho_2(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{4.3.15}$$

The first two terms, $U_E(\rho_1)$ and $U_E(\rho_2)$ are the energies of the two charge distributions by themselves, while the third term is known as the **interaction energy**.

The energy in an electric configuration can also be computed by integrating over the square of the electric field:

$$U_E = \frac{1}{2}\epsilon_0 \int d^3r |\boldsymbol{E}(\boldsymbol{r})|^2 \tag{4.3.16}$$

It is important to remember that this method of calculating the energy of a charge distribution necessarily contains the self-energy of the particles in question, as well as their interaction energy. This expression must therefore only be used with non-singular charge distributions, since point charges will lead to pathological infinite energies [4.4].

This equation is the basis of the definition of the **electric field energy density**:

$$u_E(\boldsymbol{r}) = \frac{1}{2}\epsilon_0 |\boldsymbol{E}(\boldsymbol{r})|^2 \tag{4.3.17}$$

4.3.5 Example: Electric field of a Uniformly Charged Sphere

Consider a sphere of radius R and a total charge Q uniformly distributed throughout its volume. Since the problem has spherical symmetry, we can use Gauss's law to calculate the electric field inside and outside the sphere.

First, let us chose an arbitrary surface of integration on which to apply Gauss's law. We will chose a sphere of radius s, concentric with the first sphere.

Consider first the case where $s < R$, in order to find the field within the sphere. In order to apply Gauss's law, we need to calculate how much charge is contained within this sphere. The charge density of the body is:

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi r^3} \tag{4.3.18}$$

Therefore, the charge inside the surface of integration is: $Q_s = \frac{s^3}{R^3}Q$. Gauss's law then says that:

$$\int_{sphere} \boldsymbol{E} \cdot d\vec{a} = \frac{Q_s}{\epsilon_0} \tag{4.3.19}$$

Since $\boldsymbol{E} \perp d\boldsymbol{a}$ and is constant over the entire surface, we can easily carry out the dot product on the LHS and pull $|\boldsymbol{E}|$ out of the integral. The remaining integral is just the surface area of the sphere!

$$|\boldsymbol{E}|(4\pi s^2) = \frac{Q_s}{\epsilon_0} \tag{4.3.20}$$

Plugging in for Q_s, we have:

$$|\boldsymbol{E}| = \frac{1}{4\pi\epsilon_0}\frac{Qs}{R^3} \tag{4.3.21}$$

[4.4] For more discussion, see Zangwill pg. 78.

Now, if $s > R$, $Q_s = Q$, so our task is much simpler. Following the same procedure, we see that:

$$|\boldsymbol{E}| = \frac{1}{4\pi\epsilon_0}\frac{Q}{s^2} \tag{4.3.22}$$

So, our final result is:

$$\boxed{\boldsymbol{E}_{s<R} = \frac{1}{4\pi\epsilon_0}\frac{Qs}{R^3}\hat{s}} \tag{4.3.23}$$

$$\boxed{\boldsymbol{E}_{s>R} = \frac{1}{4\pi\epsilon_0}\frac{Q}{s^2}\hat{s}} \tag{4.3.24}$$

4.4 The Scalar Potential

4.4.1 Definition of the Scalar Potential

The Helmholtz decomposition theorem tells us that any vector field can be broken into a curl free and a divergence free portion: $\boldsymbol{F} = -\nabla\phi + \nabla \times \boldsymbol{A}$. Since Maxwell's equations tell us that $\nabla \times \boldsymbol{E} = 0$, we can make an even stronger statement: that the electric field can be represented entirely by the curl free portion of the function above!:

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}) \tag{4.4.1}$$

The Helmholtz theorem states that, as long as $\boldsymbol{E}$ falls off faster than $\frac{1}{r}$ as $r \to \infty$, ϕ can be written:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi}\int_{allspace}\frac{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}dV' \tag{4.4.2}$$

However, from another of Maxwell's equations we know that $\nabla \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$, so we can rewrite this equation as:

$$\boxed{\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_{allspace}\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}dV'} \tag{4.4.3}$$

It is important to note that the physically important function $\boldsymbol{E}$ (which produces forces etc.) depends only on the *gradient* of ϕ. Therefore, since $\nabla(\phi + c) = \nabla\phi$, the scalar potential is arbitrary up to an additive constant. This is known as **gauge freedom**. Although we will often chose $c = 0$ for simple calculations, there are several other gauge choices that will end up being convenient for more complicated problems.

4.4.2 Example: Potential of a Uniformly Charged Sphere

In the previous example, we found the electric field inside and outside a uniformly charged sphere. We will now integrate these fields to find the appropriate potentials.

Outside the sphere, the electric field is simply that of a point particle, so the potential must *also* be that of a point particle. However, if this does not satisfy you, you can easily carry out the integration:

$$V(s) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{s}\frac{Q}{s^2}\hat{s} \cdot d\boldsymbol{s} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{s}} \tag{4.4.4}$$

Inside the sphere, we will need to split the integral into two parts:

$$V(s) = -\Big(\int_{\infty}^{R} \boldsymbol{E}_{s>R} \int_{R}^{s} \boldsymbol{E}_{s<R} \Big) \tag{4.4.5}$$

Plugging in our electric field from the previous example, we have:

$$V(s) = -\frac{Q}{4\pi\epsilon_0} \Big(\int_{\infty}^{R} \frac{1}{s'^2} ds' \frac{1}{2R^3} \int_{R}^{s} s'^2 ds' \Big) \tag{4.4.6}$$

Evaluating these integrals leaves:

$$V(s) = -\frac{Q}{4\pi\epsilon_0} \Big(-R + \frac{1}{2R^3}(s^2 - r^3) \Big) \tag{4.4.7}$$

This is then nicely rewritten by pulling out a $\frac{1}{2R}$:

$$\boxed{V(s) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2R} \Big(3 - \frac{s^2}{R^2} \Big)} \tag{4.4.8}$$

4.4.3 Example: Potential of a Uniformly Charged Spherical Shell (by direct integration)

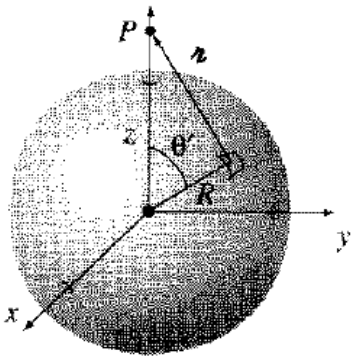


Figure 4.4.1: Uniformly charged sphere variable choice (Source: Griffiths E&M pg. 85)

We wish to calculate the potential of the pictured uniformly charged spherical shell (radius r, charge per unit area σ) by direct integration [4.5]. Note that, by symmetry, we may choose the point at which we measure the field to be along the z-axis.

Now we must find an expression for $\mathscr{r}$ in terms of r, r', and θ. Luckily the law of cosines provides just such an expression:

$$\mathscr{r}^{\,2} = r^2 + r'^2 - 2rr' \cos\theta \tag{4.4.9}$$

For a spherical shell, r' is constant, so lets set $r' = R$, and $r = s$. Therefore, the integral we are

[4.5] This problem and solution are directly from Griffiths, pg. 85.

must solve is simply:

$$V(s) = \frac{1}{4\pi\epsilon_0}\int_{surface}\frac{\sigma}{\not{r}}d\not{A}' = 2\pi\frac{1}{4\pi\epsilon_0}\int_0^\pi \frac{\sigma R^2\sin\theta}{\sqrt{s^2+R^2-2sR\cos\theta}}d\theta \tag{4.4.10}$$

Making the u-substitution $u = s^2 + R^2 - 2sR\cos\theta$ turns the integral into:

$$V(s) = \frac{1}{4\pi\epsilon_0}\frac{\pi R\sigma}{s}\int u^{-\frac{1}{2}}du \tag{4.4.11}$$

Which evaluates to:

$$V(s) = \frac{2\pi R\sigma}{s}\left[\sqrt{(R+s)^2} - \sqrt{(R-s)^2}\right] \tag{4.4.12}$$

Or, alternately, utilizing the fact that $|x| = \sqrt{x^2}$:

$$V(s) = \frac{2\pi R\sigma}{s}\Big[|R+s| - |R-s|\Big] \tag{4.4.13}$$

Now inside the sphere $s < R$, while outside the shell $s > R$. This expression then simplifies to the final answers:

$$\boxed{V_{s>R,outside}(s) = \frac{R^2\sigma}{\epsilon_0 s}} \tag{4.4.14}$$

$$\boxed{V_{s<R,inside}(s) = \frac{R\sigma}{\epsilon_0}} \tag{4.4.15}$$

4.5 Capacitance

4.5.1 Example: Capacitance of a Parallel Plate Capacitor

Imagine a set of parallel plates, each with area A, set a distance d apart. Suppose some charge is moved from one plate to the other by way of a connecting wire, so that the plates have charges of Q and $-Q$ respectively. What is the capacitance of this system?

Capacitance in general is defined as:

$$C = \frac{Q}{V} \tag{4.5.1}$$

Since we have imagined that we know the charge on the plates, what we need to do is calculate the voltage between the plates. We can do this by integrating the electric field, which we know must be $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$:

$$V = \int_0^d E dx = \frac{Qd}{A\epsilon_0} \tag{4.5.2}$$

Then, plugging in to the first equation:

$$C = \frac{A\epsilon_0}{d} \tag{4.5.3}$$

4.5.2 Energy Stored in a Charged Capacitor

In order to charge a capacitor, we first place one positive and one negative electron on each plate. Each subsequent electron that we add to these plates will then have to fight against the potential of the first electron (and so forth) in order to be placed on the plate. The total energy is then equal to the total amount of work done during this process:

$$W = \int_0^Q V(q)dq \tag{4.5.4}$$

Where $V(q)$ is the potential on the plate as a function of the charge currently on the plate. However, since we also know that for a capacitor, $C = \frac{Q}{V} \rightarrow V = \frac{Q}{C}$, we have

$$W = \int_0^Q \frac{q}{C}dq \tag{4.5.5}$$

or, evaluating the integral:

$$\boxed{W = \frac{1}{2}\frac{Q^2}{C}} \tag{4.5.6}$$

This can be rewritten in terms of V, again using $C = \frac{Q}{V}$:

$$\boxed{W = \frac{1}{2}QV = \frac{1}{2}CV^2} \tag{4.5.7}$$

4.6 Potential Theory (Poisson and Laplace Equations)

Poisson's equation states that:

$$\nabla^2 = \phi(\boldsymbol{r}) = -\frac{\rho_{free}(\boldsymbol{r})}{\epsilon_0} \tag{4.6.1}$$

In the case of a region containing zero free charge (for example a cavity or capacitor), this reduces to **Laplace's equation**:

$$\boxed{\nabla^2\phi(\boldsymbol{r}) = 0} \tag{4.6.2}$$

General solutions to this equation can be found by separation of variables in each coordinate system. The particular solution is then identified by applying two boundary conditions.

First, we require that the potential be continuous across boundaries. If this were not the case, the electric field would be infinite at the discontinuity, which is unphysical. For example, if $\phi_1(\boldsymbol{r})$ and $\phi_2(\boldsymbol{r})$ are potentials in two regions that share a boundary $\boldsymbol{r}_s$, then:

$$\boxed{\phi_1(\boldsymbol{r}_s) = \phi_2(\boldsymbol{r}_s)} \tag{4.6.3}$$

The second condition comes from the requirement that:

$$\epsilon_1\boldsymbol{E}_1 \cdot \hat{n} - \epsilon_2\boldsymbol{E}_2 \cdot \hat{n} = \sigma_f \tag{4.6.4}$$

Which provides the following condition on the potential[4.6]:

$$\boxed{\epsilon_1 \frac{\partial \phi_1}{\partial n}\bigg|_{\boldsymbol{r}_s} - \epsilon_2 \frac{\partial \phi_2}{\partial n}\bigg|_{\boldsymbol{r}_s} = \sigma_f} \tag{4.6.5}$$

4.6.1 Cartesian Solution

The solution to Laplace's equation in Cartesian symmetry is:

$$X_\alpha(x) = \begin{cases} A_0 + B_0 x & \alpha = 0 \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0 \end{cases} \tag{4.6.6}$$

$$Y_\beta(y) = \begin{cases} C_0 + D_0 y & \beta = 0 \\ C_\beta e^{\beta y} + D_\beta e^{-\beta y} & \beta \neq 0 \end{cases} \tag{4.6.7}$$

$$Z_\gamma(z) = \begin{cases} E_0 + F_0 x & \gamma = 0 \\ E_\alpha e^{\gamma z} + F_\gamma e^{-\gamma z} & \gamma \neq 0 \end{cases} \tag{4.6.8}$$

Where α^2, β^2, and γ^2 are the separation constants, and we require that:

$$\alpha^2 + \beta^2 + \gamma^2 = 0 \tag{4.6.9}$$

Which means that at least ONE of the three constants must be imaginary (unless all three are zero).

4.6.2 Cylindrical Solution

Under cylindrical symmetry, Laplace's equation separates into:

$$\frac{d^2 G}{d\phi^2} + \alpha^2 G = 0 \tag{4.6.10}$$

$$\frac{d^2 Z}{dz^2} + k^2 Z = 0 \tag{4.6.11}$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + (k^2 \rho^2 - \alpha^2) R = 0 \tag{4.6.12}$$

The first two of these equations yield exponential/linear solutions:

$$G_\alpha(\phi) = \begin{cases} C_0 + D_0 \alpha & \alpha = 0 \\ C_\alpha e^{i\alpha\phi} + D_\alpha e^{-i\alpha\phi} & \alpha \neq 0 \end{cases} \tag{4.6.13}$$

[4.6] Note that both derivatives here are written with respect to the surface normal of ONE side of the surface. Some books use both surface normals and drop the minus sign.

$$Z_k(z) = \begin{cases} E_0 + F_0 y & k = 0 \\ E_k e^{kz} + F_k e^{-kz} & k \neq 0 \end{cases} \tag{4.6.14}$$

Notice that, since α and k can be either real or imaginary as required by boundary conditions, these are essentially the same solutions as to the linear equation. The i's in the first equation are included because, in general, one of these two directions will be oscillatory.

The remaining equation, eq. 4.6.12, is the Bessel Function Equation. However, it only produces Bessel functions as solutions when $k \neq 0$:

$$R_\alpha^k(\rho) = \begin{cases} A_\alpha^k J_\alpha(k\rho) + B_\alpha^k N_\alpha(k\rho) & k^2 > 0, \alpha \neq 0 \\ A_\alpha^k I_\alpha(k\rho) + B_\alpha^k K_\alpha(k\rho) & k^2 > 0, \alpha \neq 0 \end{cases} \tag{4.6.15}$$

Where I_α and K_α are the modified Bessel functions (which will rarely appear). In words, the important thing to remember is that **if $k \neq 0$, the solution is a linear combination of Bessel functions of $k\rho$.** In the case where $k = 0$, however, the solutions are:

$$R_\alpha^0(\rho) = \begin{cases} A_0 + B_0 \log \rho & k = 0, \alpha = 0 \\ A_\alpha^0 \rho^\alpha + B_\alpha^0 \rho^{-\alpha} & k = 0, \alpha \neq 0 \end{cases} \tag{4.6.16}$$

In theory, we would perform separation of variables to find the values and signs of α and k for each problem. However, the best way to approach this type of problem is to apply the boundary conditions on ϕ and z first. Seeing which functions will fit those conditions will *tell* you the values of α and k, allowing you to pick the appropriate radial solution.

4.6.3 A Note on Scale Independence

A clever trick greatly simplifies potential problems that are scale independent in one or more directions. If a problem looks identical if you were to scale the z axis by a factor, then the solution cannot depend on z, meaning that $Z(z) =$ constant.

4.7 The Multipole Expansion (Monopoles and Dipoles)

4.7.1 The Multipole Expansion

Calculating potentials of arbitrary charge distributions can be extremely messy. However, the farther away from the charge distribution we go, the less the specific details of the charge distribution matter! The multipole expansion will allow us to approximate the field far away from the source.

For an arbitrary charge distribution $\rho(\boldsymbol{r}')$, the potential is:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\boldsymbol{r}') dV' \tag{4.7.1}$$

From the law of cosines (and figure 4.1.1) we see that

$$\mathscr{r}^{\,2} = r^2 + r'^2 - 2rr'\cos\theta \tag{4.7.2}$$

Or, letting [4.7] $\epsilon = (\frac{r'}{r})(\frac{r'}{r} - 2\cos\theta')$:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-\frac{1}{2}} \tag{4.7.3}$$

The binomial theorem tells us that we can expand the $(1+\epsilon)$ term into an infinite series, and that since $\epsilon << 1$, the series will be well approximated by the first several terms. Therefore we will write:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + ...\right) \tag{4.7.4}$$

In effect, we are now done: these three terms provide an approximation of the potential, and are known as the **monopole**, **dipole**, and **quadrupole** terms respectively. Of course the point at which we truncate the series is arbitrary, depending on how small ϵ is, and how precisely we want to specify $V(\boldsymbol{r})$. We could use only the first two terms, or add a forth (which would be the **octopole** term).

It is worth remembering that, for the nth term by itself, $V \sim \frac{1}{r^n}$. Therefore the monopole term falls off as $\frac{1}{r}$, the dipole as $\frac{1}{r^2}$, etc.

This expression can be somewhat simplified by plugging back in for ϵ and collecting like powers of $\frac{r'}{r}$. It then turns out that the coefficients of these like terms are just the Legendre polynomials! The potential can then be (exactly) written as:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\boldsymbol{r}')dV' \tag{4.7.5}$$

Or, explicitly writing the first several terms:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\boldsymbol{r}')dV' + \frac{1}{r^2}\int r'\cos\theta'\rho(\boldsymbol{r}')dV' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\boldsymbol{r}')dV' + ...\right] \tag{4.7.6}$$

4.7.2 The Monopole and Dipole Potentials

We will now consider the first several terms individually. The monopole term is simply the potential as if all of the charge of the distribution was located at a point at its center:

$$V_{mono}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \tag{4.7.7}$$

The dipole term is slightly more complicated. We have:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\boldsymbol{r}')dV' \tag{4.7.8}$$

[4.7] There is a prime on the θ because varying this angle will correspond to selecting different parts of the volume of the charge distribution, which is the primed r-variable

By noting that $\hat{r} \cdot \boldsymbol{r}' = r' \cos\theta'$, we can rewrite this as:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \cdot \int r' \rho(\boldsymbol{r}') dV' \tag{4.7.9}$$

The integral in this expression is independent of $\boldsymbol{r}$, and is often therefore packaged inside a new vector, known as the **dipole moment**:

$$\boldsymbol{p} = \int r' \rho(\boldsymbol{r}') dV' \tag{4.7.10}$$

The dipole potential can now be rewritten simply as:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \hat{r}}{r^2} \tag{4.7.11}$$

From this definition of $\boldsymbol{p}$ we also see that for a discrete distribution of n charges q_i:

$$\boldsymbol{p} = \sum_{i=1}^{n} q_i \boldsymbol{r_i'} \tag{4.7.12}$$

It is clear from this last expression that, in general *the dipole moment is dependent on the choice of origin.* It turns out [4.8] that the dipole moment is *in*dependent of the choice of origin IF the total charge $Q = \sum_{i=1}^{n} q_i = 0$!

4.7.3 The Spherical Multipole Expansion

In some situations (in fact, most situations), it is more convenient to expand $\frac{1}{\mathscr{r}}$ directly in spherical coordinates. In terms of the Legendre polynomials, we can expand[4.9]:

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\hat{r} \cdot \hat{r}'), \quad \text{for } r' < r \tag{4.7.13}$$

Notice that $\hat{r} \cdot \hat{r}' = \cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$. The Addition Theorem for Spherical Harmonics says that:

$$P_l(\cos(\gamma)) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^{*m}(\theta', \phi') Y_l^m(\theta, \phi) \tag{4.7.14}$$

So, making this substitution, we can now write the multipole expansion directly as:

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \frac{4\pi}{2l+1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{r'}{r}\right)^l Y_l^{*m}(\theta', \phi') Y_l^m(\theta, \phi), \quad \text{for } r' < r \tag{4.7.15}$$

Using this result, the spherical multipole potential can be written as:

$$\Phi(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_l^m \frac{Y_l^m(\theta, \phi)}{r^{l+1}} + B_l^m r^l Y_l^m(\theta, \phi) \right] \tag{4.7.16}$$

[4.8] Griffiths pg. 151-152

[4.9] See Zangwill pg. 107. If you're optimistic, try looking in Jackson.

Where the first term is the **exterior** multipole expansion, and the second term is the **interior** multipole expansion. The constants are the multipole moments:

$$A_l^m = \frac{4\pi}{2l+1}\int d^3r'\rho(\boldsymbol{r}')r'^l Y_l^{m*}(\theta',\phi'), \qquad B_l^m = \frac{4\pi}{2l+1}\int d^3r'\frac{\rho(\boldsymbol{r}')}{r'^{l+1}}Y_l^{m*}(\theta',\phi') \tag{4.7.17}$$

If the problem has azimuthal symmetry (like all good problems), these equations reduce to:

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[A_l\frac{P_l(\cos(\theta))}{r^{l+1}} + B_l r^l P_l(\cos(\theta))\right] \tag{4.7.18}$$

Where again the first and second terms are the exterior and interior expansions respectively. The constants can are now:

$$A_l = \int d^3r'\rho(\boldsymbol{r}')r'^l P_l(\cos(\theta')), \qquad B_l = \int d^3r'\frac{\rho(\boldsymbol{r}')}{r'^{l+1}}P_l(\cos(\theta')) \tag{4.7.19}$$

4.7.4 The Electric Field of a Dipole

Now that we have the potential of a dipole, finding the electric field is simply a matter of taking the gradient (albeit in spherical coordinates):

$$\boldsymbol{E}_{dip}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{4.7.20}$$

A few qualitative observations of this field are worth noting (and very common pGRE material):

- The field falls off as $\frac{1}{r^3}$
- When $\theta = 0$, the field points in the $\hat{\theta}$ direction.

4.7.5 Torques and Forces on Physical Dipoles

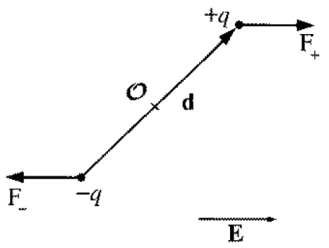


Figure 4.7.1: Torque on a physical dipole (Source: Griffiths "Introduction to Electrodynamics", pg. 164)

A physical dipole consists of two opposite charges on either end of a (mass-less, charge-less, insulating) stick. When placed in an electric field, the dipole will rotate and/or translate, depending on the specifics of the field.

Consider first the torque on a dipole (pictured in fig. 4.7.1). The torque can be directly written as:

$$\boldsymbol{N} = \left(\frac{\boldsymbol{d}}{2} \times q\boldsymbol{E}\right) + \left(\frac{-\boldsymbol{d}}{2} \times -q\boldsymbol{E}\right) = q\boldsymbol{d} \times \boldsymbol{E} \tag{4.7.21}$$

or

$$\boxed{\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}} \tag{4.7.22}$$

If $\boldsymbol{E}$ is uniform, the net force on the dipole is clearly zero. However, if $\boldsymbol{E}$ is NOT uniform, then

$$\boldsymbol{F} = \boldsymbol{F}_+ + \boldsymbol{F}_- = q(\boldsymbol{E}_+ - \boldsymbol{E}_-) = q\Delta\boldsymbol{E} \tag{4.7.23}$$

Where $\Delta\boldsymbol{E} = \boldsymbol{E}_+ - \boldsymbol{E}_-$. Assuming that the dipole is very small, we can approximate $\Delta\boldsymbol{E} = \boldsymbol{d}\cdot\nabla\boldsymbol{E}$ which then gives us the usual expression for the force on a dipole:

$$\boxed{\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E}} \tag{4.7.24}$$

Given these equations, we can now calculate the potential energy of a pure dipole $\boldsymbol{p}$ in an electric field $\boldsymbol{E}$. Imagine bringing the dipole in from infinity to the origin. If take a path perpendicular to $\nabla\boldsymbol{E}$, we will do no work during this stage. Now, we will rotate from it's current position at the origin (which will be making a right angle with $\boldsymbol{E}$, since we moved in along the gradient) to an angle θ. The work required to do this is

$$\int_{\frac{\pi}{2}}^{\theta} pE\sin\theta' d\theta' = -pE\cos\theta = -\boldsymbol{p}\cdot\boldsymbol{E} \tag{4.7.25}$$

So, therefore:

$$\boxed{U_{dip} = -\boldsymbol{p}\cdot\boldsymbol{E}} \tag{4.7.26}$$

4.8 Polarizability and Dielectrics

4.8.1 Deriving the Bound Surface and Volume Charges

For a dipole moment per unit volume $P(\boldsymbol{r}')$, we have the following potential:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\boldsymbol{P}(\boldsymbol{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2} dV' \tag{4.8.1}$$

Noting that $\nabla'\left(\frac{1}{\imath}\right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2}$ (derived by explicit differentiation in Griffiths), we can rewrite this expression as:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}(\boldsymbol{r}')\cdot\nabla'\left(\frac{1}{\imath}\right) dV' \tag{4.8.2}$$

Now, integrating by parts (and using the product rule on the first term):

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V \nabla'\cdot\left(\frac{\boldsymbol{P}(\boldsymbol{r}')}{\imath}\right) dV' - \int_V \frac{1}{\imath}(\nabla'\cdot\boldsymbol{P}(\boldsymbol{r}'))dV'\right] \tag{4.8.3}$$

Applying the divergence theorem to the first term yields:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\oint_S \left(\frac{\boldsymbol{P}(\boldsymbol{r}')}{\mathscr{r}}\right)\cdot d\vec{a}' - \int_V \frac{1}{\mathscr{r}}(\nabla'\cdot\boldsymbol{P}(\boldsymbol{r}'))dV'\right] \tag{4.8.4}$$

We will now define the surface and bound charges in order to simplify this expression. Note that the first term looks like the potential of a surface charge distribution. We will therefore define:

$$\boxed{\sigma_b = \boldsymbol{P}(\boldsymbol{r}')\cdot\hat{n}} \tag{4.8.5}$$

Where $\hat{n}$ is the surface normal, parallel to $d\boldsymbol{a}'$. The second term, on the other hand, looks like the potential of a strange volume charge distribution, so we will define:

$$\boxed{\rho_b = -\nabla\cdot\boldsymbol{P}(\boldsymbol{r}')} \tag{4.8.6}$$

We can use these "bound charges" the same way as any charges, calculating electric fields etc. However, they do allow us to easily rewrite the potential of our object:

$$V_{dip}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}}da' - \int_V \frac{\rho_b}{\mathscr{r}}dV'\right] \tag{4.8.7}$$

4.8.2 Polarizability

When a system of charges that are free to move (such as particles in an atom or charges on the surface of a conductor) is subjected to an electric field, the position of the changes will be rearranged to try and counteract the external field. For example, in an atom, the electron will be pushed towards one side of the electron cloud. The strength of this response (and therefore the electric field it creates) is called the **polarizability** of the system.

When an object is polarized in 1D, it acquires an electric dipole moment:

$$\boldsymbol{p} = \alpha\boldsymbol{E} \tag{4.8.8}$$

Where $\boldsymbol{E}$ is the incident electric field, and α is the polarizability of the system (a constant).

For general 3D systems, the polarizability α is replaced by a corresponding constant tensor:

$$p_i = \alpha_{ij}E_j \tag{4.8.9}$$

This tensor can be diagonalized to find the principle axes of the system's polarizability: much like the similar process for an inertia tensor.

We will generally be more interested in the polarization *density* in a material, defined as:

$$\underline{\mathrm{P}} = \frac{d\boldsymbol{p}}{dV} \tag{4.8.10}$$

I have no idea why the usual convention about lower case letters and density was reversed here, but it unfortunately was so watch out!

4.8.3 The “Fake" Field

There is a useful calculation trick for finding the electric field produced by the polarization of an object derived in Zangwill, pg. 162. The “fake field" $\mathcal{E}(r)$, is the field of the object if, instead of some polarization, it had a uniform charge of $\rho = 1$. This field can then be easily calculated using the normal tools of electrostatics. Once $\mathcal{E}(r)$ has been found, the electric field *due to the polarization of the object* is:

$$\boldsymbol{E}_P(r) = -(\boldsymbol{P} \cdot \nabla)\mathcal{E}(r) \tag{4.8.11}$$

4.9 Fields in Matter

4.9.1 D and H

When working in regions where materials are electrically or magnetically polarization (dielectrics, magnetizable materials, etc.) it is possible to define modified versions of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields **that are created only by free charges and currents**:

$$\boxed{\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}} \tag{4.9.1}$$

$$\boxed{\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}} \tag{4.9.2}$$

If the matter's polarization response is linear and isotropic[4.10], these equations can be further simplified. In the most general 3Dd anisotropic case[4.11]:

$$P_i = \sum_j \epsilon_0 \chi_{ij} E_j \tag{4.9.3}$$

Where χ_{ij} is the electric susceptibility tensor. If the material is isotropic, then this tensor becomes a scalar:

$$\boldsymbol{P} = \epsilon_0 \chi_E \boldsymbol{E} \tag{4.9.4}$$

If the system is linear, then, we assume that $\boldsymbol{P} \propto \boldsymbol{E}$, so that χ_E is constant. Then:

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \epsilon_0 \chi_E \boldsymbol{E} \tag{4.9.5}$$

So, defining ϵ such that:

$$\boxed{\chi_E = \frac{\epsilon}{\epsilon_0} - 1} \tag{4.9.6}$$

We arrive at:

$$\boxed{\boldsymbol{D} = \epsilon \boldsymbol{E}} \tag{4.9.7}$$

[4.10] Linear

[4.11] Notice that the ϵ_0 that appears here will not be replaced by an analogous μ_0 in the version for M. This asymmetry is produced by our desire for the form of χ_E and χ_M to be the same.

Similarly, if we define:

$$\boxed{\chi_M = \frac{\mu}{\mu_0} - 1} \tag{4.9.8}$$

Then:

$$\boxed{\mu \boldsymbol{H} = \boldsymbol{E}} \tag{4.9.9}$$

Note that μ is seemingly on the wrong side of the second equation. This is the result of a historical misunderstanding in which $\boldsymbol{H}$ was considered the physical field (rather than $\boldsymbol{B}$) for a time. Unfortunately, the convention stuck before the mistake was uncovered.

4.10 Boundary Conditions

We now have all the tools available to state the most general boundary conditions on $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. For memorization sake, it is easiest to remember the boundary conditions on $\boldsymbol{D}$ and $\boldsymbol{H}$:

$$\begin{aligned}
(\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \hat{n} &= \sigma_f \\
(\boldsymbol{D}_1 - \boldsymbol{D}_2) \times \hat{n} &= (\boldsymbol{P}_1 - \boldsymbol{P}_2) \times \hat{n} \\
(\boldsymbol{H}_1 - \boldsymbol{H}_2) \cdot \hat{n} &= -(\boldsymbol{M}_1 - \boldsymbol{M}_2) \cdot \hat{n} \\
(\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \hat{n} &= \boldsymbol{K}_f
\end{aligned}$$

Note which side the $\hat{n}$'s are on in the cross products, as switching that is an easy way to make a sign error. To recover the $\boldsymbol{E}$ and $\boldsymbol{B}$ conditions, $\sigma_f \to \sigma$, $\boldsymbol{K}_f \to \boldsymbol{K}$, $\boldsymbol{M} = \boldsymbol{P} = 0$, and the relevant ϵ_0's and μ_0's appear[4.12]:

$$\begin{aligned}
(\boldsymbol{E}_1 - \boldsymbol{E}_2) \cdot \hat{n} &= \frac{\sigma}{\epsilon_0} \\
(\boldsymbol{E}_1 - \boldsymbol{E}_2) \times \hat{n} &= 0 \\
(\boldsymbol{B}_1 - \boldsymbol{B}_2) \cdot \hat{n} &= 0 \\
(\boldsymbol{B}_1 - \boldsymbol{B}_2) \times \hat{n} &= \mu_0 \boldsymbol{K}
\end{aligned}$$

Integrating these conditions then produces boundary conditions on the scalar potentials $\phi = -\int \boldsymbol{E} \cdot d\boldsymbol{l}$ and $\psi = -\int \boldsymbol{H} \cdot d\boldsymbol{l}$.

4.11 The Method of Images

The method of images relies on the fact that, for a given charge distribution, there is only one possible potential (that is, the solution to Laplace's equation is unique). Given this, if a solution can be found that matches the given boundary conditions, it must be correct, regardless of how it was found.

We can exploit this trick by replacing a difficult problem with a simpler one that shares the same boundary conditions, usually by adding additional charges that lie outside the region of interest. For example, a the potential of a point charge over a conducting plane can be found by replacing the plane with a second "mirror" point charge below the plane. The resulting solution

[4.12] ϵ_0, not ϵ, since if we are using $\boldsymbol{E}$ we're not in a dielectric!

is obviously only valid *above* the plane.

Deciding where to place these image charges is more of an art than a science. However, the general guideline is to start by trying to get the correct potential on each surface.

4.11.1 Some Useful Image Charges

There are several image charge distributions that occur commonly enough to be worth memorizing:

- **Point charge above a conducting plane**. The appropriate image charge is an equal and opposite point charge placed an equal distance below the plane.
- **Point charge a distance s from a conducting sphere**.

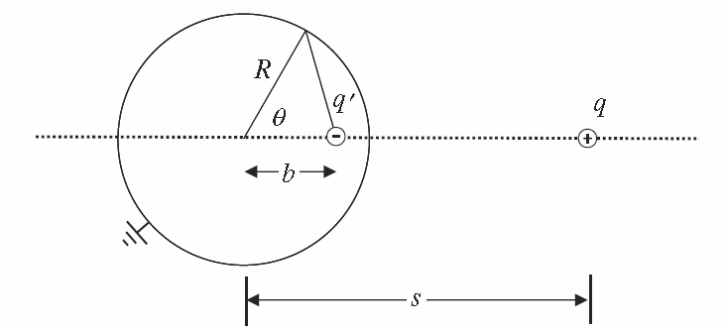


Figure 4.11.1: Image charge for a conducting sphere (Source: Zangwill E&M: pg. 246)

The appropriate image charge is placed a distance $b = \frac{R^2}{s}$ from the center of the sphere, with a charge $q' = -\frac{R}{s}q$.

If the conducting sphere is held at a potential other than zero, add an additional image charge at the center of the sphere to mimic this potential.

In particular, if the sphere holds a charge Q, the image charge at its center should be $q'' = Q + \frac{qR}{s}$ (taking into account the other image charge).

4.12 Green Functions

The Green function for a problem is a mathematical object that allows us to easily find the potential for a wide range of possible situations. The goal is to find a function $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ for the problem such that

$$\nabla^2 \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{\epsilon_0}\delta(\boldsymbol{r}, \boldsymbol{r}') \tag{4.12.1}$$

If this is true, then Green's Theorem:

$$\int_V f\nabla^2 g - g\nabla^2 f d^3r = \int_S \hat{n} \cdot (f\nabla g - g\nabla f) dS \tag{4.12.2}$$

Reduces, with the choices $f = \phi$ and $g = G$ (remembering that $\nabla^2\phi = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}$):

$$\phi(\boldsymbol{r}) = \int_V \rho(\boldsymbol{r}')\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')d^3r' - \epsilon_0 \int_S \phi(\boldsymbol{r}')\frac{\partial \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'}dS' + \epsilon_0 \int_S \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')\frac{\partial \phi(\boldsymbol{r}')}{\partial n'}dS' \tag{4.12.3}$$

In principle, if we can find $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ for a given geometry, this equation now gives us the potential of the charge distribution $\rho(\boldsymbol{r})$.

In practice, we usually deal with two main types of boundary conditions on $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$: **Dirichlet**, where $\boldsymbol{G}(\boldsymbol{r}_S, \boldsymbol{r}') = 0$ (along the surface), or **Neumann**, where $\frac{\partial \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')}{dn'} = -\frac{1}{\epsilon_0 A}$.

In the first (and most common) of these cases, the Dirichlet boundary condition causes the last integral to be zero, simplifying the expression above to:

$$\boxed{\phi(\boldsymbol{r}) = \int_V \rho(\boldsymbol{r}')\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')d^3r' - \epsilon_0 \int_S \phi(\boldsymbol{r}')\frac{\partial \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')}{\partial n'}dS'} \tag{4.12.4}$$

While for the Neumann condition, the second integral becomes $\langle\phi\rangle_S$:

$$\boxed{\phi(\boldsymbol{r}) = \langle\phi\rangle_S + \int_V \rho(\boldsymbol{r}')\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')d^3r' + \epsilon_0 \int_S \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')\frac{\partial \phi(\boldsymbol{r}')}{\partial n'}dS'} \tag{4.12.5}$$

4.12.1 Physical Interpretation

The Green function for a problem is the potential created by a single point charge (the delta potential described in equation 4.12.1). This potential can be conceptually split into two parts: G_0 (the field of the point charge) and F, the field of all other charges in the system the arise in *response* to the point charge. As such:

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = G_0 + F \tag{4.12.6}$$

There is a **very important connection to the method of images** here. For example, take the classic image charge problem of a point charge near a conducting plane. The potential of the point charge is G_0, while the potential of the *image* charge placed below the plane is F! The result is that the Green function for this problem can be determined directly using the method of images, without any complicated mathematics.

4.12.2 Specific Green Functions

There are many mathematical methods for coaxing Green functions out of problems, many of which are well covered in Zangwill. However, there are a few well known Green functions that are worth knowing:

- **Point charge a distance a from a conducting (uncharged) sphere**:

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - \frac{1}{|\frac{r'}{a}\boldsymbol{r} - \frac{a}{r'}\boldsymbol{r}'|} \tag{4.12.7}$$

  We can derive this equation by knowing the image charge for a conducting sphere discussed earlier. If a point charge q is brought a distance a (on axis) from a conducting sphere of radius R, the overall potential in space (from BOTH the real charge and the image charge) is:

$$\phi(r) = \frac{q}{r - s} + \frac{-q\frac{R}{s}}{r - \frac{R^2}{s}} \tag{4.12.8}$$

If we let $q = 1$ and rearrange the second term, we get:

$$\phi(r) = \frac{1}{r-s} + \frac{R}{rs - R^2} = \frac{1}{r-s} + \frac{R}{s(R - \frac{sR^2}{s^2})} \tag{4.12.9}$$

Which is Jackson eq. 2.16. This can then be simply rearranged into the on-axis version of the Green function again (where $r' = s$):

$$\phi(r) = \frac{1}{r-s} + \frac{R}{\frac{r}{R}s - \frac{R}{s}s)} \tag{4.12.10}$$

4.13 Magnetic Multipole Expansion

4.13.1 Magnetic Dipoles

Physically, magnetic dipoles are loops of current. The **magnetic dipole moment** (usually symbolized by m or μ) is defined from the vector potential multipole expansion as:

$$\boldsymbol{m} = \frac{1}{2}\int_V d^3r \boldsymbol{r}(r) \times \boldsymbol{j}(r) \tag{4.13.1}$$

Where $\boldsymbol{m}$ is oriented normal to the plane of the current loop. For a loop of area A carrying a current I, this simplifies enormously to:

$$\boldsymbol{m} = I\boldsymbol{A} \tag{4.13.2}$$

An effective dipole can also be generated by a charged particle moving in circles. In this case, a quick classical calculation [4.13] lets us write the effective dipole moment as

$$\boldsymbol{m}_L = \frac{q}{2m}\boldsymbol{L} \tag{4.13.3}$$

Quantum mechanics introduces a couple correction factors to this equation, which produce the following equation for a spinning quantum mechanical particle:

$$\boldsymbol{m}_S = \frac{g\mu_B}{\hbar}\boldsymbol{S} \tag{4.13.4}$$

Where μ_B is the Bohr magneton, and g is the appropriate Landau g-factor. Sometimes these factors are grouped together into one proportionality constant γ, called the **gyromagnetic ratio**:

$$\boldsymbol{m} = \gamma\boldsymbol{J} \tag{4.13.5}$$

4.13.2 Forces on a Magnetic Dipole

In general, the force on a magnetic dipole is

$$F = \nabla(\boldsymbol{m} \cdot \boldsymbol{B}) \tag{4.13.6}$$

However, in the (common) case that $\boldsymbol{m}$ is spatially invariant, this simplifies to $F = (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}$.

[4.13] Zangwill pg. 340.

It is important to note that the force on a magnetic dipole is zero *unless* either the dipole moment or (more likely) the field itself has some gradient. Generally, it is the *gradient* of a magnetic field that exerts force on a dipole.

When placed in an external magnetic field, magnetic dipoles will experience a torque that tends to align the magnetic dipole moment with the external field lines:

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B} \tag{4.13.7}$$

The potential energy of a magnetic dipole in a field is

$$U = -\boldsymbol{m} \cdot \boldsymbol{B} \tag{4.13.8}$$

However, there is an **important, subtle point** here. With electric dipoles, since the charge (and therefore internal energy) of the dipoles didn't change, we could find the force on a dipole by taking the gradient of U. However, for magnetic dipoles, the current in the dipole CAN be changed by moving the dipole in the external magnetic field. Normally, we add an additional requirement that the current remain fixed, but this modification means that energy may leave or enter the system in this way!

It is possible[4.14] to do a rigorous calculation to find the *total* energy of the dipole system, including that of the currents. It comes out that the magnetic potential energy is in fact the *negative* of the magnetic total energy.

4.13.3 Magnetic Field of a Magnetic Dipole

The magnetic vector potential of a magnetic dipole is [4.15]:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}, \qquad r >> R \tag{4.13.9}$$

Therefore,

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \left[\boldsymbol{m}(\nabla \cdot \frac{\boldsymbol{r}}{r^3}) - (\boldsymbol{m} \cdot \nabla) \frac{\boldsymbol{r}}{r^3} \right] \tag{4.13.10}$$

The first term reduces is a delta function, $4\pi\delta(\boldsymbol{r})$, since it is zero for all non-zero r, but non zero if we integrate over a spherical volume and then use Gauss's law. The remaining term works out nicely in index notation:

$$m_i \frac{\partial}{\partial r_i} \frac{r_j}{r^3} = m_i \left(\frac{\delta_{ij}}{r^3} - \frac{3}{2} \frac{2 r_j r_i}{r^5} \right) = \frac{\boldsymbol{m}}{r^3} - \frac{3\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{m})}{r^5} = \frac{\boldsymbol{m}}{r^3} - \frac{3\hat{r}(\hat{r} \cdot \boldsymbol{m})}{r^3} \tag{4.13.11}$$

Therefore, the final field becomes

$$\boxed{\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{r}(\hat{r} \cdot \boldsymbol{m}) - \boldsymbol{m}}{r^3}} \tag{4.13.12}$$

[4.14] Zangwill pg. 387.

[4.15] Getting here from the beginning of the multipole expansion takes a few identities and some work, so I've left it out. You can find it worked out in Zangwill, pg. 337.

Notice that this field is essentially *exactly* the same as that of the electric dipole in form! This means that, qualitatively, the two fields have the same shape. It also makes it a lot easier to memorize.

4.14 Magnetostatics

Magnetostatics is defined by the requirement that $\nabla \cdot \boldsymbol{j} = 0$: no currents are created or destroyed.

4.14.1 The Lorentz Force

The force on a charged particle moving through a magnetic field is given by the Lorentz force:

$$\boldsymbol{F} = q\vec{v} \times \boldsymbol{B} \tag{4.14.1}$$

For a current (which is, after all, just a continuous stream of charged particles), this generalizes to **the force of one current distribution on another**

$$\boldsymbol{F}_{1\to 2} = \int_{V_2} \boldsymbol{j}_2(\boldsymbol{r}) \times \boldsymbol{B}_1(\boldsymbol{r}) d^3r \tag{4.14.2}$$

4.14.2 Ohms Law and Potential Theory with Steady Currents

Ohm's law is normally written (by engineers or anyone working in a lab) as:

$$V = IR \tag{4.14.3}$$

However, when working with electric fields, it is usually easier to take a spatial integral and rewrite $r = \frac{1}{\sigma}$ where σ is the conductivity and r is the resistivity. The relationship between resistivity and resistance for a wire is:

$$R = \frac{rL}{a} \tag{4.14.4}$$

Where L is the length of the wire and a is its cross-sectional area.

A perfect insulator has a conductivity of zero. Rewriting Ohms law using the conductivity yields:

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{4.14.5}$$

σ is a property of a given material. Therefore, Ohms law allows us to find the current generated in a given conductor by some external electric field.

Combined with the central assumptions of magnetostatics, Ohms law allows us to apply potential theory to situations with steady currents. For spatially-constant σ, $\nabla \cdot \boldsymbol{j} = 0$ and $\boldsymbol{j} = \sigma \boldsymbol{E}$ implies that $\sigma \nabla \cdot \boldsymbol{E} = 0$, or, in turn: $\rho = 0$ (by Gauss's law) and, since $\nabla^2 \phi = 0$ (since $\boldsymbol{E} = -\nabla\phi$). This last condition means that we have the same Laplace equation for the potential as in electrostatics, and can therefore apply all of our potential theory techniques.

This results in two boundary conditions that must be met by an electric potential in a region

with steady currents:

$$\hat{n}_2 \cdot (\boldsymbol{j}_1 - \boldsymbol{j}_2) = j_{1,\perp} - j_{2,\perp} = \sigma_1 \frac{\partial \phi_1}{\partial n}|_S - \sigma_2 \frac{\partial \phi_2}{\partial n}|_S = 0 \tag{4.14.6}$$

$$\hat{n}_2 \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = E_{1,\parallel} - E_{2,\parallel} = \phi_1|_S - \phi_2|_S = 0 \tag{4.14.7}$$

4.14.3 Resistors

Resistance is somewhat analogous to capacitance, in the sense that it is a ratio between the potential over a system and some other quantity. The primary difference is that, in addition to geometry, resistance also depends on the material properties of the system.

Much like capacitance, the easiest way to find the resistance of some system is to place an arbitrary potential across it and calculate the current that arises. Once this is found, Ohms law can be used to calculate the resistance.

4.14.4 The Biot-Savart Law

In magnetostatics, it is true by assumption that $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$. Specifying both the curl and divergence of $\boldsymbol{B}$ in this way makes it possible, through Helmholtz's theorem, to specify a unique $\boldsymbol{B}$ for a given $\boldsymbol{j}$. The formula that emerges is the Biot-Savart law:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_V d^3 r' \frac{\boldsymbol{j}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \tag{4.14.8}$$

This general formula holds for any charge distribution. If the current in question is restricted to a 2D or 1D surface, the equation can be simplified accordingly:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_S dS \frac{\boldsymbol{K}(\boldsymbol{r}_S) \times (\boldsymbol{r} - \boldsymbol{r}_S)}{|\boldsymbol{r} - \boldsymbol{r}_S|^3} \tag{4.14.9}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_l d\vec{l} \frac{I \times (\boldsymbol{r} - \boldsymbol{l})}{|\boldsymbol{r} - \boldsymbol{l}|^3} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{l} \times (\boldsymbol{r} - \boldsymbol{l})}{|\boldsymbol{r} - \boldsymbol{l}|^3} \tag{4.14.10}$$

4.14.5 Ampere's Law

Applying the Stokes theorem to $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ yields:

$$\oint_C d\vec{l} \cdot \boldsymbol{B} = \mu_0 \int_S d\vec{S} \cdot \boldsymbol{j} \tag{4.14.11}$$

This law is the magnetostatic equivalent to Gauss's law. It says that the magnetic field along a curve C is equal to the integral over all of the current that passes through the surface bounded by C.

Applying Ampere's law is similar to using Gauss's law. We identify a suitable curve C based on the symmetry of the problem: we want the curve to be everywhere parallel or perpendicular to $\boldsymbol{B}$. It is often useful to also exploit areas of zero magnetic field when choosing C.

4.14.6 The Vector Potential

Since $\nabla \cdot \boldsymbol{B} = 0$, we can write $\boldsymbol{B}$ in terms of a vector function $\boldsymbol{A}$, defined by the relation:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{4.14.12}$$

Much like the electric potential, the magnetic vector potential is arbitrary up to the gradient of a scalar function, since $\nabla \times \nabla f = 0$:

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla f \tag{4.14.13}$$

We can derive a general expression for $\boldsymbol{A}$ in terms of currents by starting with the other Maxwell equation for $\boldsymbol{B}$, $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$:

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{j} \implies \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{j} \tag{4.14.14}$$

We can further simplify if we chose f to put us in the Coulomb gauge, where by definition $\nabla \cdot \boldsymbol{A} = 0$. Then the equation reduces to $\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}$. By analogy with Poisson's equation for the *electrical potential* of a point charge (where $\rho \to \frac{\epsilon_0 \mu_0 \boldsymbol{j}}{4\pi}$) gives us a general formula for the $\boldsymbol{A}$ created by some system of currents:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_V d^3 r' \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{4.14.15}$$

4.14.7 Inductance

Picture two wire loops positioned next to one another. If a current begins in one loop, it will create magnetic flux through the other, inducing a current there as well. How *much* current is induced per a given current depends on the geometry of the situation, much like the voltage across a capacitor for a given charge depends only on the geometry of the capacitor. This motivates us to define a similar object to capacitance current-carrying conductors. This quantity is the **inductance**.

Consider first the **self inductance** of a loop of wire with a current (which determines the amount of back-current produced in the loop as you try to establish a magnetic field). This inductance is given by

$$L = \frac{\phi}{I} \tag{4.14.16}$$

Where ϕ is the flux through the loop and I is the current. More generally, self inductance for non-loops (say, the self-inductance of a wire) is defined by:

$$L = \frac{1}{I^2} \int_V d^3 r \boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}) \tag{4.14.17}$$

When written this way, it becomes clear (by comparison with the magnetic energy equation) that

$$U_B = \frac{1}{2} L I^2 \tag{4.14.18}$$

Multiple conductors can share a **mutual inductance**, where each generates currents in the others. The inductance of such a system is a matrix, M_{ik}, where the self inductances described above make up the diagonal: M_{ii}. For a set of N conductors, we now have:

$$M_{ik} = \sum_{k=1}^{N} \frac{\phi_i}{I_k} \tag{4.14.19}$$

and

$$M_{ik} = \frac{\mu_0}{4\pi} \frac{1}{I_i I_k} \int d^3r \int d^3r' \frac{\boldsymbol{j}_i(r) \cdot \boldsymbol{j}_k(r')}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{4.14.20}$$

It is important to note that, by a kind of reciprocity relationship, this matrix must be symmetric: the inductance loop 1 has with loop 2 must be the same as loop 2 with loop 1, since inductance only depends on the geometry. In other words:

$$M_{ik} = M_{ki} \tag{4.14.21}$$

4.14.8 Magnetic Energy

The total energy contained in a magnetic field is

$$U_B = \frac{1}{2\mu_0} \int_V |\boldsymbol{B}|^2 d^3r \geq 0 \tag{4.14.22}$$

We can U_B is positive definite by contradiction: if it were not, it would be energetically favorable for currents to form spontaneously (since zero magnetic field would not be the ground state).

Sometimes we are only concerned with the interaction energy between two current-carrying objects (i.e. the energy that would be released if they were taken to infinity). This energy can be calculated as

$$V_B = \int \boldsymbol{j}_1(r) \cdot \boldsymbol{A}_2(r) d^3r = \int \boldsymbol{j}_2(r) \cdot \boldsymbol{A}_1(r) d^3r \tag{4.14.23}$$

Notice the reciprocity relationship displayed here: the interaction energy can just as easily be calculated using the potential of 1 and the current of 2 as the potential of 2 and the current of 1

4.14.9 Magnetic Scalar Potential

Since $\nabla \times \boldsymbol{B} = 0$ when $\boldsymbol{j} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$ always, we can draw an analogy between this system and the set of equations that describe the electrical field in a region free of charge: $\nabla \times \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{E} = 0$. Therefore, **within the current-free region** we can define a **magnetic scalar potential** that has the same form as the electrical scalar potential:

$$\boldsymbol{B} = -\nabla \psi \tag{4.14.24}$$

Therefore, since $\nabla \cdot \boldsymbol{B} = 0$

$$\nabla^2 \psi = 0 \tag{4.14.25}$$

This is the same Laplace equation we had for electrostatics, so we can apply **all** of the tools of potential theory that we developed there! Bessel functions and all.

The boundary conditions on the scalar potential are very similar to those for electrostatics:

The Maxwell equations for $\boldsymbol{B}$ produce a different set of boundary conditions than we had with electrostatics:

$$\hat{n} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2)\big|_{surface} = (B_{1,\perp} - B_{2,\perp})\big|_{surface} = 0 \tag{4.14.26}$$

$$\hat{n} \times (\boldsymbol{B}_1 - \boldsymbol{B}_2)\big|_{surface} = (B_{1,\parallel} - B_{2,\parallel})\big|_{surface} = \mu_0 \boldsymbol{K}(r)|_{surface} \tag{4.14.27}$$

4.14.10 Simple Magnetic Materials and the Auxiliary Field, H

When a magnetizable material is exposed to a magnetic field, its magnetic domains align to produce a magnetic dipole moment per unit volume, $\boldsymbol{M}$. This in turn combines to produce an overall dipole moment for the object[4.16]:

$$\boldsymbol{m} = \int \boldsymbol{M} dV \tag{4.14.28}$$

The degree to which magnetizable materials magnetize is characterized by their magnetic permeability, μ. This plays an analogous role to ϵ for electric permittivity. Sometimes μ is given as a dimensionless ratio with the vacuum permittivity:

$$\kappa_m = \frac{\mu}{\mu_0} \tag{4.14.29}$$

When dealing with magnetizable materials, it is often most convenient to work with the auxiliary field, $\boldsymbol{H}$, which is defined to be:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \tag{4.14.30}$$

In general the response of a given material to an incident magnetic field can be very complicated. However, we generally model materials as behaving linearly to an incident field. Such linear materials are said to be **paramagnetic** if $\chi_m > 0$, which **re-enforce** the incident $\boldsymbol{B}$ field, or **diamagnetic** if $\chi_m < 0$, which **decreases** the incident $\boldsymbol{B}$ field.

In such a linear material. for some incident auxiliary field $\boldsymbol{H}$, the magnetization per unit volume is given by:

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \tag{4.14.31}$$

Where χ_m is yet another expression for μ:

$$\chi_m = \kappa_m - 1 = \frac{\mu}{\mu_0} - 1 \tag{4.14.32}$$

[4.16] Don't ask me why big M is the dipole moment DENSITY while little m is the TOTAL dipole moment.

This allows us to write eq. 4.14.30 *for such linear materials* in an even simpler form:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} \tag{4.14.33}$$

4.14.11 Bound and Free Currents

When taken together, the aligned magnetic dipoles of a magnetized material produce a magnetic field identical to that of a system of volume and surface currents. By analogy with electric polarizability, we will refer to these currents as **bound currents**, to differentiate them from the **free currents**, which are the REAL currents in the material. The magnitude and direction of these currents are determined by the following equations:

$$\boldsymbol{j}_M = \nabla \times \boldsymbol{M} \tag{4.14.34}$$

$$\sigma_M = \boldsymbol{M} \cdot \hat{n}\big|_s \tag{4.14.35}$$

4.14.12 The Auxiliary Magnetic Scalar Potential and the “Fake" Magnetic Charge

Inside or around magnetizable materials, it is often more convenient to defined the magnetic scalar potential as:

$$\boldsymbol{H} = -\nabla \psi_M \tag{4.14.36}$$

Where the “M" subscript differentiates this potential from the other magnetic scalar potential defined earlier. This potential has its own set of general boundary conditions, which can be found in Zangwill, pg. 417. However, when the materials involved are simple, linear magnetic materials, the boundary conditions simplify to the following:

$$\psi_1\big|_s = \psi_2\big|_s \tag{4.14.37}$$

$$\mu_1 \frac{d\psi_1}{dn}\big|_s = \mu_2 \frac{d\psi_2}{dn}\big|_s \tag{4.14.38}$$

As a **purely mathematical** trick, it is often convenient to calculate ψ_M in terms of a fictitious “magnetic surface charge" and “magnetic volume charge":

$$\rho^* = -\nabla \cdot \boldsymbol{M} \tag{4.14.39}$$

$$\sigma^* = \boldsymbol{M} \cdot \hat{n}\big|_s \tag{4.14.40}$$

Once these have been determined, the potential can be found using potential equations from electrostatics. What used to be $\boldsymbol{E}$ is now $\boldsymbol{H}$, so we can (for example) find $\boldsymbol{H}$ using Gauss's law:

$$\nabla \cdot \boldsymbol{H}_M = \rho^* \tag{4.14.41}$$

It's worth emphasizing again that this is **just a math trick** that we can use because the equations turn out to be the same, and we are much happier dealing with electric fields. There's no such thing as a magnetic charge!!

4.15 Electrodynamics

4.15.1 Time Varying Scalar and Vector Potentials

When fields and sources vary in time, it is possible for charges to create magnetic fields. This in turn intertwines the roles of the scalar and vector potentials.

Plugging $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ into Maxwell's $\nabla \times \boldsymbol{B}$ equation yields:

$$0 = \nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) \tag{4.15.1}$$

Since, in general, $\nabla \times \nabla f = 0$ for some scalar f, the quantity in the parentheses can be written as the gradient of a scalar function, which is the scalar potential we know and love:

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi \tag{4.15.2}$$

Rearranging, we now have an expression for the electric field represented by these time-varying potentials:

$$\boxed{\boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}} \tag{4.15.3}$$

Plugging this result, (and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$) into the remaining Maxwell equations yields a system of two (difficult) coupled equations that *theoretically* contain all the physics of the dynamic system.

$$\boxed{\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = \frac{-\rho}{\epsilon_0}} \tag{4.15.4}$$

$$\boxed{\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \right) = -\mu_0 \boldsymbol{j}} \tag{4.15.5}$$

These equations are simplified substantially by choosing the correct gauge, as discussed in the section on gauge freedom.

4.15.2 Conservation of Charge and its Consequences

The conservation of charge is normally stated mathematically as:

$$\nabla \cdot \boldsymbol{j} = \frac{d\rho}{dt} \tag{4.15.6}$$

Where $\boldsymbol{j}$ is the current density, and ρ is the charge distribution. This relation can be easily derived by starting with the more intuitive expression that the total change in charge in a region is due to the sum of the currents leaving the region:

$$\frac{dQ}{dt} = \int \boldsymbol{j} \cdot d\boldsymbol{A} \tag{4.15.7}$$

Transforming $Q \to \int_V \rho d^3r$ and applying the divergence theorem in reverse to the right hand side:

$$\int \nabla \cdot \boldsymbol{j} dV = \frac{d}{dt} \int \rho dV \tag{4.15.8}$$

Since these integrals can now be combined, we have

$$\int \left(\nabla \cdot \boldsymbol{j} - \frac{d\rho}{dt} \right) dV = 0 \tag{4.15.9}$$

We then argue that the integrand itself must in general be zero:

$$\boxed{\nabla \cdot \boldsymbol{j} = \frac{d\rho}{dt}} \tag{4.15.10}$$

4.15.3 The Displacement Current

Charge conservation states that:

$$\nabla \cdot \boldsymbol{j} = \frac{d\rho}{dt} \tag{4.15.11}$$

However, differentiating Gauss's law simply gives:

$$\frac{\partial \rho}{\partial t} = \epsilon_0 \nabla \cdot \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.15.12}$$

Combining these two equations:

$$\nabla \cdot \boldsymbol{j} = \epsilon_0 \nabla \cdot \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.15.13}$$

This equality suggests that some type of current is created by a changing electric field. This current is called the **displacement current** for historical reasons. The displacement current is thus defined as:

$$\boldsymbol{j}_D = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.15.14}$$

4.15.4 Polarization Current

Just as real moving charges constitute a current (and can create a magnetic field), moving bound charges *also* create an effective current, called the **Polarization Current**. This can be found by applying the conservation of *bound* charge[4.17]:

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \boldsymbol{j}_b = 0 \tag{4.15.15}$$

Since

$$\rho_b = -\nabla \cdot \boldsymbol{P} \tag{4.15.16}$$

[4.17] I am using the subscript b to denote "bound". Zangwill uses p in the same expressions for "polarization".

We can rewrite this expression as

$$\nabla \cdot \boldsymbol{j}_b = \nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t} \tag{4.15.17}$$

The most general solution to this equation is:

$$\boldsymbol{j}_b = \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \Lambda \tag{4.15.18}$$

Where Λ is some vector function. It turns out[4.18] that this function is actually an effective *magnetization* caused by the polarization current. Therefore, the equation is normally written as:

$$\boldsymbol{j}_b = \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M}_b \tag{4.15.19}$$

4.15.5 Faraday's Law

Farady's law for dynamic fields is:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{4.15.20}$$

Faraday's law states that, when the magnetic flux (field $\times$ area) changes through a conducting loop, a voltage will be induced in the loop:

$$V = -\frac{d\phi}{dt} \tag{4.15.21}$$

Usually, the V in this equation is written as a script E for the Electromotive Force. However, since this force is measured in volts and always takes the form of a voltage, I see no need for its separate existence.

4.16 Slowly Varying Fields and Currents: Quasi-statics

We characterize a field or current as "slowly varying" if its characteristic length scale and oscillation frequency obey the inequality[4.19].

$$\omega^2 l^2 << c^2 \tag{4.16.1}$$

This can be viewed as a requirement that *changes in the system happen slowly compared to the time for light signals to propagate through the system.*

When making these sorts of estimates, it is useful to make the approximations

$$\nabla \approx \frac{1}{l}, \qquad \frac{\partial}{\partial t} \approx \omega \tag{4.16.2}$$

[4.18] See Zangwill pg. 459.
[4.19] Zangwill handwaves his way to this expression on pg. 468.

4.16.1 Quasi-Electrostatics and Quasi-Magnetostatics

In the Quasi-Electrostatic regime, we assume that $\frac{\partial \boldsymbol{E}}{\partial t} = 0$. This knocks out a term in Maxwell's equations.
In Quasi-Magnetistatics we similarly make the assumption that $\frac{\partial \boldsymbol{B}}{\partial t} = 0$.

4.17 Gauge Freedom

As previously noted, there is an inherent ambiguity in our definitions of the scalar and vector potentials:

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t}, \qquad \boldsymbol{A}' = \boldsymbol{A} + \nabla \Lambda \tag{4.17.1}$$

We are free to chose whatever arbitrary Λ that we like above. *This choice is equivalent to specify the value of:* $\nabla \cdot \boldsymbol{A}$.

It is important to keep straight which gauge different equations are written in![4.20]

4.17.1 The Coulomb Gauge

In the Coulomb gauge, we set:

$$\nabla \cdot \boldsymbol{A} = 0 \tag{4.17.2}$$

This is a natural choice, as terms of $\nabla \cdot \boldsymbol{A}$ appear in both of the master equations for the time varying potentials. With this choice, those equations simplify to:

$$\boxed{\nabla^2 \phi = \frac{-\rho}{\epsilon_0}} \tag{4.17.3}$$

$$\boxed{\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{j} + \frac{1}{c^2}\nabla \frac{\partial \phi}{\partial t}} \tag{4.17.4}$$

Once you *have* the potentials, the corresponding fields are given by:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{4.17.5}$$

4.17.2 The Lorentz Gauge

In the Lorentz gauge we set:

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{4.17.6}$$

[4.20] Zangwill puts subscripts on EVERY A and ϕ to remind you of this. I think that's cumbersome, so I'm leaving them off. Goodness knows, when it comes to actually calculating anything on paper, you'll be dropping them too.

This serves to decouple the master equations, yielding two somewhat complicated but symmetric equations:

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{-\rho}{\epsilon_0}} \tag{4.17.7}$$

$$\boxed{\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu_0\boldsymbol{j}} \tag{4.17.8}$$

This form of the equations lends itself to both wave problems (i.e. radiation) and relativistic calculations.

Once you *have* the potentials, the corresponding fields are given by:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{4.17.9}$$

4.18 Energy and Momentum in E&M Fields

4.18.1 Energy Flow and the Poynting Vector

The mechanical work done by a current against an electric field is $\boldsymbol{F}\cdot\boldsymbol{v}$, or:

$$\frac{dW_{mech}}{dt} = \int_V (\rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B})\cdot\boldsymbol{v}d^3r \tag{4.18.1}$$

Since $\boldsymbol{j} \parallel \boldsymbol{v}$, $(\boldsymbol{j}\times\boldsymbol{B})\cdot\boldsymbol{v} = 0$. Also, $\rho\boldsymbol{v} = \boldsymbol{j}$. Thus, we can rewrite

$$\frac{dW_{mech}}{dt} = \int_V (\boldsymbol{j}\cdot\boldsymbol{E})d^3r \tag{4.18.2}$$

But, since one of Maxwell's equations tells us that $\mu_0\boldsymbol{j} = \nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}$, this can be rewritten (by just substituting for $\boldsymbol{j}$ in $\boldsymbol{j}\cdot\boldsymbol{E}$):

$$\frac{dW_{mech}}{dt} = \int_V \left(\frac{1}{\mu_0}\nabla\times\boldsymbol{B} - \epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\cdot\boldsymbol{E}d^3r \tag{4.18.3}$$

A vector identity, combined with Maxwell's equation for $\nabla\times\boldsymbol{E}$ shows that: $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = -\boldsymbol{B}\frac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{E}(\nabla\times\boldsymbol{B})$. This allows the equation above to be rearranged as:

$$\frac{dW_{mech}}{dt} = \int_V \left(\frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) + \frac{1}{\mu_0}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right)d^3r \tag{4.18.4}$$

If you're clever, you notice that:

$$\int_V \frac{1}{\mu_0}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} + \epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}d^3r = \frac{\partial}{\partial t}\int_V \frac{1}{2}\epsilon_0(\boldsymbol{E}\cdot\boldsymbol{E} + c^2\boldsymbol{B}\cdot\boldsymbol{B})d^3r = \frac{\partial}{\partial t}U_{EM} \tag{4.18.5}$$

We can also rewrite one of the terms as a surface integral of a vector function called the **Poynting Vector**:

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \tag{4.18.6}$$

Applying the Stokes theorem to the $\boldsymbol{E} \times \boldsymbol{B}$ term, we can now rewrite the entire equation as:

$$\frac{d}{dt}(U_{mech} + U_{EM}) = \frac{dU_{tot}}{dt} = -\int_S \boldsymbol{S} \cdot d\boldsymbol{A} \tag{4.18.7}$$

This equation has a profound interpretation. Energy can obviously be transferred between electromagnetic and mechanical energy, all on the left hand side of the equation. However, it is also possible for the *total* energy to change, if the Poynting flux term is non-zero! This term must represent energy being radiated from the system.

$\boldsymbol{S}$ can be interpreted as a current density of electromagnetic energy.

4.18.2 Electromagnetic Momentum

Electromagnetic fields can store and carry momentum. The momentum density contained in a field is:

$$\boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \epsilon_0(\boldsymbol{E} \times \boldsymbol{B}) \tag{4.18.8}$$

For a single particle in a field[4.21], the total momentum of the particle due to the field is $p = q\boldsymbol{A}$. This expression often pops up when writing the Hamiltonian of such a particle.

Fields can also store angular momentum. The angular momentum density is sensibly defined by:

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{g} \tag{4.18.9}$$

4.19 Electromagnetic Waves

4.19.1 The Wave Equation

Maxwell's equations in free space are:

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{4.19.1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{4.19.2}$$

By taking the curl of $\nabla \times \boldsymbol{E}$, we can write:

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \times -\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) \tag{4.19.3}$$

Substituting in the other Maxwell equation for $\nabla \times \boldsymbol{B}$:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{4.19.4}$$

[4.21] Zangwill pg. 515 for details.

Since $\nabla \times (\nabla \times \boldsymbol{f}) = \nabla(\nabla \cdot \boldsymbol{f}) - \nabla^2 \boldsymbol{f}$, we can rewrite this as:

$$\boxed{\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0} \tag{4.19.5}$$

This is the wave equation for the electric field. Following the exact same process but starting with the $\nabla \times B$ Maxwell equation yields the same equation for $\boldsymbol{B}$:

$$\boxed{\nabla^2 \boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0} \tag{4.19.6}$$

Notice how the existence of these wave equations (and therefore their solutions) is due to the mixing of $\boldsymbol{E}$ and $\boldsymbol{B}$ terms in the time-dependent part of Maxwell's equations. In a deep sense, that's the only reason we have light.

4.19.2 Plane Waves

The most important solution to the wave equation is the plane wave: a spatial structure that translates with time in only one direction. It can be shown[4.22] that *any* function $\boldsymbol{w}(z,t) = \boldsymbol{g}(z-ct) + \boldsymbol{f}(z+ct)$ (for arbitrary $\boldsymbol{g}$ and $\boldsymbol{f}$) will be a solution.

This waveform can be generalized to any direction (rather than just z) by making the substitution $z = \boldsymbol{k} \cdot \boldsymbol{r}$, where $\boldsymbol{k}$ is the **wave vector**. $\boldsymbol{k}$ points in the direction of propagation of the wave, and has a magnitude related to the wave's speed. We will also recast $c|\boldsymbol{k}| = \omega$.

It is pretty easy to see that, when evolved in time, this waveform will move with a speed c in the $\boldsymbol{k}$ direction. This is known as the **phase velocity** (since it is defined by watching a point of constant phase move).

The fields corresponding to these solutions look like:

$$\boldsymbol{E} = \boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t), \qquad \boldsymbol{B} = \frac{1}{c}\hat{k} \times \boldsymbol{E} \tag{4.19.7}$$

Notice the potential for confusion between $\boldsymbol{E}$ and $\boldsymbol{E}_0$. While both are vectors, only the latter is constant. We see from these equations that

$$|\boldsymbol{E}| = c|\boldsymbol{B}| \tag{4.19.8}$$

Therefore, it will suffice for us to work almost exclusively with the electric field, knowing that we can easily reproduce the corresponding magnetic field.

Without loss of generality, we will chose to work in a basis of sinusoidal functions, such that

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) \tag{4.19.9}$$

To make the calculus easier, we will chose to work with complex waves, *with the understanding that the actual wave we are talking about is only the real part of our equation.* We can therefore express $\boldsymbol{E} = Re[\tilde{E}]$, where:

$$\tilde{E} = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \tag{4.19.10}$$

[4.22] Zangwill pg. 539.

If a wave is completely represented by one such sinusoidal function, it is said to be a **monochromatic plane wave**. If not, any solution to the wave equation *can* be represented as a Fourier series over such plane waves.

Waves carry energy, and the energy carried in a monochromatic plane wave is given simply by the usual E&M energy equation:

$$u_{EM} = \frac{1}{2}\epsilon_0(|Re(\tilde{E})|^2 + c^2|Re(\tilde{B})|^2) = \frac{1}{2}\epsilon_0|Re(\tilde{E})|^2 \tag{4.19.11}$$

Over a time average:

$$\langle u_{EM}\rangle = \frac{1}{2}\epsilon_0|Re(\tilde{E_0})|^2 \tag{4.19.12}$$

Similar expressions can then be worked out for the momentum and Poynting flux of the wave:

$$\langle \boldsymbol{g}\rangle = \frac{\langle u_{EM}\rangle}{c}\hat{k} \tag{4.19.13}$$

$$\langle \boldsymbol{S}\rangle = c^2\langle \boldsymbol{g}\rangle = c\langle u_{EM}\rangle\hat{k} \tag{4.19.14}$$

Since the Poynting flux is an energy density *current*, we can define the **energy velocity** as:

$$\boldsymbol{v}_E = \frac{\langle \boldsymbol{S}\rangle}{\langle u_{EM}\rangle} = c\hat{k} \tag{4.19.15}$$

Which confirms that we have, in fact, described a light wave!

4.19.3 Polarization

We have shown that, for any given E-field, a corresponding B-field of $\boldsymbol{B} = \hat{k} \times \boldsymbol{E}$ can be a freely propagating electromagnetic wave. This means that the direction of the electric field is arbitrary, and can in fact change in time as long as the B-field also changes.

We define the **polarization** of a light wave by the motion of the tip of the E-field vector. In the most general case, we will allow this vector to trace out an ellipse in the transverse plane.

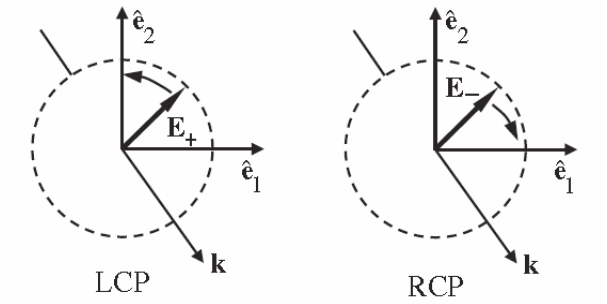


Figure 4.19.1: Coordinate system and a sign convention for circular polarization. Source: Zangwill pg. 547.

In the basis $\hat{e}_1$ and $\hat{e}_2$ as illustrated in figure 4.19.1, the most general electric field can be

expressed as:

$$Re(\tilde{E}) = A\cos(\phi+\delta_1)\hat{e}_1 + B\cos(\phi+\delta_2)\hat{e}_2 = E_1\hat{e}_1 + E_2\hat{e}_2 \tag{4.19.16}$$

Some rearrangement of this equation shows that the components E_1, E_2 of $\boldsymbol{E}$ obey the equation of an ellipse paramaterized by the phase difference between the $\hat{e}_2$ and $\hat{e}_1$ components $\delta = \delta_2 - \delta_1$:

$$\left(\frac{E_1}{A}\right)^2 + \left(\frac{E_2}{B}\right)^2 - 2\left(\frac{E_1}{A}\right)\left(\frac{E_2}{B}\right)\cos(\delta) = \sin^2(\delta) \tag{4.19.17}$$

This state is known as **elliptical polarization**.

This equation simplifies substantially if we assume that the $\hat{e}_1$ and $\hat{e}_2$ components are completely out of phase, i.e. $\delta = \delta_2 - \delta_1 = m\pi, \quad (m = 0, 1, 2, ...)$. We can parameterize the wave in terms of the angle between $\boldsymbol{E}$ and $\hat{e}_1$: θ.

$$\tilde{E} = E_0(\cos(\theta)\hat{e}_1 + \sin(\theta)\hat{e}_2)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{4.19.18}$$

This state is called **linear polarization**, as the tip of $\boldsymbol{E}$ traces back and forth across the line defined by θ (note that θ is fixed for a given wave, so the coefficients of the unit vectors above are constant).

A second simplification occurs when $\delta = \delta_2 - \delta_1 = \frac{\pi}{2}m, \quad (m = \pm1, \pm2, ...)$. In this case, the ellipse becomes a circle, which can be described by:

$$\tilde{E}_\pm = E_0\left(\frac{\hat{e}_1 \pm i\hat{e}_2}{\sqrt{2}}\right)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{4.19.19}$$

This is known as **circular polarization**. Based on this equation, it is natural to transform into a basis of circular coordinates:

$$\hat{e}_+ = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2), \qquad \hat{e}_- = \frac{1}{\sqrt{2}}(\hat{e}_1 - i\hat{e}_2) \tag{4.19.20}$$

Notice that the choice of $+$ and $-$ for these vectors is arbitrary. This leads to a troublesome **sign convention** that differs even between sub-fields of physics. In these notes (consistent with Zangwil) $+$ is taken to be counter-clockwise as see looking INTO the light ray.

In this basis, the wave can be conveniently written as:

$$\tilde{E_\pm} = (\hat{e}_\pm)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{4.19.21}$$

This circular basis is an equally good basis for expressing an arbitrary wave.

4.20 Wave Packets

Plane waves provide a simple basis of solutions to the wave equation, but are obviously unphysical (because they extend to infinity). This problem can be solved by assembling a **wave packet** by combining a continuum of plane waves, weighted by a vector **envelope function** of frequency

(or, equivalently, $\boldsymbol{k}$), $\boldsymbol{\varepsilon}(\boldsymbol{k})$:

$$\tilde{E} = \frac{1}{(2\pi)^3}\int \boldsymbol{\varepsilon}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}d\vec{k} \tag{4.20.1}$$

This general vector wave packet is very difficult to use, in general we will work with a similar **scalar wave packet**, assembled from solutions to the scalar wave equation:

$$u(\boldsymbol{r},t) = \frac{1}{(2\pi)^3}\int \varepsilon(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}d\vec{k} \tag{4.20.2}$$

Again, $\varepsilon(\boldsymbol{k})$ is an envelope function that defines the shape of the wave packet at $t = 0$: however in this case it is a scalar. We have switched notation from $\tilde{E}$ to $u(\boldsymbol{r},t)$ to indicate that this function *cannot* actually be an electric field (because it is a scalar).

Incidentally, it can be shown (Zangwill pg. 554) that $u(\boldsymbol{r})$ and $\varepsilon(\boldsymbol{k})$ are a Fourier transform pair, meaning that:

$$\Delta x_i \Delta k_i \geq \frac{1}{2} \tag{4.20.3}$$

This "uncertainty relationship" is characteristic of all Fourier transform pairs. In this case, it states that a *a wave packet can be made arbitrarily small, at the cost of including more and more frequencies.* Just as for the position-momentum pair in QM, the maximally certain state possible is a Gaussian envelope function, making this the standard choice.

4.20.1 Group Velocity

In general, $\omega(\boldsymbol{k})$. This means that different plane waves in a packet may travel at different velocities (depending on the properties of the material). If we assume that a wavepacket is mostly localized in frequency space (includes a narrow band of frequencies), we can justify approximating ω as a Taylor series:

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_0) + \left.\frac{\partial\omega}{\partial k_i}\right|_{\boldsymbol{k}=\boldsymbol{k}_0}(\boldsymbol{k}-\boldsymbol{k}_0)_i + ... \tag{4.20.4}$$

We will define the **group velocity** of the wave packet as the second coefficient in this expansion:

$$v_g = \left.\frac{\partial\omega}{\partial k_i}\right|_{\boldsymbol{k}=\boldsymbol{k}_0} = \left.\nabla_k\omega(\boldsymbol{k})\right|_{\boldsymbol{k}=\boldsymbol{k}_0} \tag{4.20.5}$$

This is the effective speed at which the wave packet propagates. Notice that, in vacuum:

$$\omega = c\boldsymbol{k} \quad \rightarrow v_g = c \tag{4.20.6}$$

In a medium, this velocity will in general be a function of $\boldsymbol{k}$.

4.21 Waves in Matter

Waves in matter can be conceptually subdivided into simple waves, in which all frequencies travel at the same velocity, i.e. the group velocity is constant, and waves in **dispersive media**, where the group velocity is a function of frequency.

4.21.1 Simple Media

Simple media are characterized by the equations:

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H} \tag{4.21.1}$$

For convenience, we will introduce two collections of these constants. The **index of refraction**:

$$\boxed{n = c\sqrt{\mu\epsilon} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}} \tag{4.21.2}$$

and the **intrinsic impedance**:

$$Z = \sqrt{\frac{\mu}{\epsilon}} \tag{4.21.3}$$

For a given medium, then, Z and n contain the same information. This means you will commonly need to translate from one to the other, which is easily done[4.23]:

$$\boxed{Z = \frac{\mu}{nc}} \tag{4.21.4}$$

In many cases, it may be assumed that $\mu = \mu_0$. This is often justified, as magnetic effects for most materials are much weaker than their electric counterparts. Such materials are called **non-magnetic**.

4.21.2 Simple Plane Waves

In simple media, it is typical to chose to describe waves in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$. If no sources are present, Maxwell's equations are then:

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{H} = 0 \tag{4.21.5}$$

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}, \qquad \nabla \times \boldsymbol{B} = \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.21.6}$$

If we apply these equations to a typical monochromatic plane wave:

$$\tilde{E} = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \qquad \tilde{H} = \tilde{H}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \tag{4.21.7}$$

We produce a set of four constraints. The first two state that both $\boldsymbol{E}$ or $\boldsymbol{H}$ are perpendicular to the direction of propagation of the wave:

$$\boldsymbol{k} \cdot \boldsymbol{E} = 0, \qquad \boldsymbol{k} \cdot \boldsymbol{H} = 0 \tag{4.21.8}$$

The second two set restrictions on the relationship between $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\boldsymbol{k} \times \boldsymbol{E} = \omega\mu \boldsymbol{H}, \qquad \boldsymbol{k} \times \boldsymbol{H} = \omega\epsilon \boldsymbol{E} \tag{4.21.9}$$

[4.23] Note that this expression is equivalent to $Z = \frac{n}{\epsilon c}$. This version is preferable, since $|\boldsymbol{H}| = \frac{1}{Z}|\boldsymbol{E}|$, so we would like to have n in the numerator of that expression when n is complex later.

Combining these latter two equations (by taking the curl of one and then plugging in the other) gives:

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) = -\omega^2 \mu \epsilon \boldsymbol{E} \tag{4.21.10}$$

The triple cross product identity then produces:

$$\boldsymbol{k} \cdot \boldsymbol{k} = \mu \epsilon \omega^2 \quad \rightarrow \quad \omega(\boldsymbol{k}) = \frac{c}{n} k \tag{4.21.11}$$

Therefore, the group velocity $\frac{\partial \omega}{\partial k}$ is just $v_g = \frac{c}{n}$. Media with higher indices of refraction have the effect of slowing down the wave.

The second set of conditions also gives us a relation between $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\boldsymbol{H} = \frac{1}{Z} \hat{k} \times \boldsymbol{E} \tag{4.21.12}$$

Therefore, just like in vacuum, we can often think about waves only in terms of the electric field, simply recovering $\boldsymbol{H}$ at the end of the calculation.

As usual, Maxwell's equations require the following conditions at boundaries (with no sources)(directions relative to the surface normal):

$$\boldsymbol{D}_{1,\perp} = \boldsymbol{D}_{2,\perp}, \qquad \boldsymbol{B}_{1,\perp} = \boldsymbol{B}_{2,\perp} \tag{4.21.13}$$

$$\boldsymbol{E}_{1,\parallel} = \boldsymbol{E}_{2,\parallel}, \qquad \boldsymbol{H}_{1,\parallel} = \boldsymbol{H}_{2,\parallel} \tag{4.21.14}$$

4.21.3 Specular Reflection and Snell's Law

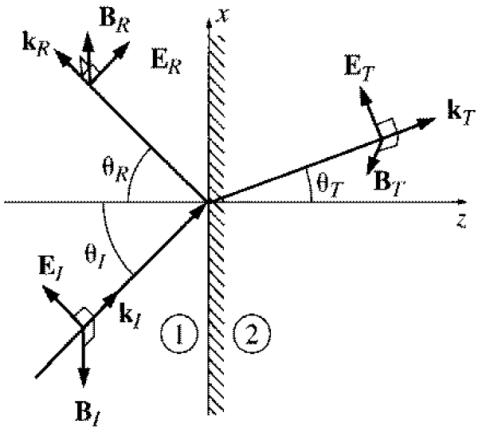


Figure 4.21.1: Reflection and refraction at a boundary. Source: Griffiths E&M, pg. 389.

Imagine a light wave incident on a boundary between two regions with different indices of refraction. An incident ray will produce both a transmitted and a refracted ray. Together, these three rays lie in a plane, which is defined as the **plane of incidence**. Matching the incident,

transmitted, and reflected waves at the boundary produces an equation like:

$$()e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega_I t)} + ()e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega_R t)} = ()e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega_T t)}, \quad (at\ z = 0) \tag{4.21.15}$$

Where the empty parentheses stand for hither-to-be-determined constants. In order to have any hope of satisfying the boundary conditions given by Maxwell's equations *everywhere* on this plane, two conditions must be satisfied:

- $\omega_I = \omega_R = \omega_T$. If this was not the case, the solutions could only ever match for one point in time.
- $\boldsymbol{k}_I \cdot \boldsymbol{r}|_{z=0} = \boldsymbol{k}_R \cdot \boldsymbol{r}|_{z=0} = \boldsymbol{k}_T \cdot \boldsymbol{r}|_{z=0}$. Letting $\boldsymbol{r} = \langle 1, 0, 0 \rangle$ or something similar makes it clear that this statement also implies that $k_{I,x} = k_{R,x} = k_{T,x}$, and $k_{I,y} = k_{R,y} = k_{T,y}$.

The condition that $k_{I,x} = k_{R,x} = k_{T,x}$ immediately implies that $\sin(\theta_I) = \sin(\theta_R)$, since $|\boldsymbol{k}_I| = |\boldsymbol{k}_R|$. This is equivalent to **the Law of Specular Reflection**:

$$\theta_I = \theta_R \tag{4.21.16}$$

Applying the same reasoning to the transmitted wave produces:

$$|\boldsymbol{k}_I| \sin(\theta_I) = |\boldsymbol{k}_R| \sin(\theta_R) \tag{4.21.17}$$

Since $k = \frac{\omega n}{c}$, this produces **Snell's Law**:

$$n_1 \sin(\theta_I) = n_2 \sin(\theta_R) \tag{4.21.18}$$

4.21.4 Total Internal Reflection

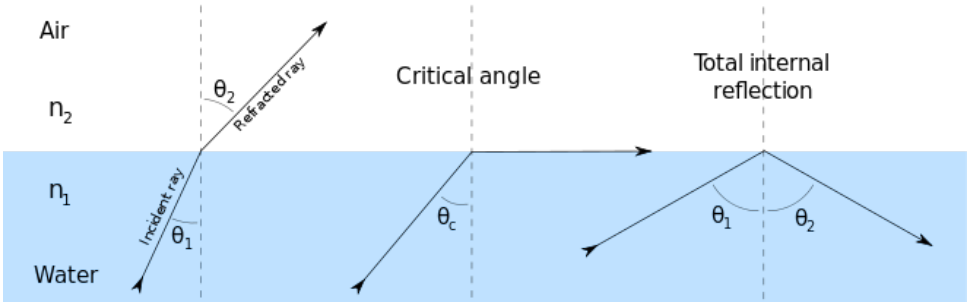


Figure 4.21.2: Total internal reflection. Source: Wikipedia.

Total internal reflection occurs when a wave travels from a region of higher n to one of lower n at a steep angle. Snell's law predicts that, after a critical angle denoted θ_c where the wave is refracted horizontal to the surface, the wave can actually be refracted *back* into the original material. The critical angle can be calculated by setting $\theta_T = \frac{\pi}{2}$ in Snell's law, which yields:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n1}\right) \tag{4.21.19}$$

4.21.5 Polarized Light at Boundaries (The Fresnel Equations)

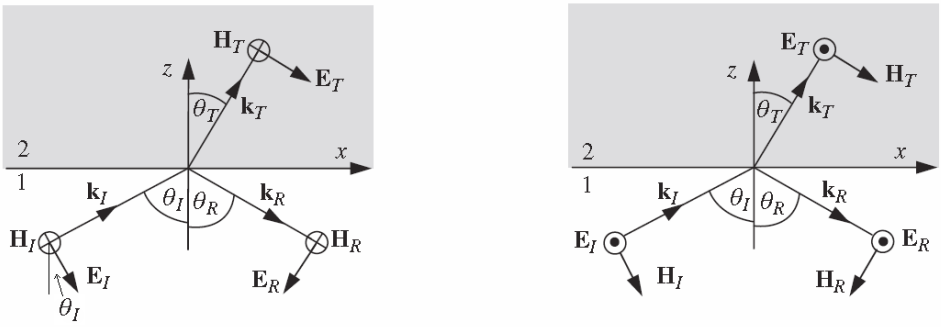


Figure 4.21.3: Reflection and refraction of a polarized wave at a boundary. Source: Zangwill pg. 591.

When considering the interaction between a polarized light wave and a boundary, we must match boundary conditions for both $\boldsymbol{E}$ and $\boldsymbol{H}$. We will treat two possible situations separately:

- **p-polarized, TM (transverse magnetic), or $\|$:** $\boldsymbol{E}$ is parallel to the plane of incidence (left panel in figure 4.21.3).
- **s-polarized, TE (transverse electric), or $\perp$:** $\boldsymbol{E}$ is perpendicular to the plane of incidence (right panel in figure 4.21.3).

We will treat the p case (the s-case follows by the same logic). The requirement that $\boldsymbol{E}_{1,\|} = \boldsymbol{E}_{2,\|}$ directly yields (based on figure 4.21.3):

$$E_I \cos(\theta_I) + E_R \cos(\theta_R) = E_T \cos(\theta_T) \tag{4.21.20}$$

While the condition that $\boldsymbol{H}_{1,\|} = \boldsymbol{H}_{2,\|}$ means that:

$$H_I + H_R = H_T \tag{4.21.21}$$

But since $\boldsymbol{H} = \frac{1}{Z}\hat{k} \times \boldsymbol{E}$, this second equation can be rewritten as[4.24]

$$\frac{1}{Z_1}(E_I + E_R) = \frac{1}{Z_2}E_T \tag{4.21.22}$$

Combining these two equations (and changing from Z to n) generates **Fresnel's Equations** for the reflection and transmission coefficients:

$$\boxed{R_p = \left(\frac{E_R}{E_I}\right)_p = \left(\frac{n_2 \cos\theta_I - n_1 \cos\theta_T}{n_1 \cos\theta_T + n_2 \cos\theta_I}\right)^2} \tag{4.21.23}$$

$$\boxed{T_p = \left(\frac{E_T}{E_I}\right)_p = \left(\frac{2n_1 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}\right)^2} \tag{4.21.24}$$

[4.24] Both sides also get a $\hat{\boldsymbol{k}}\times$, but we can cancel those and consider just the parts written below.

Following the same process for s-polarized waves yields:

$$\boxed{R_s = \left(\frac{E_R}{E_I}\right)_s = \left(\frac{n_1 \cos\theta_I - n_2 \cos\theta_T}{n_1 \cos\theta_I + n_2 \cos\theta_T}\right)^2} \tag{4.21.25}$$

$$\boxed{T_s = \left(\frac{E_T}{E_I}\right)_s = \left(\frac{2n_1 \cos\theta_I}{n_1 \cos\theta_I + n_2 \cos\theta_T}\right)^2} \tag{4.21.26}$$

4.21.6 Simple Conducting Matter

Simple conducting matter is simple matter that also allows for currents to be driven by the electric field, so that in addition to $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, we also produce a (free) current $\boldsymbol{j}_f = \sigma \boldsymbol{E}$. In simple conducting matter ϵ, μ, and σ are all constant: in particular they have no dependence on the frequency of the incident wave. $\boldsymbol{E}$ and $\boldsymbol{H}$ must still be perpendicular to $\boldsymbol{k}$ (no dispersion).

The fourth of Maxwell's equations now reads:

$$\nabla \times \boldsymbol{B} = \mu\left(\boldsymbol{j} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t}\right) = \mu\left(\sigma \boldsymbol{E} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{4.21.27}$$

If we take $\boldsymbol{E}$ to be a plane wave, we can re-write this expression as:

$$\nabla \times \boldsymbol{B} = \mu\left(\frac{i\sigma}{\omega} + \epsilon\right)\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \tilde{\epsilon} \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.21.28}$$

Where we have defined (for our convenience) a **complex dielectric constant**:

$$\tilde{\epsilon} = \epsilon + \frac{i\sigma}{\omega} \tag{4.21.29}$$

Or:

$$\boxed{\frac{\tilde{\epsilon}}{\epsilon} = 1 + \frac{i\sigma}{\epsilon\omega}} \tag{4.21.30}$$

Correspondingly, we must also now define a complex index of refraction and impedance:

$$\tilde{n}(\omega) = c\sqrt{\mu\tilde{\epsilon}(\omega)} \tag{4.21.31}$$

$$\tilde{Z}(\omega) = \sqrt{\frac{\mu}{\tilde{\epsilon}(\omega)}} \tag{4.21.32}$$

The dispersion relation can now be written in the same form as that in free space (that name is a bit misleading here: this fact means that there IS no dispersion!):

$$\boldsymbol{k} = \tilde{n}\frac{\omega}{c}\hat{k} \tag{4.21.33}$$

It is important to remember that these quantities are not physical. The physical variables (ϵ, μ, and σ) are still constant! These complex quantities are simply a convenient way of writing Maxwell's equations for this situation.

It maybe concerning that, in both of these equations, we appear to be taking the square root of an imaginary number, which is an odd procedure. In order to avoid having $\sqrt{i}$ in our equations, we can compute $\tilde{n}$ by writing:

$$\tilde{n} = \alpha + i\beta \tag{4.21.34}$$

Such that

$$\tilde{n}^2 = \alpha^2 + 2i\alpha\beta - \beta^2 = (\alpha^2 - \beta^2) + i(2\alpha\beta) \tag{4.21.35}$$

Then, we can match real and imaginary parts to find a solution for $\tilde{n}$:

$$(\alpha^2 - \beta^2) + i(2\alpha\beta) = \tilde{n}^2 = c^2\mu\left(\epsilon + \frac{i\sigma}{\omega}\right) \tag{4.21.36}$$

Solutions to this system are:

$$\mathrm{Re}(\tilde{n}) = \alpha = c\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{\frac{1}{2}} \tag{4.21.37}$$

$$\mathrm{Im}(\tilde{n}) = \beta = c\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{\frac{1}{2}} \tag{4.21.38}$$

Since $\boldsymbol{k} = \tilde{n}\frac{\omega}{c}\hat{k}$, the equation for a plane wave can now be written:

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{\omega}{c}\mathrm{Im}(\tilde{n})\hat{k}\cdot\boldsymbol{r}} e^{i(\frac{\omega}{c}\mathrm{Re}(\tilde{n})\hat{k}\cdot\boldsymbol{r} - \omega t)} \tag{4.21.39}$$

The first exponential is completely real, meaning that it will decay rather than oscillate within the material. This disipation of energy is known as Joule heating. The value $Im(\tilde{n}) = \beta$ is therefore a measure of how strongly the wave is absorbed.

The amount of energy lost due to Joule heating can be calculated by considering the energy lost as work done to create free current in the presence of the electric field:

$$\frac{dW}{dt} = \int_V d^3r \boldsymbol{j}_f \cdot \boldsymbol{E} = \sigma \int_V d^3r |\boldsymbol{E}|^2 \tag{4.21.40}$$

4.21.7 Waves in Dispersive Media

A dispersive medium is one in which the variables $\epsilon(\omega)$, $\mu(\omega)$ and $\sigma(\omega)$ are dependent on frequency. In general, these quantities will actually become tensors, which can cause a plane wave to physically disperse (spread out) as it travels through the material. In simpler calculations, we will assume that these tensors are diagonal and have the same value along each axis:

$$\epsilon(\omega)^{\leftrightarrow} \to \tilde{\epsilon}(\omega) I \tag{4.21.41}$$

Where I is the identity matrix. In this case, the wave will still disperse in 1D, since the speed of light in the material will be different for different frequencies of the light wave.

The current induced by the (already complex) Electric field is now written in terms of a

complex conductance:

$$\tilde{j} = \tilde{\sigma}(\omega)\tilde{E} \tag{4.21.42}$$

Here, it turns out[4.25] that $\tilde{\sigma}(\omega)$ is a Fourier transform pair with the "actual" conductance, which is a function of time:

$$\tilde{\sigma}(\omega) = \int_{-\infty}^{\infty} \sigma(t) e^{i\omega t} dt, \qquad \sigma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\sigma}(\omega) e^{-i\omega t} d\omega \tag{4.21.43}$$

The same is generally true for the host of other complex variables we must introduce: each corresponds via a Fourier transform to a "real" time dependent function[4.26]:

$$\tilde{P} = \epsilon_0 \tilde{\chi}(\omega)\tilde{E}, \qquad \tilde{M} = \tilde{\chi}_M(\omega)\tilde{H} \tag{4.21.44}$$

$$\tilde{D} = \tilde{\epsilon}(\omega)\tilde{E}, \qquad \tilde{B} = \tilde{\mu}(\omega)\tilde{H} \tag{4.21.45}$$

Suppose that we decide to calculate the polarization current $\tilde{j}_P$ based on this oscillating electric field:

$$\tilde{j}_P = \frac{\partial \tilde{P}}{\partial t} \tag{4.21.46}$$

Since $\tilde{P}$ is the Fourier transform of $P(t)$, the time derivative can be replaced by an $-i\omega$:

$$\tilde{j}_P = -i\omega \tilde{P} = -i\omega \epsilon_0 \tilde{\chi}(\omega)\tilde{E} \tag{4.21.47}$$

But since $\tilde{P} = \epsilon_0 \tilde{\chi}(\omega)\tilde{E}$, we know that:

$$\tilde{P} = \frac{i\tilde{\sigma}}{\omega} \tag{4.21.48}$$

Then, since:

$$\tilde{D} = \epsilon_0 \tilde{E} + \tilde{P} = \tilde{\epsilon}(\omega)\tilde{E} \tag{4.21.49}$$

We can solve for a new relation for $\tilde{\epsilon}(\omega)$):

$$\boxed{\tilde{\epsilon}(\omega) = \epsilon_0 + \frac{i\tilde{\sigma}(\omega)}{\omega}} \tag{4.21.50}$$

Notice that **this relation is slightly different from the one for simple conducting matter!** Namely, the ϵ from before has become an ϵ_0. This does not mean that $\epsilon(\omega) = \epsilon_0$: merely that the information about $\epsilon(\omega)$ has been bundled up inside $\tilde{\sigma}(\omega)$.

We could also use Maxwell's equations to show that $\boldsymbol{j} = \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M}$ implies:

$$\boldsymbol{j} = \frac{\chi + \chi_M + \chi\chi_M}{\chi} \frac{\partial \boldsymbol{P}}{\partial t} = \frac{\chi + \chi_M + \chi\chi_M}{\chi_M(1+\chi)} \nabla \times \boldsymbol{M} \tag{4.21.51}$$

[4.25] Zangwill pg. 625.

[4.26] For this reason, Zangwill replaces the $\tilde{F}$ with a $\hat{F}$ for these functions. I'm sticking with $\tilde{F}$, because I think the change causes more confusion than it's worth.

These equations lead to an important conceptual result: **when describing time-dependent currents in a material, we can choose to model the currents as entirely polarization currents (using $\tilde{\epsilon}$) or magnetization currents (using $\tilde{\chi}_M$)).**

For low frequency systems, we will often use both $\tilde{\epsilon}$ and $\tilde{\mu} = \mu_0(1 + \tilde{\chi}_M)$, while at higher frequencies we will set $\boldsymbol{M} = 0$, treating ALL response of the material as a dielectric (and thus grouping it into $\tilde{\epsilon}$).

Having established that we can describe dispersive media through a combination of complex, frequency dependent functions, we now want to find physical models that will provide us with these functions.

4.21.8 The Drude Model

The Drude model pictures a material as being made up of free charges (neutralized by some other oppositely charged particles) that experience a simple drag force proportional to their velocity. Their equation of motion is thus:

$$m\dot{\boldsymbol{v}} = q\underline{\mathrm{E}} - \frac{m}{\tau}\boldsymbol{v} \tag{4.21.52}$$

Where τ is the damping constant. Imagine a single particle being hit by a plane wave. Ignoring the transient initial solution, the particle will oscillate at the same frequency as the wave driving it. Making this assumption allows us to solve this equation:

$$\boldsymbol{v} = \frac{qE_0}{m}\frac{1}{\frac{1}{\tau} - i\omega}e^{i\omega t} \tag{4.21.53}$$

Then, since $\tilde{j}(\omega) = nq\boldsymbol{v} = \tilde{\sigma}(\omega)\boldsymbol{E}$, we can solve for $\tilde{\sigma}(\omega)$:

$$\tilde{\sigma}(\omega) = \frac{nq^2\tau}{m(1 - i\omega\tau)} = \frac{\sigma_0}{1 - i\omega\tau} \tag{4.21.54}$$

For any dispersive medium, $\tilde{\epsilon}(\omega) = \epsilon_0 + \frac{i\tilde{\sigma}(\omega)}{\omega}$. Therefore, plugging in $\tilde{\sigma}(\omega)$:

$$\frac{\tilde{\epsilon}(\omega)}{\epsilon_0} = \left(1 - \frac{\omega_p^2\tau^2}{1 + \omega^2\tau^2}\right) + i\left(\frac{\omega_p^2\tau}{\omega}\frac{1}{1 + \omega^2\tau^2}\right) \tag{4.21.55}$$

Where ω_p^2 is the natural oscillation frequency of the plasma, known as the **plasma frequency**:

$$\boxed{\omega_p^2 = \frac{nq^2}{\epsilon_0 m}} \tag{4.21.56}$$

In the low frequency/high damping limit, $\omega\tau << 1$:

$$\frac{\tilde{\epsilon}(\omega)}{\epsilon_0} \approx 1 + i\frac{\omega_p^2\tau}{\omega} \tag{4.21.57}$$

While for the high frequency/low damping limit, $\omega\tau >> 1$:

$$\boxed{\frac{\tilde{\epsilon}(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2}} \tag{4.21.58}$$

Notice that, in the high frequency limit, $\tilde{\epsilon}(\omega)$ is entirely real, meaning that a wave will not decay: it will be *undamped*. In fact, the wave is undamped for all $\omega > \omega_p$.

When a plasma is cold and/or dilute, it will suffer very little damping, meaning that $\tau >> 1$, and therefore that the second limit can be used. This scenario tends to come up much more often in exams.

4.21.9 The Lorentz Model

The Lorentz Model pictures a dielectric material as being made up of individual atoms, each of which has a immovable nucleus and an electron which may vibrate as if on a damped spring with frequency ω_0 and damping constant Γ. The equation of motion for a particle is thus:

$$m\ddot{\boldsymbol{r}} = q\boldsymbol{E} - m\Gamma\dot{\boldsymbol{r}} + m\omega_0^2\boldsymbol{r} \tag{4.21.59}$$

In terms of these variables (and the plasma frequency defined in the previous section):

$$\frac{\tilde{\epsilon}(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma} \tag{4.21.60}$$

4.21.10 The Appleton Model of a Magnetized Plasma

The Appleton model is similar to the Drude model, but without collisions and allowing for an external magnetic field. Therefore, it is ideal for describing a cold (relatively collision-less) plasma.

In this model, the presence of the magnetic field allows for coupling between the x and y (transverse) directions. For this reason, it is no longer sufficient to use a vector permitivity, and we must revert to the most general definition of ϵ as a tensor:

$$\boldsymbol{D} = \tilde{\epsilon} \cdot \boldsymbol{E} \tag{4.21.61}$$

The Appleton model then says that, in terms of the plasma frequency and the cyclotron frequency $\omega_c = \frac{qB_0}{m}$:

$$\tilde{\epsilon}(\omega) = \epsilon_0 \begin{pmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & i\frac{\omega_p^2\omega_c}{\omega(\omega^2 - \omega_c^2)} & 0 \\ -i\frac{\omega_p^2\omega_c}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{pmatrix} \tag{4.21.62}$$

4.22 Waveguides and Transmission Lines

Waveguide's and transmission lines are both structures that control the propagation of electromagnetic waves. Several examples of tremendous practical importance are power transmission lines, coax data cables, and fiber optics.

Fields propagating through such a system are classified as **TE (Transverse Electric**, **TM (Transverse Magnetic**, or **TE (Transverse Electric and Magnetic**, where "transverse" here means that the given field vector is always perpendicular to the direction of the wave's propagation through the waveguide. The primary distinction between waveguides and transmission lines is that **waveguides can only support TE and TM waves, while transmission lines are usually TEM**[4.27].

In any waveguide or transmission line the fields must satisfy the usual conductor boundary conditions at the surface:

$$\boldsymbol{E}_{\parallel}\big|_s = 0 \tag{4.22.1}$$

$$\boldsymbol{H}_{\perp}\big|_s = 0 \tag{4.22.2}$$

4.22.1 TEM Waves (Transmission Lines)

Suppose that a TEM wave propagates in the *hatz* direction. The simplest fields will look like a plane wave along $\hat{z}$, but will have more complex behavior in the transverse plane determined by the geometry of the conductor. We will therefore chose to separate:

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_{\perp} e^{i(kz-\omega t)} \\ \boldsymbol{H} &= \boldsymbol{H}_{\perp} e^{i(kz-\omega t)} \end{aligned} \tag{4.22.3}$$

For some geometries, it will be simple enough (or sometimes necessary) to use these fields along with the free space Maxwell equations and conductor boundary conditions to find the fields. However, if the walls of the conductor are nicely perpendicular to the direction of propagation, it is convenient to create a *separate* boundary value problem in the transverse plane.

To do so, we will substitute $\boldsymbol{E} = \boldsymbol{E}_{\perp}$ and $\boldsymbol{B} = \boldsymbol{B}_{\perp}$ into the source free Maxwell equations. By splitting up the vector derivative into perpendicular and parallel components, $\nabla = \nabla_{\perp} + \hat{z}\frac{\partial}{\partial z}$, we can write Maxwell's curl equations as:

$$\begin{aligned} \nabla_{\perp} \times \boldsymbol{E}_{\perp} + \hat{z} \times \frac{\partial \boldsymbol{E}_{\perp}}{\partial z} &= -\mu \frac{\partial \boldsymbol{H}_{\perp}}{\partial t} \\ \nabla_{\perp} \times \boldsymbol{H}_{\perp} + \hat{z} \times \frac{\partial \boldsymbol{H}_{\perp}}{\partial z} &= \epsilon \frac{\partial \boldsymbol{E}_{\perp}}{\partial t} \end{aligned} \tag{4.22.4}$$

Notice that the $\nabla_{\perp}$ components above are in the $\hat{z}$ direction, while all other components are in the transverse plane. We can therefore take only the $\hat{z}$ components to write:

$$\begin{aligned} \nabla_{\perp} \times \boldsymbol{E}_{\perp} &= 0 \\ \nabla_{\perp} \times \boldsymbol{H}_{\perp} &= 0 \end{aligned} \tag{4.22.5}$$

Applying a similar process to the divergence Maxwell equations produces:

$$\begin{aligned} \nabla_{\perp} \cdot \boldsymbol{E}_{\perp} &= 0 \\ \nabla_{\perp} \cdot \boldsymbol{H}_{\perp} &= 0 \end{aligned} \tag{4.22.6}$$

[4.27] It seems to be possible for transmission lines to carry a TE or TM wave under some circumstances, but this is an unusual case.

At this point, these four equations (in addition to the normal boundary conditions for conducting matter) completely specify a 2D boundary value problem for the traverse fields.

Returning to the $\hat{z}$ and transverse separated Maxwell curl equations above (eq. 4.22.4), we examine the equality of the remaining transverse terms:

$$\begin{aligned} \hat{z} \times \frac{\partial \boldsymbol{E}_\perp}{\partial z} &= -\mu \frac{\partial \boldsymbol{H}_\perp}{\partial t} \\ \hat{z} \times \frac{\partial \boldsymbol{H}_\perp}{\partial z} &= \epsilon \frac{\partial \boldsymbol{E}_\perp}{\partial t} \end{aligned} \tag{4.22.7}$$

If we further assume that even the transverse fields are plane waves ($\boldsymbol{E}_\perp \propto e^{i(kz-\omega t)}$, $\boldsymbol{B}_\perp \propto e^{i(kz-\omega t)}$) [4.28], these equations become:

$$\begin{aligned} \hat{z} \times k\boldsymbol{E}_\perp &= -\mu\omega \boldsymbol{H}_\perp \\ \hat{z} \times k\boldsymbol{H}_\perp &= \epsilon\omega \boldsymbol{E}_\perp \end{aligned} \tag{4.22.8}$$

This system of equations implies that:

$$\omega = \frac{1}{\sqrt{\mu\epsilon}} h \tag{4.22.9}$$

Plugging this relation back into the transverse terms above (eq. 4.22.8), we can rewrite the $\hat{z} \times \boldsymbol{E}_\perp$ equation to be:

$$\hat{z} \times \boldsymbol{E}_\perp = \sqrt{\frac{\mu}{\epsilon}} \boldsymbol{H}_\perp \tag{4.22.10}$$

This equation is analogous to $\boldsymbol{B} = \frac{1}{c}\hat{k} \times \boldsymbol{E}$ did in free space.

4.22.2 Conducting Plane(s)

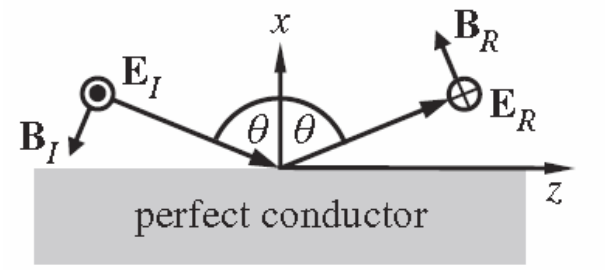


Figure 4.22.1: TE wave incident on a conducting plane. Source: Zangwill pg. 673

Before considering waveguides, it is constructive to examine the interaction between an E&M wave and a conducting plane (which can be thought of as one face of a rectangular waveguide).

Imagine a TE wave incident at an angle θ from the surface normal on a conducting plane (Fig. 4.22.1). At the surface, boundary conditions demand that $\boldsymbol{E}_\perp\big|_s = 0$. However, this condition applies to the *total* field at this point, which includes both the incident *and* reflected

[4.28] Zangwill writes k as h for some reason in this section.

fields. Two valid fields that satisfy this condition would therefore be:

$$\begin{aligned} \boldsymbol{E}_I &= \hat{y}E_0 \exp(ik(z\sin\theta - x\cos\theta - ct)) \\ \boldsymbol{E}_R &= -\hat{y}E_0 \exp(ik(z\sin\theta + x\cos\theta - ct)) \end{aligned} \tag{4.22.11}$$

The $\boldsymbol{B}$ field is then fixed by $\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t}$. Notice that, at $x = 0$, these fields exactly cancel everywhere along the plane, satisfying the boundary condition. Notice, also, by the nature of the cosine function that represents the real part of this field, that there are infinitely many such zeros, spaced out by the relation:

$$kx\cos\theta = m\pi, \quad m \in \mathbb{Z}^+ \tag{4.22.12}$$

We are therefore free to place a *second* conducting plate at any of these values of x. Doing so will cause the wave to bounce back and forth indefinitely between the plates, acting as a sort of 2D waveguide. Notice that it is not possible to have a TEM wave exhibit this behavior: that wave would be normally incident on the plate, and therefore never propagate down the waveguide.

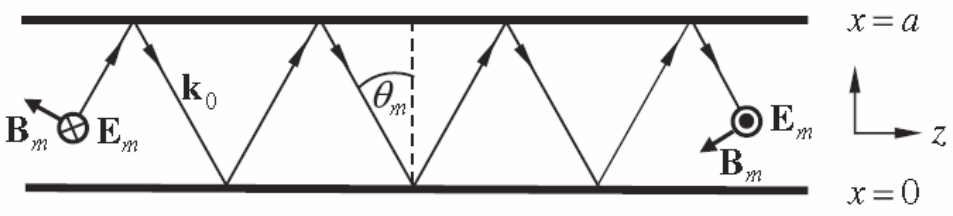


Figure 4.22.2: TE wave between two conducting plates. Source: Zangwill pg. 673

Conversely, then, if we *place* a plate at $x = a$ as shown in Fig. 4.22.2, we are imposing a requirement that:

$$k = m\frac{1}{\cos\theta_m}\frac{\pi}{a}, \quad m \in \mathbb{Z}^+ \tag{4.22.13}$$

Since $0 < \theta_m < \frac{\pi}{2}$ as drawn in the figure, $0 < \cos\theta_m < 1$ and $k = \frac{\omega}{c}$[4.29], we see that the following relationship governs the possible values of ω_m for a given plate placement $x = a$:

$$\omega_m > m\frac{\pi}{a} \tag{4.22.14}$$

The boundary value here defines the **cutoff frequency** of this waveguide:

$$\omega_c = m\frac{\pi}{a} \tag{4.22.15}$$

The physical meaning of this frequency is illuminated by rewriting the total field inside the waveguide in terms of the

$$\boldsymbol{E}_I + \boldsymbol{E}_R = -2E_0 \sin\left(\frac{m\pi x}{a}\right) e^{i(kz\sin\theta - \omega t)} \tag{4.22.16}$$

[4.29] Here I am assuming that the inside of the waveguide is vacuum so that the wave speed is c. If the waveguide is filled with matter, substitute the appropriate wavespeed in terms of the index of refraction.

Notice that we can rewrite

$$k^2 \sin^2\theta = k^2(1-\cos^2\theta) = \frac{\omega^2}{c^2} - m^2\left(\frac{\pi}{a}\right)^2 = {h_m}^2 \tag{4.22.17}$$

For convenience, we will refer to this entire expression as ${h_m}^2$. Then $k\sin\theta_m = h_m$, so:

$$\boldsymbol{E} = -2E_0 \sin\left(\frac{m\pi x}{a}\right) e^{i(zh_m - \omega t)} \tag{4.22.18}$$

Notice now that if h_m is **real**, this wave will **propagate** (since the total z dependence will remain imaginary). However, if h_m is **imaginary**, the wave will **die out** as an evanescent wave (as the z dependence becomes real).

Therefore, h_m real corresponds to $\omega_m > m\frac{\pi}{a} = \omega_p$. Therefore **only waves with frequencies greater than the cutoff frequency ω_p can propagate through the waveguide.**

4.22.3 Conducting Tubes

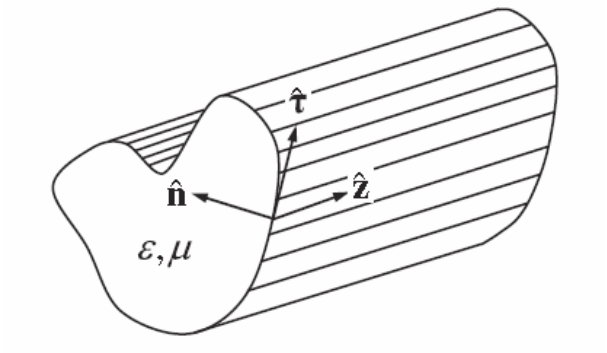


Figure 4.22.3: A generalized conducting tube. Source: Zangwill pg. 675.

The most common type of waveguide is the conducting tube. A generalized conducting tube is shown in figure 4.22.3 to be an infinitely long tube with conducting, parallel walls and a uniform cross section. Such a tube can support either TE or TM fields, but *not* TEM fields. Translation symmetry along the axis of the tube guarantees that $E_z = \hat{z}\cdot\boldsymbol{E}$ is constant along the z axis (but not necessarily zero), and the same is true for $H_z = \hat{z}\cdot\boldsymbol{H}$. If we then assume that the field propogates as a plane wave down the z axis,it is natural to write:

$$\begin{aligned} \boldsymbol{E} &= (\boldsymbol{E}_\perp(\boldsymbol{r}_\perp) + \hat{z}E_z(\boldsymbol{r}_\perp))e^{i(kz-\omega t)} \\ \boldsymbol{H} &= (\boldsymbol{H}_\perp(\boldsymbol{r}_\perp) + \hat{z}H_z(\boldsymbol{r}_\perp))e^{i(kz-\omega t)} \end{aligned} \tag{4.22.19}$$

Notice that for TE waves, $E_z = 0$, while for TM waves $H_z = 0$. Maxwell's curl equations are:

$$\begin{aligned} \nabla\times\boldsymbol{E} &= i\omega\mu\boldsymbol{H} \\ \nabla\times\boldsymbol{H} &= -i\omega\epsilon\boldsymbol{E} \end{aligned} \tag{4.22.20}$$

Again writing $\nabla = \nabla_\perp + \hat{z}\frac{\partial}{\partial z}$ and using the fields defined above,

$$\nabla\times\boldsymbol{E} = (\nabla_\perp\times\boldsymbol{E}_\perp + ik\hat{z}\times\boldsymbol{E}_\perp - \hat{z}\times\nabla_\perp E_z)e^{i(kz-\omega t)} \tag{4.22.21}$$

A similar expression can be found for $\boldsymbol{H}$. Notice in the above equation that the first $\nabla_\perp \times \boldsymbol{E}_\perp$ must point along $\hat{z}$ (perpendicular to $\boldsymbol{E}_\perp$), while the remaining two terms must be transverse (perpendicular to $\hat{z}$). Therefore, substituting these curl expressions into Maxwell's equations allows us to split them up by components:

$$\begin{aligned} \nabla_\perp \times \boldsymbol{E}_\perp &= i\omega\mu H_z \hat{z} \\ i\omega\mu \boldsymbol{H}_\perp - ik\hat{z} \times \boldsymbol{E}_\perp &= -\hat{z} \times \nabla_\perp E_z \end{aligned} \tag{4.22.22}$$

$$\begin{aligned} \nabla_\perp \times \boldsymbol{H}_\perp &= -i\omega\epsilon E_z \hat{z} \\ i\omega\epsilon \boldsymbol{E}_\perp + ik\hat{z} \times \boldsymbol{H}_\perp &= -\hat{z} \times \nabla_\perp H_z \end{aligned} \tag{4.22.23}$$

If we take E_z and H_z to be givens, this system of equations completely defines $\boldsymbol{E}_\perp$ and $\boldsymbol{H}_\perp$ (each of which has two components). Rearrangement (by taking the cross product $\hat{z}$ with the second and fourth equations) yields:

$$\boxed{\begin{aligned} \boldsymbol{E}_\perp &= \frac{ik}{\gamma^2}\nabla_\perp E_z - \frac{i\omega\mu}{\gamma^2}\hat{z} \times \nabla_\perp H_z \\ \boldsymbol{H}_\perp &= \frac{ik}{\gamma^2}\nabla_\perp H_z + \frac{i\omega\epsilon}{\gamma^2}\hat{z} \times \nabla_\perp E_z \end{aligned}} \tag{4.22.24}$$

Where for convenience

$$\gamma^2 = \mu\epsilon\omega^2 - k^2 \tag{4.22.25}$$

Yet more rearrangement (now taking the curl of both of *these* equations) transforms these equations into a pair of Helmholtz equations:

$$\boxed{\begin{aligned} (\nabla_\perp{}^2 + \gamma^2)E_z &= 0 \\ (\nabla_\perp{}^2 + \gamma^2)H_z &= 0 \end{aligned}} \tag{4.22.26}$$

So far, these equations are totally general for any mode. However, assuming that $E_z = 0$ or $H_z = 0$ *greatly* simplifies both of the boxed sets of equations (eq. 4.22.24 and eq. 4.22.26) to a single Helmholtz equation equation for the $\hat{z}$ direction and a pair of equations to determine the traverse fields. Some final rearrangement produces this system:

- **TE:**

$$\begin{aligned} E_z &= 0 \\ (\nabla_\perp{}^2 + \gamma^2)H_z &= 0 \\ \boldsymbol{H}_\perp &= \frac{ik}{\gamma^2}\nabla_\perp H_z \\ \boldsymbol{E}_\perp &= -\frac{\omega\mu}{k}\hat{z} \times \boldsymbol{H}_\perp \end{aligned} \tag{4.22.27}$$

- **TM:**

$$\begin{aligned} H_z &= 0 \\ (\nabla_\perp{}^2 + \gamma^2)E_z &= 0 \\ \boldsymbol{E}_\perp &= \frac{ik}{\gamma^2}\nabla_\perp E_z \\ \boldsymbol{H}_\perp &= \frac{\omega\epsilon}{k}\hat{z}\times\boldsymbol{E}_\perp \end{aligned} \tag{4.22.28}$$

This result has reduced the entire problem of finding the fields to finding their $\hat{z}$ components. The Helmholtz equations for these components are a boundary value problem determined by the condition that $\hat{n}\times\boldsymbol{E}\big|_s = 0$ everywhere on the surface of the tube. Zangwill shows[4.30] that for our generalized tube geometry, this condition reduces to:

$$\boxed{E_z\big|_s = 0 \quad \text{(TM Waves)}}, \qquad \boxed{\left.\frac{\partial H_z}{\partial n}\right|_s = 0 \quad \text{(TE Waves)}} \tag{4.22.29}$$

This tidy system of equations given out initial fields justifies our choice of TE and TM as a "basis": any field can be written as $\boldsymbol{E} = \boldsymbol{E}_{TM} + \boldsymbol{E}_{TE}$.

Explicitly calculating the TE and TM components of our fields in this manner yield a useful result:

$$\begin{aligned} Z\boldsymbol{H}_{TE} &= \boldsymbol{E}_{TM} \\ \boldsymbol{E}_{TE} &= -Z\boldsymbol{H}_{TM} \end{aligned} \tag{4.22.30}$$

Where Z is the impedance $Z = \sqrt{\frac{\mu}{\epsilon}}$.

Another very important result[4.31] is the cutoff frequency for a generalized tube waveguide:

$$\boxed{\omega_{c,i} = \frac{\gamma_i}{\sqrt{\mu\epsilon}} = \gamma_i\frac{c}{n}} \tag{4.22.31}$$

Where i indexes the modes of the waveguide (each mode has its own cutoff frequency).

These equations alone do not prove that TEM modes cannot propagate through a conducting tube. Setting $H_z = 0$ in our original transverse Maxwell equations tells us that $\nabla_\perp \times \boldsymbol{E}_\perp = 0$, while Maxwell's divergence equation requires that $\nabla_\perp \cot \boldsymbol{E}_\perp = 0$. Together these describe a boundary value problem $\nabla_\perp^2\phi = 0$ where $\boldsymbol{E}_\perp = -\nabla\phi$. Since the boundary is a conductor, ϕ must be constant and therefore $\boldsymbol{E}_\perp = 0$. A similar argument with the other set of Maxwell equations **setting** $E_z = 0$ shows that $\boldsymbol{H}_\perp = 0$. Therefore, if $E_z = H_z = 0$, **no wave can propagate**.

4.22.4 Example: Rectangular Tube Waveguides

For the case of a rectangular tube with sides of length a and b, the Helmholtz equation boundary value problem reduces to:

$$\text{TM:} \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right]E_z = 0, \qquad \text{and} \quad E_z|_s = 0 \tag{4.22.32}$$

[4.30] pg. 677.
[4.31] Derived in Zangwill pg. 679

$$\text{TE:} \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right] H_z = 0, \qquad \text{and} \quad \left.\frac{\partial H_z}{\partial n}\right|_s = 0 \tag{4.22.33}$$

Which produces fields that look like:

$$\text{TE:} \quad E_z^{mn}(x,y) = E \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right), \qquad m,n \in \mathbb{Z}^+ \tag{4.22.34}$$

$$\text{TM:} \quad H_z^{mn}(x,y) = H \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right), \qquad m,n \in \mathbb{Z}^+ \tag{4.22.35}$$

With eigenvalues $\gamma_{mn} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

4.22.5 Example: Cylindrical Tube Waveguides

For the case of a cylindrical tube of radius R, the Helmholtz equation boundary value problem reduces to:

$$\text{TM:} \quad \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \gamma^2\right] E_z = 0, \qquad \text{and} \quad E_z|_s = 0 \tag{4.22.36}$$

$$\text{TE:} \quad \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \gamma^2\right] H_z = 0, \qquad \text{and} \quad \left.\frac{\partial H_z}{\partial n}\right|_s = 0 \tag{4.22.37}$$

The solutions to these Helmholtz equations in cylindrical coordinates are cylindrical Bessel functions (of the first kind, in order to enforce finiteness of the field at the origin) in ρ and exponential in ϕ:

$$\text{TM:} \quad E_z^{mn}(\rho,\phi) = E J_m(\gamma_{mn}^{TM}\rho) e^{\pm im\phi}, \qquad m = 0, 1..., \quad n = 1, 2... \tag{4.22.38}$$

$$\text{TE:} \quad H_z^{mn}(\rho,\phi) = H J_m(\gamma_{mn}^{TE}\rho) e^{\pm im\phi}, \qquad m = 0, 1..., \quad n = 1, 2... \tag{4.22.39}$$

Where the absence of of $n = 0$ is due to the non-existence of $J_{0,0}$. The eigenvalues are defined as:

$$\gamma_{mn}^{TM} = \frac{u_{mn}}{R}, \qquad \gamma_{mn}^{TE} = \frac{w_{mn}}{R} \tag{4.22.40}$$

Where u_{mn} are the zeros of the Bessel function J_{mn}, and w_{mn} are the zeros of the *derivative* of the Bessel function: J'_{mn} (the derivative appears because the TE boundary condition is a derivative).

4.22.6 Flow of Energy Through a Waveguide

The time averaged Poynting flux through a waveguide is given by:

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\text{Re}(\boldsymbol{E}_\perp \times \boldsymbol{H}_\perp^*) \cdot \hat{z} \tag{4.22.41}$$

For both TE and TM waves, this energy propagates at the group velocity.

4.23 Fields of Moving Particles (Radiation)

As a charged particle moves, its electric field is deformed and it generates a magnetic field (as a sort of one-particle current). These fields are further deformed if the particle accelerates. During this acceleration, many of the field lines may remain attached to the particle. However, some field lines cross themselves, forming closed field *loops* in a phenomenon known as field line re-connection. These loops are now self-sufficient radiation fields, and will propagate away from the particle to infinity. This is the birth of an electromagnetic wave.

This process is somewhat analogous to a wet dog shaking off water droplets. Most of the water remains attached to the dog by surface tension. However, occasionally some water forms a droplet, which is flung free of the fur into empty space, now held together by *its own* surface tension.

While it is true that all radiation is created by accelerating charges, the reverse is not true. Under special circumstances, multiple particles may move in a way that conspires to cancel out the radiating components of their fields as they accelerate, therefore producing no radiation.

4.23.1 Retardation

When describing radiation, we are concerned with the field of an accelerating charge very far away (at infinity, in fact). Since these fields (and therefore the information they carry) propagate at finite speed (the speed of light), the field at some point $\boldsymbol{r}$ is determined, not by the acceleration $\ddot{\boldsymbol{r}}$ of the source charge *now*, but it's acceleration some time ago: t', called the **retarded time**[4.32].

$$t' = t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c} \tag{4.23.1}$$

The quantity $|\boldsymbol{r} - \boldsymbol{r}'| = \boldsymbol{\imath}$ is can be expressed exactly using the law of cosines (see fig. 4.1.1 for a reminder of the definition of each vector):

$$|\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{r'^2 + r^2 - 2rr'\cos\theta} = \sqrt{(r - r'\cos\theta)^2 + r'^2\sin^2\theta} \tag{4.23.2}$$

In the limit where $r' << r$, we can approximate $r' \approx 0$ in the above expression. **Notice that, if we are calculating a radiation field, this is approximation is actually exact**. We can chose to examine the radiation field at infinity, where this condition is satisfied exactly.

$$|\boldsymbol{r} - \boldsymbol{r}'| \approx \boldsymbol{r} \qquad \text{(1st order)} \tag{4.23.3}$$

In which case, the retarded time becomes

$$t' = t - \frac{\boldsymbol{r}}{c} \qquad \text{(1st order)} \tag{4.23.4}$$

Even at infinity, this first order approximation is only valid if the source can be treated as a point source without loss of important physics. This is sometimes NOT the case, as interference between different parts of the source can create interference that change

[4.32] There is, theoretically, also an *advanced time*, $t' = t + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}$. However, this time is not causal, and is therefore not used.

the radiation fields. In this, case we will use a 2nd order approximation. Letting only $r'^2 = 0$ in eq. 4.23.2:

$$|\boldsymbol{r} - \boldsymbol{r}'| \approx \boldsymbol{r} - \boldsymbol{r}' \cos\theta = \boldsymbol{r} - \hat{r} \cdot \boldsymbol{r}' \qquad \text{(2nd order)} \tag{4.23.5}$$

Such that, to 2nd order:

$$t' = t - \frac{\boldsymbol{r}}{c} + \frac{\hat{r} \cdot \boldsymbol{r}'}{c} \qquad \text{(2nd order)} \tag{4.23.6}$$

It can be shown[4.33] that the retarded *potentials* are exactly what we might hope they would be:

$$\phi_{ret}(\boldsymbol{r}, t) = \phi(\boldsymbol{r}, t_{ret}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}', t_{ret})}{\mathscr{r}} d^3r' \tag{4.23.7}$$

$$\boldsymbol{A}_{ret}(\boldsymbol{r}, t) = \boldsymbol{A}(\boldsymbol{r}, t_{ret}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r}, t_{ret})}{\mathscr{r}} d^3r' \tag{4.23.8}$$

However, **THE SAME IS NOT TRUE FOR THE FIELDS** (in short, because the derivatives to produce them are complicated by the implicit dependence of t_{ret} on $\boldsymbol{r}$). The fields must be obtained by $\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. This calculation can be done for a general current and charge density to produce **Schott's Formulae (or Jefimenko's Equations)**[4.34].

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \left[\frac{\rho_{ret} \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} + \frac{\boldsymbol{\mathscr{r}}}{c\mathscr{r}^2} \frac{\partial \rho_{ret}}{\partial t} - \frac{1}{c^2 \mathscr{r}} \frac{\partial \boldsymbol{j}_{ret}}{\partial t} \right] \tag{4.23.9}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \left[\frac{\boldsymbol{j}_{ret}}{\mathscr{r}^3} + \frac{1}{c\mathscr{r}^2} \frac{\partial \boldsymbol{j}_{ret}}{\partial t} \right] \times \boldsymbol{\mathscr{r}} \tag{4.23.10}$$

Where $\boldsymbol{\mathscr{r}} = |\boldsymbol{r} - \boldsymbol{r}'|$ and $t_{ret} = t - \frac{\mathscr{r}}{c}$. In actual calculations, it is usually easier to just calculate the retarded potentials, and find the fields directly from them.

4.23.2 Fields of a Moving Point Charge

Suppose that a point charge q moves along some path. The Schott/Jefimenko equations (eq. 4.23.9 and eq. 4.23.10) are not easily applied to this situation, since the particle does not really constitute a current or a charge density. Instead, we must return to the retarded potential:

$$\phi_{ret}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}', t_{ret})}{\mathscr{r}} d^3r' \tag{4.23.11}$$

At a given observation point in time and space, the field produced by the charge is created by only *one* time and position on the particles trajectory. Therefore, $\mathscr{r}$ is fixed as we evaluate the integral (although its functional dependence changes from $|\boldsymbol{r} - \boldsymbol{r}'|$ to $|\boldsymbol{r} - \boldsymbol{r}'(\boldsymbol{r}, t_{ret})|$).

$$\phi_{ret}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{\mathscr{r}} \int \rho(\boldsymbol{r}', t_{ret}) d^3r' \tag{4.23.12}$$

[4.33] Griffiths E&M pg. 424.
[4.34] Derived in Zangwill, pg.726. or Griffiths E&M pg. 427.

For a stationary charge, the remaining integral would be simply equal to q. However, Lorentz contraction of the particle actually *distorts* the charge of the particle[4.35]. Lorentz contraction along an axis introduces a factor of $\frac{1}{1-\frac{v}{c}}$. However, since we are viewing the particle from an arbitrary *angle*, this is modified to be $\frac{1}{1-\hat{\boldsymbol{\imath}}\cdot\frac{\boldsymbol{v}}{c}}$.

so that:

$$\phi_{ret}(\boldsymbol{r},t)=\frac{1}{4\pi\epsilon_0}\frac{1}{\imath}\left[\int\rho(\boldsymbol{r}',t_{ret})d^3r'\right]=\frac{1}{4\pi\epsilon_0}\frac{1}{\imath}\left[\frac{q}{1-\hat{\boldsymbol{\imath}}\cdot\frac{\boldsymbol{v}}{c}}\right] \tag{4.23.13}$$

Similarly,

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\int\frac{\rho(\boldsymbol{r}',t_{ret})\boldsymbol{v}(t_{ret})}{\imath}d^3r=\frac{\mu_0}{4\pi}\frac{\boldsymbol{v}(t_{ret})}{\imath}\int\rho(\boldsymbol{r}',t_{ret})d^3r=\frac{\mu_0}{4\pi}\frac{qc\boldsymbol{v}}{(\imath\ c-\boldsymbol{\imath}\cdot\boldsymbol{v})} \tag{4.23.14}$$

The final results are called the **Liénard-Wiechert potentials**:

$$\begin{aligned}\phi_{ret}(\boldsymbol{r},t)&=\frac{1}{4\pi\epsilon_0}\frac{1}{\imath}\frac{q}{1-\hat{\boldsymbol{\imath}}\cdot\frac{\boldsymbol{v}}{c}}\\ \boldsymbol{A}(\boldsymbol{r},t)&=\frac{\mu_0}{4\pi}\frac{qc\boldsymbol{v}}{(\imath\ c-\boldsymbol{\imath}\cdot\boldsymbol{v})}\end{aligned} \tag{4.23.15}$$

Notice that:

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\boldsymbol{v}}{c^2}\phi(\boldsymbol{r},t) \tag{4.23.16}$$

These results then lead, through a tremendously intimidating several pages of vector algebra[4.36] to the fields generated by the motion of the point charge. Letting $\boldsymbol{u}\equiv c\hat{\boldsymbol{\imath}}-\boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t)&=\frac{q}{4\pi\epsilon_0}\frac{\imath}{(\boldsymbol{\imath}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u}+\boldsymbol{\imath}\times(\boldsymbol{u}\times\boldsymbol{a})]\\ \boldsymbol{B}(\boldsymbol{r},t)&=\frac{1}{c}\hat{\boldsymbol{\imath}}\times\boldsymbol{E}(\boldsymbol{r},t)\end{aligned} \tag{4.23.17}$$

An important example is that of a particle moving with constant velocity . In this case:

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q}{4\phi\epsilon_0}\frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}\frac{1-\frac{v^2}{c^2}}{(1-\frac{v^2\sin^2\theta}{c^2})^{\frac{3}{2}}} \tag{4.23.18}$$

In this case, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are as shown in figure 4.23.1.

4.23.3 Time-Dependent Electric Dipoles

Consider a time dependent dipole moment $\boldsymbol{p}(t)$ (the following discussion will be valid for *any* dipole moment). The associated charge distribution is (if we place the dipole at the origin):

$$\rho(\boldsymbol{r},t)=-\boldsymbol{p}(t)\cdot\nabla\delta(\boldsymbol{r})=-\nabla\cdot(\boldsymbol{p}(t)\delta(\boldsymbol{r})) \tag{4.23.19}$$

[4.35] It turns out this is true even for a point particle, since point particles are really just a limit of a finite charge distribution.

[4.36] Griffiths E&M pg.436-437, if you really must.

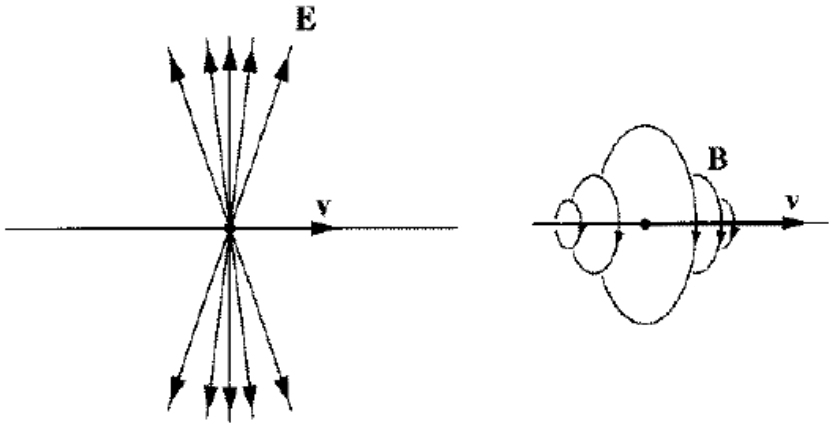


Figure 4.23.1: Radiation pattern point charge with a constant velocity (comparable to c) point charge. (Source: Griffith's E&M, pg. 440)

and therefore, by the charge continuity equation:

$$\boldsymbol{j}(\boldsymbol{r},t) = \dot{\boldsymbol{p}}(t)\delta(\boldsymbol{r}) \tag{4.23.20}$$

Evaluating the retarded potentials with these functions yields:

$$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{\dot{\boldsymbol{p}}(t_{ret})\cdot\boldsymbol{r}}{cr^2} + \frac{\boldsymbol{p}(t_{ret})\cdot\boldsymbol{r}}{r^3}\right] \tag{4.23.21}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}(t_{ret})}{r} \tag{4.23.22}$$

The resulting fields are then:

$$\boxed{\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{r}(\hat{r}\cdot\boldsymbol{p}_{ret}) - \boldsymbol{p}_{ret}}{r^3} + \frac{3\hat{r}(\hat{r}\cdot\dot{\boldsymbol{p}}_{ret}) - \dot{\boldsymbol{p}}_{ret}}{cr^2} + \frac{3\hat{r}(\hat{r}\cdot\ddot{\boldsymbol{p}}_{ret}) - \ddot{\boldsymbol{p}}_{ret}}{c^2 r}\right]} \tag{4.23.23}$$

$$\boxed{\boldsymbol{B}(\boldsymbol{r},t) = -\frac{\mu_0}{4\pi}\hat{r}\times\left[\frac{\dot{\boldsymbol{p}}_{ret}}{r^2} + \frac{\ddot{\boldsymbol{p}}_{ret}}{cr}\right]} \tag{4.23.24}$$

Notice that the terms in both of these fields have different power's of r in their denominators. These terms are each dominant in a particular region of space. For example, since $\boldsymbol{S} \propto \boldsymbol{E}\times\boldsymbol{B}$, only the $\frac{1}{r}$ terms can contribute to radiation, as all others will die off as $\boldsymbol{r}\to\infty$.

If the source varies with a characteristic frequency ω:

- **The Near Field**: $\propto \frac{1}{r^3}$, $\quad r\omega << c$.

  The near field is the closest to the source, including the source itself. It's field contributions is the $\boldsymbol{p}_{ret}$ term of $\boldsymbol{E}$ ($\boldsymbol{B}$ is zero in this region).

- **The Intermediate Field**: $\propto \frac{1}{r^2}$, $\quad r\omega \approx c$.

  The intermediate field includes the $\dot{\boldsymbol{p}}_{ret}$ terms.

- **The Far Field / Radiation Zone**: $\propto \frac{1}{r}, \quad r\omega >> c.$

  The intermediate field extends to infinity, and includes the $\ddot{\boldsymbol{p}}_{ret}$ terms. It is *only* these components of the fields that contribute to radiation.

  Here, it is useful to define some notation:

$$\boldsymbol{E}_{rad}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\frac{3\hat{r}(\hat{r}\cdot\ddot{\boldsymbol{p}}_{ret}) - \ddot{\boldsymbol{p}}_{ret}}{c^2 r}, \qquad \boldsymbol{B}_{rad}(\boldsymbol{r},t) = -\frac{\mu_0}{4\pi}\hat{r}\times\left[\frac{\ddot{\boldsymbol{p}}_{ret}}{cr}\right] \tag{4.23.25}$$

The power radiated by the dipole per unit solid angle can be written in terms of the Poynting flux *of only the radiation zone terms*:

$$\frac{dP}{d\Omega} = r^2\hat{r}\cdot\boldsymbol{S} = r^2\hat{r}\cdot\frac{1}{\mu_0}(\boldsymbol{E}_{rad}\times\boldsymbol{B}_{rad}) \tag{4.23.26}$$

Substituting in the fields found earlier gives the radial power distribution for a generalized dipole moment:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}|\hat{r}\times\ddot{\boldsymbol{p}}_{ret}|^2 \tag{4.23.27}$$

Note that, if the direction of $\ddot{\boldsymbol{p}}_{ret}$ is fixed, the cross product produces a $\sin^2\theta$ factor. Integrating over the solid angle produces the total radiated power:

$$P(t) = \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\boldsymbol{p}}_{ret}|^2}{3c^3} \tag{4.23.28}$$

This equation is very similar to (but not *the same as* the Larmor formula for the power radiated from a point charge. While conceptually separate, the Larmor formula can be reproduced here by setting $\boldsymbol{p} = q\boldsymbol{r}$.

4.23.4 Fields in the Radiation Zone (Time-domain)

In general, the vector potential in the radiation zone (letting $\mathscr{r} \approx r$ is:

$$\boldsymbol{A}_{rad}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\int d^3r'\boldsymbol{j}_{ret} \tag{4.23.29}$$

The general fields produced by this current are then:

$$c\boldsymbol{B}_{rad}(\boldsymbol{r},t) = -\hat{r}\times\frac{\partial\boldsymbol{A}_{rad}}{\partial t} \tag{4.23.30}$$

$$\boldsymbol{E}_{rad}(\boldsymbol{r},t) = -\hat{r}\times c\boldsymbol{B}_{rad}(\boldsymbol{r},t) \tag{4.23.31}$$

So, since clearly $|\boldsymbol{E}_{rad}| = c|\boldsymbol{B}_{rad}|$:, we have:

$$\frac{dP}{d\Omega} = r^2\hat{r}\cdot\boldsymbol{S} = \frac{r^2}{c\mu_0}|\boldsymbol{E}_{rad}|^2 \tag{4.23.32}$$

4.23.5 Fields in the Radiation Zone (Time-domain)

While the fields given in the previous section are valid, it is often easier (for inherently periodic problems) to work with their Fourier transform into the frequency domain. Both the current and vector potential can be written as a Fourier transform:

$$\begin{aligned} \boldsymbol{j}(\boldsymbol{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{j}(\boldsymbol{r}|\omega)e^{-i\omega t}d\omega \\ \boldsymbol{A}_{rad}(\boldsymbol{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{A}_{rad}(\boldsymbol{r}|\omega)e^{-i\omega t}d\omega \end{aligned} \tag{4.23.33}$$

This allows us to write[4.37]:

$$\boldsymbol{A}_{rad}(\boldsymbol{r}|\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\boldsymbol{j}(\boldsymbol{k}|\omega) \tag{4.23.34}$$

Where $\boldsymbol{j}(\boldsymbol{k}|\omega)$ is the Fourier transform in *both* space and time of the source current density.

The resulting fields are:

$$\begin{aligned} \boldsymbol{E}_{rad}(\boldsymbol{r}|\omega) &= -i\omega\hat{\boldsymbol{r}}\times[\hat{\boldsymbol{r}}\times\boldsymbol{A}_{rad}(\boldsymbol{r}|\omega)] \\ c\boldsymbol{B}_{rad}(\boldsymbol{r}|\omega) &= i\omega\hat{\boldsymbol{r}}\times\boldsymbol{A}_{rad}(\boldsymbol{r}|\omega) \end{aligned} \tag{4.23.35}$$

4.23.6 The Larmor Formula

Consider an accelerating point charge with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}_0(t)$, which we will interpret as a current:

$$\boldsymbol{j}(\boldsymbol{r},t) = q\boldsymbol{v}(t)\delta(\boldsymbol{r}-\boldsymbol{r}_0(t)) \tag{4.23.36}$$

Taking the first order retardation approximation, $\boldsymbol{j}_{ret} = \boldsymbol{j}(\boldsymbol{r}, t-\frac{r}{c})$. Evaluating the field and then plugging into the general power distribution formula gives:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2|\boldsymbol{a}_{ret}|^2}{16\pi^2 c}\sin^2\theta \tag{4.23.37}$$

Which, integrated over the solid angle, is the **Larmor formula**:

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2|\boldsymbol{a}_{ret}|^2}{3c^3} \tag{4.23.38}$$

4.23.7 Generalized Cartesian Multipole Radiation

The general $\boldsymbol{E}$ and $\boldsymbol{B}$ fields of a certain current distribution $\boldsymbol{j}$ are determined by time derivatives of the vector potential:

$$\boldsymbol{A}_{rad}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r}\int d^3r'\boldsymbol{j}_{ret} \tag{4.23.39}$$

If we introduce a new quantity, the **radiation vector**:

$$\boldsymbol{\alpha}(\boldsymbol{r},t) = \frac{\partial}{\partial t}\int d^3r'\boldsymbol{j}(\boldsymbol{r'}, t-\frac{r}{c}+\frac{1}{c}\hat{\boldsymbol{r}}\cdot\boldsymbol{r'}) \tag{4.23.40}$$

[4.37] Zangwill pg. 736

Now, suppose we are only interested in an approximation of fields at a point far from the source (as in the electric potential multipole expansion). In this case, we can approximate:

$$\begin{aligned}\boldsymbol{j}_{ret} &= \boldsymbol{j}(\boldsymbol{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\boldsymbol{r}'}{c}) \\ &\approx \vec{j}(\boldsymbol{r}', t - \frac{r}{c}) + \frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}{c}\frac{\partial}{\partial t}\vec{j}(\boldsymbol{r}', t - \frac{r}{c}) + \frac{1}{2}\left(\frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}{c}\right)^2\frac{\partial^2}{\partial t^2}\boldsymbol{j}(\boldsymbol{r}', t - \frac{r}{c}) + ... \end{aligned} \tag{4.23.41}$$

Just like the terms of the electric scalar potential multipole expansion, each of these terms corresponds to a particular "moment". Zangwill shows[4.38] that $\boldsymbol{\alpha}$ can now be written (up to the three terms in $\boldsymbol{j}$ written above) in terms of the electric dipole moment $\boldsymbol{p}$, the magnetic dipole moment $\boldsymbol{m}$, and the electric quadrupole moment $\boldsymbol{Q}$:

$$\boldsymbol{\alpha}(\boldsymbol{r}, t) \approx \frac{d^2}{dt^2}\boldsymbol{p}_{ret} + \frac{1}{c}\frac{d^2}{dt^2}\boldsymbol{m}_{ret}\times\hat{\boldsymbol{r}} + \frac{1}{c}\frac{d^3}{dt^3}\boldsymbol{Q}_{ret}\cdot\hat{\boldsymbol{r}} \tag{4.23.42}$$

By dividing the field in this way, we can easily compute the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields for each moment. By convention, the following labels are used to refer to the moments:

- E1 $\rightarrow$ Electric Dipole
- M1 $\rightarrow$ Magnetic Dipole
- E2 $\rightarrow$ Electric Quadrupole

4.23.8 Electric and Magnetic Dipole Radiation

The radiation zone fields for the electric dipole are[4.39]:

$$\begin{aligned}\boldsymbol{E}_{E1} = -\hat{\boldsymbol{r}}\times c\boldsymbol{B}_{E1} = \frac{\mu_0}{4\pi}\frac{\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\ddot{\boldsymbol{p}}_{\boldsymbol{ret}}) - \ddot{\boldsymbol{p}}_{\boldsymbol{ret}}}{r} = \frac{\mu_0}{4\pi}\frac{[\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\boldsymbol{p})]}{r} \\ \boldsymbol{B}_{E1} = -\frac{\mu_0}{4\pi c}\frac{\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}_{\boldsymbol{ret}}}{r}\end{aligned} \tag{4.23.43}$$

The angular power distribution is:

$$\left(\frac{dP}{d\Omega}\right)_{E1} = \frac{\mu_0}{16\pi^2 c}|\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}_{\boldsymbol{ret}}|^2 \tag{4.23.44}$$

The corresponding radiated power looks *very similar* to the Larmor formula, but is in fact more general:

$$P_{E1}(t) = \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\boldsymbol{p}}_{\boldsymbol{ret}}|^2}{3c^3} \tag{4.23.45}$$

The Larmor formula can be retrieved by setting $\boldsymbol{p} = q\boldsymbol{r}$.

In the (extremely common) case of a time harmonic dipole moment $\boldsymbol{p}(t) = \boldsymbol{p}e^{-i\omega t}$, the dipole

[4.38] In sections throughout section 20.7.
[4.39] Zangwill pg. 745.

fields simplify to:

$$\begin{aligned} \boldsymbol{E}_{E1} &= -\frac{\mu_0}{4\pi}\omega^2[\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{p})]\frac{e^{i(kr-\omega t)}}{r} \\ \boldsymbol{B}_{E1} &= \frac{\mu_0}{4\pi}\frac{\omega^2}{c}(\hat{\boldsymbol{r}} \times \boldsymbol{p})\frac{e^{i(kr-\omega t)}}{r} \end{aligned} \tag{4.23.46}$$

Everything for the magnetic dipole is the same, but with $\boldsymbol{p} \to \frac{\boldsymbol{m}}{c}$, and switching $\boldsymbol{E}$ and $\boldsymbol{M}$ such that:

$$\boldsymbol{E}_{M1} \to -c\boldsymbol{B}_{E1}, \qquad \boldsymbol{B}_{M1} \to \frac{1}{c}\boldsymbol{E}_{E1} \tag{4.23.47}$$

This is because nature feels bad about how many radiation formulas you already have to memorize.

4.24 Electromagnetic Scattering

In the electromagnetic scattering problem, we are interested in the interaction between an incoming light wave and some object that absorbs the light and re-radiates it. The nature of this interaction is described by the angular redistribution of power:

$$\frac{d\sigma}{d\Omega} = \frac{\text{Power scattered into } d\Omega}{\text{Total incident power}} \tag{4.24.1}$$

Which can be expressed in a number of useful ways:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{r^2\hat{\boldsymbol{r}} \cdot \langle \boldsymbol{S}_{rad} \rangle}{|\langle \boldsymbol{S}_{inc} \rangle|} = \frac{\left\langle \frac{dP}{d\Omega} \right\rangle}{|\langle \boldsymbol{S}_{inc} \rangle|} = \frac{\left\langle \frac{dP}{d\Omega} \right\rangle}{\frac{1}{2}\epsilon_0 c|\boldsymbol{E}_{inc}|^2}} \tag{4.24.2}$$

The total field can be split into an incident wave and a scattered wave:

$$\boldsymbol{E} = \boldsymbol{E}_{inc} + \boldsymbol{E}_{rad} = E_0\left(\hat{\boldsymbol{e}}_0 e^{i\boldsymbol{k}_0 \cdot \boldsymbol{r}} + \frac{e^{ikr}}{r}\boldsymbol{f}(\boldsymbol{k})\right)e^{-i\omega t} \tag{4.24.3}$$

Where we have absorbed all of the angular information about the scattering pattern into the **scattering amplitude** $\boldsymbol{f}(\boldsymbol{k})$.

Since the radiated field is by definition in the radial direction, $\hat{\boldsymbol{r}} \cdot \langle \boldsymbol{S}_{rad} \rangle = \langle \boldsymbol{S}_{rad} \rangle$. Therefore[4.40]:

$$\frac{d\sigma}{d\Omega} = \frac{r^2\langle \boldsymbol{S}_{rad} \rangle}{|\langle \boldsymbol{S}_{inc} \rangle|} = \frac{r^2|E_{rad}|^2}{|E_0|^2} = |\boldsymbol{f}(\boldsymbol{k})|^2 \tag{4.24.4}$$

In general, this differential cross section will depend on the polarization of the incoming wave. If we are interested in scattered waves with polarization $\boldsymbol{\epsilon}$, then $\hat{\boldsymbol{r}} = \hat{\boldsymbol{\epsilon}}$, so:

$$\frac{d\sigma}{d\Omega} = \frac{r^2|\boldsymbol{\epsilon}^* \cdot E_{rad}|^2}{|E_0|^2} = |\boldsymbol{\epsilon}^* \cdot \boldsymbol{f}(\boldsymbol{k})|^2 \tag{4.24.5}$$

We have previously derived expressions for the electric field radiated by an arbitrary time

[4.40] Remember, these fields are in vacuum, so $\boldsymbol{E} = c\boldsymbol{B}$.

harmonic current density. Inserting this expression yields[4.41]:

$$\frac{d\sigma_{scatt}}{d\Omega} = \left(\frac{k}{4\pi\epsilon_0 E_0 c}\right)^2 \left|\hat{\boldsymbol{k}} \times \int d^3r' \boldsymbol{j}(\boldsymbol{r}'|\omega) e^{i\boldsymbol{k}\cdot\boldsymbol{r}'}\right|^2 \tag{4.24.6}$$

4.24.1 Thomson Scattering

Thomson scattering is a light wave scattering off of a single free electron. In the weak field limit, the electric force on the electron dominates, so its equation of motion is:

$$m\ddot{\boldsymbol{r}}_0 = -eE_0 e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}_0 - \omega t)} \tag{4.24.7}$$

This motion creates a current density, which in turn radiates. Using the radiation vector potential to find $\boldsymbol{E}_{rad}$, we can calculate that:

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 |\hat{\boldsymbol{k}}_0 \times \hat{\boldsymbol{e}}_0|^2 = r_e^2(1 - |\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{e}}_0|^2) \tag{4.24.8}$$

Not coincidentally, r_e is the **classical electron radius**: $r_e = \frac{e^2}{4\pi\epsilon_0 mc^2}$. This result was derived assuming a polarized incident wave. However, we are generally interested in the *unpolarized* cross section. This can be calculated by averaging over two polarization basis vectors:

$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2}\sum_{m=1}^{2} r_e^2(1 - |\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{e}}_m|^2) \tag{4.24.9}$$

In this case:

$$\frac{d\sigma_\perp}{d\Omega} = r_e^2(1 - \cos^2(0)) = r_e^2 \tag{4.24.10}$$

While

$$\frac{d\sigma_\parallel}{d\Omega} = r_e^2(1 - \sin^2\theta) = r_e^2\cos^2\theta \tag{4.24.11}$$

Remember that θ is measured from the z-axis. Therefore:

$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2}r_e^2(1 + \cos^2\theta) \tag{4.24.12}$$

4.24.2 Rayleigh Scattering

Rayleigh scattering is scattering of a plane wave off of a small (compared to the wavelength) dielectric or conducting object. In this scenario, the radiation of the object is primarily time harmonic electric and magnetic dipole radiation. Thus we can write:

$$\boldsymbol{E}_{rad} = (\boldsymbol{E}_{E1} + \boldsymbol{E}_{M1})|_{\hat{\boldsymbol{r}}=\hat{\boldsymbol{k}}} = \left(\frac{\mu_0}{4\pi r}\right)^2 \omega^4 \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{p}) + \hat{\boldsymbol{k}} \times \boldsymbol{m}\right] \tag{4.24.13}$$

So that, then:

$$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0 c}\right)^2 \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{k}} \times \boldsymbol{p}) + \hat{\boldsymbol{k}} \times \boldsymbol{m}\right]^2 \tag{4.24.14}$$

[4.41] Zangwill pg. 777. Zangwill uses the frequency space representation: you could also use the time basis.

For a non-magnetic ($\boldsymbol{m} = 0$) simply polarizable object where $\boldsymbol{p} = \alpha\epsilon_0 E_0 \hat{\boldsymbol{e}}_0$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2\alpha}{4\pi}\right)^2 (1 - |\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{e}}_0|^2)^2 \tag{4.24.15}$$

Even without integrating, we can see immediately that $\sigma \propto k_0^4 \propto \lambda^{-4}$. This strong dependence of the cross section on wavelength is the reason that the sky is blue during the day and red at sunset. During the day, the light we see coming from the sky is reflected sunlight, which means it is dominated by low wavelength (blue) light. At sunset, when we look straight *through* the atmosphere towards the sun, blue light is preferentially scattered away from us and we see predominantly long wavelength (red) light that is *not* scattered.

4.24.3 The Born Approximation

Calculating the polarization and currents within a polarization material in an external field is rigorously an iterative process. The incident field creates some polarization, which in turn cancels out part of the field, changing the polarization etc. ad infinitum.

In the **Born approximation** we will truncate this process after the first iteration. Subsequent steps are negligible when the object is only **weakly polarization**, so the Born Approximation is valid in this limit.

Note that this is *more* general than Rayleigh scattering, which assumed that the object was both weakly polarizable AND small compared to the wavelength. The Born approximation makes no assumption about the size of the object.

If the scattering object is simply polarizable, then $\boldsymbol{P} = \epsilon_0 \chi \boldsymbol{E}_{inc}$. Assuming the incident field is a plane wave, the polarization current is:

$$\boldsymbol{j} = -i\omega\epsilon_0\chi(\omega)\boldsymbol{E}_{inc} = -i\omega(\omega)\epsilon_0\chi\hat{\boldsymbol{e}}_0 E_0 e^{i(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega t)} \tag{4.24.16}$$

Plugging this into the formula derived earlier for the differential cross section of a time harmonic current (eq. 4.24.6)[4.42]:

$$\frac{d\sigma_{scatt}}{d\Omega} = \left(\frac{k_0^2}{4\pi}\right)^2 |\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_0|^2 \left| \int d^3r' \chi(\omega) e^{i(\boldsymbol{k}_0 - \boldsymbol{k})\cdot\boldsymbol{r}'} \right|^2 \tag{4.24.17}$$

$|\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_0|^2$ sets the same angular dependence as found in the previous sections. Often, it is easiest to evaluate this integral by introducing a new variable:

$$\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}_0 \tag{4.24.18}$$

Where $\boldsymbol{k}_0$ is the wave vector of the incident wave, and $\boldsymbol{k}$ is the wave vector of the scattered wave. You can then solve the problem in terms of q and substitute back in at the end.

That means that this cross section again represents only the cross section a polarized incident wave. In order to find the unpolarized cross section, we again carry out the averaging procedure:

[4.42] Zangwill writes $\chi(\boldsymbol{r}, \omega)$, but I'm not terrible concerned with situations where the permitivity varies within the object

$$|\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_0|^2 \to \frac{1}{2}\sum_{m=1}^{2}(1 - |\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{e}}_m|^2) \tag{4.24.19}$$

If the electric susceptibility is constant with respect to ω (the low frequency limit, as inertia of charges in the material do not come into play), this simplifies to:

$$\frac{d\sigma_{scatt}}{d\Omega} = \left(\frac{k_0^2}{4\pi}\right)^2 |\hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_0|^2 (V\chi)^2 \tag{4.24.20}$$

4.24.4 The Optical Theorem

Consider[4.43] a plane wave ψ_{inc} incident on an object. Far away, the total wave in space ψ looks like:

$$\psi(\boldsymbol{r}) \approx e^{ikr} + f(\theta)\frac{e^{ikr}}{r} \tag{4.24.21}$$

Higher order terms in r^{-1} may exist, but are negligible far away. Also far away, we can approximate r by a Taylor series:

$$r = \sqrt{x^2 + y^2 + z^2} \approx z + \frac{x^2 + y^2}{2z} \tag{4.24.22}$$

Substituting these into ψ and taking the mod square to find the intensity:

$$|\psi|^2 \approx 1 + \frac{f(\theta)}{z}e^{\frac{ik}{2z}(x^2+y^2)} + \frac{f^*(\theta)}{z}e^{\frac{-ik}{2z}(x^2+y^2)} + \frac{|f(\theta)|^2}{z^2} \tag{4.24.23}$$

In the far field approximation we are working in, the final z^{-2} term is negligible. Since $c + c^* = 2\mathrm{Re}(c)$ for any complex number c, we can then write:

$$|\psi|^2 \approx 1 + 2\mathrm{Re}\left(\frac{f(\theta)}{z}e^{\frac{ik}{2z}(x^2+y^2)}\right) \tag{4.24.24}$$

Now, we want to find the total intensity incident on some surface area A in the x-y plane: $\int |\psi|^2 dxdy$. This is easy enough if we assume that the sheet occupies most of the solid angle above the source, so that we can approximate the integrals as $\int dx \to \int_{-\infty}^{\infty} dx$. If we additionally approximate $\theta \approx 0$[4.44], the integrals become:

$$\begin{aligned}\int |\psi|^2 dxdy &\approx A + 2\mathrm{Re}\left(\frac{f(0)}{z}\int_{-\infty}^{\infty} e^{\frac{ik}{2z}x^2}dx \int_{-\infty}^{\infty} e^{\frac{ik}{2z}y^2}dy\right) \\ &= A + 2\mathrm{Re}\left(\frac{f(0)}{z}\frac{2\pi i z}{k}\right) \\ &= A - \frac{4\pi}{k}\mathrm{Im}(f(0))\end{aligned} \tag{4.24.25}$$

[4.43] Derivation due to Wikipedia: `http://en.wikipedia.org/wiki/Optical_theorem`

[4.44] These two assumptions are somewhat contradictory. They would make the most sense in a limit where the screen actually occupies only a small solid angle, but for some reason most of the wave is concentrated in this area.

If there was no object in the way, we would expect $\int |\psi|^2 dA = A$. Therefore, the total[4.45] cross section of the object must be:

$$\boxed{\sigma_{tot} = \frac{4\pi}{k}\text{Im}(f(0))} \tag{4.24.26}$$

5 Thermodynamics

5.1 The Laws of Thermodynamics

5.1.1 The Zeroth Law

The so-called zeroth law of thermodynamics states that if two bodies are in thermal equilibrium with a third body, then all three are together in equilibrium. This is so eminently reasonable that they didn't want to make it a "4th Law of Thermodynamics", so it became the zeroth.

5.1.2 The First Law

The first law of thermodynamics is a simple statement of conservation of energy. Since objects that contain heat can do work, it follows that **heat is a form of energy**. This is normally summed up with an energy conservation law:

$$\Delta U = Q - W \tag{5.1.1}$$

Where W is the work done *by* the system, and as such represents energy lost, while Q is heat added *to* the system.

5.1.3 The Second Law

The second law states that **equilibrium states correspond to states of maximum entropy.** This means that, in any naturally occurring process, entropy must be increasing. This modern statement is equivalent to two historical statements that are still sometimes useful[5.1]:

- **Lord Kelvin:**

  A transformation whose only final result is to transform work into heat extracted from a source which is at the same temperature throughout is impossible.

- **Clausius:**

  If heat flows by conduction from body A to another body B, then a transformation whose only final result is to transfer heat from B to A is impossible.

5.1.4 The Third Law

The entropy of a perfect crystal at $0K$ has zero entropy.

[4.45] Including both scattering and any absorption.
[5.1] Copied here from pg. 30-31 of Fermi's Thermodynamics.

5.2 Ideal Gases

An ideal gas is a theoretical gas that experiences no inter-particle interactions. Its behavior is characterized by the ideal gas equation:

$$PV = NkT \tag{5.2.1}$$

Where N stands for the number of molecules in the gas, and k is the Boltzmann constant ($1.38 \times 10^{-23} \frac{J}{K}$).

Chemists (obsessed as they are with large quantities of stuff) tend to rewrite Nk as nR, where n is the number of moles of gas, and R is the universal gas constant $8.314 \frac{J}{K \cdot mol}$. The two definitions are perfectly equivalent.

Another fundamental property of an ideal gas (or any simple thermodynamics system) is that it may be described completely using only three parameters (or two, once the ideal gas law has been applied). For example, the state of an ideal gas can be specified by V, P, and T. The ideal gas law then allows us to express these three in terms of only two.

For an ideal gas in particular, internal energy is only a function of T, so we can write $U(T)$.

5.2.1 Velocity of Particles in an Ideal Gas

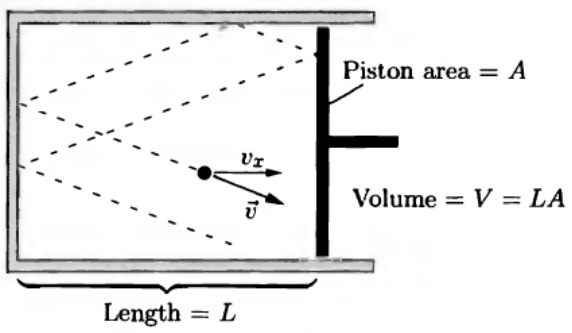


Figure 5.2.1: Source: Schroder's Thermal Physics, pg. 10

The RMS velocity of a molecule in an ideal gas can be calculated using a simple thought experiment (Fig. 5.2.1).

Imagine a molecule trapped in a box with a piston as one side. The molecule exerts an average pressure on the side of the box:

$$\bar{P} = \frac{\bar{F}_{x,onpiston}}{A} = \frac{-\bar{F}_{x,onmolecule}}{A} = \frac{-m\frac{\Delta v_x}{\Delta t}}{A} \tag{5.2.2}$$

Δt is the time between collisions, which can be reasonably written as $\Delta t = \frac{2L}{v_x}$. At the wall, $\Delta v_x = -2v_x$ (since the molecule turns completely around). Plugging in, we obtain

$$\bar{P} = \frac{-m(-2v_x)}{A(\frac{2L}{v_x})} = \frac{mv_x^2}{V} \tag{5.2.3}$$

This can be re-written as $\bar{P}V = mv_x^2$. If we now consider an ideal gas of N such particles (such

that v_x now becomes $\bar{v}_x$: the average over all the particles velocities), the ideal gas law gives us:

$$NkT = Nm\bar{v}_x^2 \tag{5.2.4}$$

or

$$kT = m\bar{v}_x^2 \tag{5.2.5}$$

Now, the same argument applies in the y and z directions, so $3kT = m\bar{v}^2$. Therefore:

$$v_{rms} = \sqrt{\bar{v^2}} = \sqrt{\frac{3kT}{m}} \tag{5.2.6}$$

This is the average speed of a molecule in an ideal gas.

5.2.2 Energy of an Ideal Gas

Since an ideal gas experiences no inter-particle interactions, its energy is simply a sum of the kinetic energies (also called "thermal energies") of each of its constituent molecules.

The thermal energy of a single molecule depends on its degrees of freedom, f. For a single degree of freedom, the equipartition theorem of statistical mechanics says that:

$$KE = \frac{1}{2}kT \tag{5.2.7}$$

Therefore, the energy for f degrees of freedom is:

$$KE = \frac{f}{2}kT \tag{5.2.8}$$

A monoatomic molecule moving back and forth in 1D has $f = 1$; the same molecule in 3D has $f = 3$. Allow a diatomic molecule to rotate adds two degrees of freedom (one for each axis around which rotation is permitted). Allowing vibration in a diatomic molecule adds another 2 degrees of freedom[5.2]. Therefore, diatomic gasses have $f = 7$.

The thermal energy of an ideal gas of N particles is then given by:

$$U_{thermal} = \frac{f}{2}NkT \tag{5.2.9}$$

5.3 The PV Plane

Work done by the expansion or contraction of a gas can be readily written as $W = PdV$. Therefore, it is often convenient to graph thermodynamic processes as functions $P(V)$ on a P vs. V plane, such that the area under a given line on the graph represents the work done by that segment of the process.

[5.2] According to Schroeder, one is for the kinetic energy of the vibration, and the other for the potential energy.

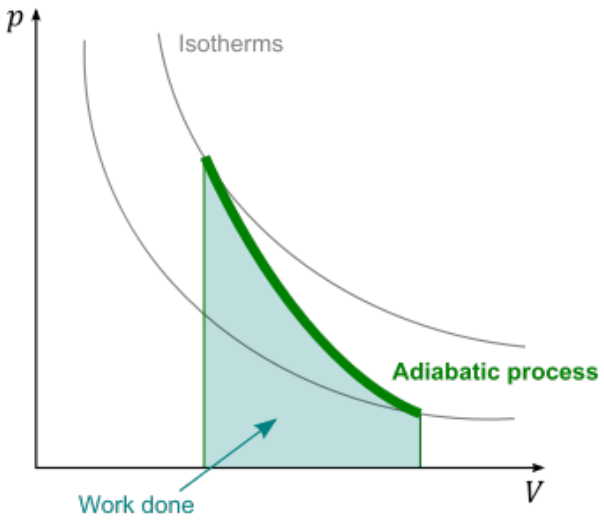


Figure 5.3.1: Two thermal process graphed on the PV plane. Source: Wikipedia.

5.4 Isotherms and Adiabats

Gases that expand or contract with certain conditions held constant follow particular curves on the PV plane and are described easily by certain mathematical expressions.

5.4.1 Isotherms

Isothermic expansion or contraction occurs when the volume of a gas (and thus its pressure) changes while the **temperature is held constant**. The ideal gas law tells us that, since for such a process NkT is constant:

$$PV = \text{constant} \tag{5.4.1}$$

Therefore, the graph of an isotherm on the PV plane goes as $P(V) \propto \frac{1}{V}$ (see Fig. 5.3.1).

The ideal gas law can also be used to calculate the work done across an isotherm:

$$W = \int_{V_a}^{V_b} PdV = \int_{V_a}^{V_b} \frac{nkT}{V} dV = nkT\text{Log}\left(\frac{V_2}{V_2}\right) \tag{5.4.2}$$

Notice how the above process depends on the fact that T is constant with respect to V.

Since the energy of an ideal gas is ONLY a function of its temperature ($U(T)$) and NOT its volume, $\Delta U = 0$ for an isotherm. This implies in turn that:

$$Q = \Delta W \tag{5.4.3}$$

5.4.2 Adiabats

Adiabatic expansion or compression is defined by the absence of heat transfer: $Q = 0$. Under this condition, we have from the 1st law that $dU + dW = dQ = 0$. Since $U(T)$ is only a function of T, this is equivalent to:

$$\frac{\partial U}{\partial T} dT + PdV = 0 \tag{5.4.4}$$

Eliminating P through the ideal gas law and rearranging,

$$\frac{\partial U}{\partial T}\left(\frac{1}{T}\right)dT + \frac{nk}{V}dV = 0 \tag{5.4.5}$$

$\frac{\partial U}{\partial T}$ happens to be the heat capacity at constant volume C_V. However, since for an ideal gas $U(T) = \frac{f}{2}nkT$:

$$\frac{\partial U}{\partial T} = \frac{f}{2}nk \tag{5.4.6}$$

So this expression can be rewritten as

$$\frac{f}{2}\left(\frac{1}{T}\right)dT + \frac{1}{V}dV = 0 \tag{5.4.7}$$

Integrating (and moving $\frac{f}{2}$ to the V term) yields:

$$\text{Log}(T) + \frac{2}{f}\text{Log}(V) = \text{constant} \tag{5.4.8}$$

Then, exponentiation finally leaves:

$$TV^{\frac{2}{f}} = \text{constant} \tag{5.4.9}$$

This is one form of the adiabat law. In order to be graphed on the PV plane, we can rewrite this expression in terms of P and V using the ideal gas law (absorbing nk into the constant side):

$$PV^{\frac{2}{f}+1} = \text{constant} \tag{5.4.10}$$

This exponent is known as the adiabatic index, γ, and is normally rewritten as:

$$\gamma = \frac{2}{f} + 1 = \frac{f+2}{f} \tag{5.4.11}$$

It also happens that:

$$\gamma = \frac{C_p}{C_V} \tag{5.4.12}$$

In terms of this exponent, then, the adiabat law can be rewritten in a number of useful ways:

$$\boxed{PV^{\gamma} = \text{constant}} \tag{5.4.13}$$

$$\boxed{TV^{\gamma-1} = \text{constant}} \tag{5.4.14}$$

$$\boxed{TP^{\frac{1-\gamma}{\gamma}} = \text{constant}} \tag{5.4.15}$$

The first of these equations tells us that $P(V) \propto \frac{1}{V^{\gamma}}$. Since in general $\gamma > 1$, **adiabats are steeper than isotherms**. This difference allows the construction of adiabatic/isothermal cyclic engines like the Carnot cycle.

5.5 Heat Engines

A heat engine is a device that takes in heat from a high-temperature reservoir, turns some into work, and dumps the rest into a low-temperature reservoir. We will also discuss refrigeration cycles, which function on a similar principle: taking outside work to move heat from a low-temperature reservoir to a high temperature reservoir.

5.5.1 Efficiency and Efficiency Limits

The concept of efficiency seeks to quantify how "good" a particular engine is. This is most naturally represented by the ratio between the *benefit* of a cycle and its *cost*:

$$\eta_E = \frac{W}{Q_h} \tag{5.5.1}$$

Where Q_h is the heat transferred from the hot reservoir (and the subscript E distinguishes this from the efficiency for a refrigeration cycle, which is discussed next). Since $W = Q_h - Q_c$ by conservation of energy, this can be readily written as:

$$\boxed{\eta_E = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}} \tag{5.5.2}$$

For a refrigeration cycle, we have a different balance of priorities:

$$\eta_R = \frac{Q_c}{W} \tag{5.5.3}$$

Again using $W = Q_h - Q_c$, we can rewrite this as:

$$\boxed{\eta_R = \frac{1}{\frac{Q_h}{Q_c} - 1}} \tag{5.5.4}$$

Now, the second law states that the net entropy of a closed system can only increase or stay constant (never decrease). Therefore, in an engine, *the entropy discharged into the cold reservoir must be greater or equal to the entropy taken from the hot reservoir*:

$$S_c \geq S_h \tag{5.5.5}$$

Since $S = \frac{Q}{T}$, this means that

$$\frac{Q_c}{T_c} \geq \frac{Q_h}{T_h} \tag{5.5.6}$$

or, rearranged,

$$\frac{Q_c}{Q_h} \geq \frac{T_c}{T_h} \tag{5.5.7}$$

The greatest possible engine efficiency occurs at the smallest possible $\frac{Q_c}{Q_h}$, so the *highest efficiency possible* for an engine is:

$$\boxed{\eta_C = 1 - \frac{T_c}{T_h}} \tag{5.5.8}$$

C here stands for Carnot, since this efficiency is theoretically achieved by the Carnot Cycle.

A similar entropy argument can be made for refrigeration cycles. In this case, the condition is reversed (since the flow of heat is reversed):

$$S_c \leq S_h \tag{5.5.9}$$

Which, by the same steps, leads to the maximum possible refrigeration efficiency:

$$\boxed{\eta_{R,max} = \frac{1}{\frac{T_h}{T_c} - 1}} \tag{5.5.10}$$

5.5.2 The Carnot Cycle

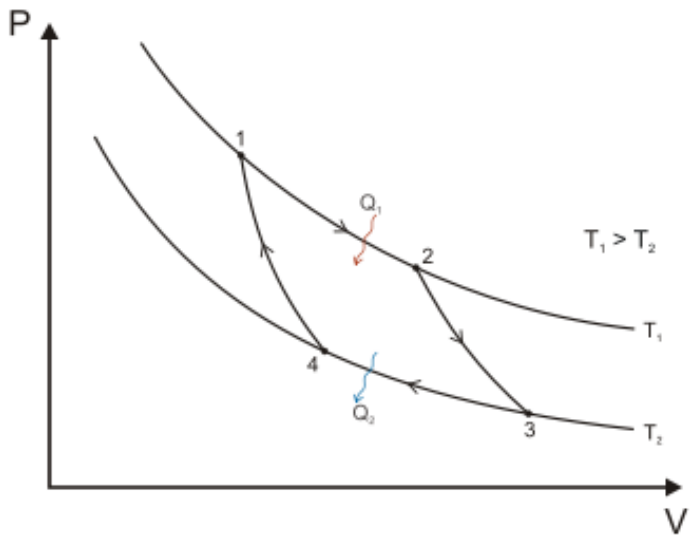


Figure 5.5.1: The Carnot Cycle. Source: Wikipedia

The Carnot Cycle[5.3] is probably the simplest heat engine cycle. It consists of two isotherms, connected by adiabats. The four legs of the cycle are as follows:

1. **1 to 2 (Isotherm):** The system absorbs heat from the hot reservoir, and the working gas expands. This is the first half of the out-stroke of the engine.

[5.3] Here's a great article about the Carnot cycle from Oberlin College: `http://www.oberlin.edu/physics/dstyer/P111/Carnot.pdf`

2. **2 to 3 (Adiabat):** The system is removed from the hot reservoir and allowed to cool. The working gas continues to expand, now adiabatically.

3. **3 to 4 (Isotherm):** The system is placed in contact with the cold reservoir, causing the working gas to contract. This does negative work, but is necessary to return the system to the beginning of the cycle.

4. **4 to 1 (Adiabat):** The system is removed from the cold reservoir and warms up slightly as it decreases even more in volume until uniting again with the top isotherm.

During each isotherm, $dU = 0$ (since $U(T)$), and therefore $Q = W$. Work is also done on the adiabatic stretches of the cycle, however it turns out that $W_{2\to3} = W_{4\to1}$, so the net work done by the cycle is just:

$$\Delta W = W_{1\to2} + W_{3\to4} \tag{5.5.11}$$

Where $W_{3\to4}$ is negative.

5.6 Intensive vs. Extensive Quantities

An **intensive** quantity is one that does not change if you change the size of the system. Examples are temperature, density, etc.

An **extensive** quantity DOES change if you change the size of the system. Examples are mass, and volume.

An intensive quantity can be constructed by taking the ratio of two extensive quantities, the result of which is scale-invariant.

Thermodynamic systems usually (always?) are described by two intensive and two extensive quantities.

5.7 The Chemical Potential

The chemical potential is the quantity that is equal when two systems (that are allowed to exchange particles) are in equilibrium. In other words, in a state of equilibrium:

$$\mu_A = \mu_B \tag{5.7.1}$$

The chemical potential can be defined with respect to any of the thermodynamic potential energies:

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = \left.\frac{\partial G}{\partial N}\right|_{T,P} = \left.\frac{\partial H}{\partial N}\right|_{S,P} = \left.\frac{\partial U}{\partial N}\right|_{S,V} \tag{5.7.2}$$

Out of these, perhaps the most often used is:

$$\boxed{\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V}} \tag{5.7.3}$$

Roughly speaking, the chemical potential represents the amount of energy required to introduce a new particle into the system.

- If $\mu > 0$, it takes energy to add particles to the system, and new particles will have to be 'forced' in.
- If $\mu < 0$, the system is 'accepting' of new particles, and they will join spontaneously.
- If $\mu = 0$, it costs no energy to either leave or enter the system. A practical example is photons in a black body, which can freely enter or exit the gas.

If we imagine that the bath of particles 'outside' the system is at $\mu = 0$, then this leads to the sensible conclusion that particles move from regions of higher chemical potential to regions of lower chemical potential.

5.8 Thermodynamic Potentials

A thermodynamic potential is a scalar function that completely describes a system. Just like potential energies, only *differences* in thermodynamic potentials are physically meaningful.

Equilibrium states are often found by minimizing the internal energy, U. However, under different conditions, it can be more convenient to instead minimize one of the other thermodynamic potentials (which will also yield an equilibrium state).

Most generally, each of these potentials includes a dN term to account for systems where the particle number is not fixed. If the number *is* fixed, this term is of course zero, which is often how the relations are expressed in textbooks.

5.8.1 The Internal Energy (or Fundamental Thermodynamic Relation

$$dU = TdS - pdV + \sum_i \mu_i dN_i \tag{5.8.1}$$

The internal energy is the energy it would take to create a system out of nothing. The $-pdV$ term represents work that would have to be done against the external atmosphere to create room for the system.

The primary importance of the thermodynamic potentials is that they easily generate many useful partial derivative expressions. For example, from the fundamental thermodynamic relation:

- Constant volume $\rightarrow T = \frac{\partial U}{\partial S}|_{V,N}$

5.8.2 The Enthalpy

$$H = U + pV \tag{5.8.2}$$

Applying the fundamental thermodynamic relation for dU:

$$dH = TdS + VdP + \sum_i \mu_i dN_i \tag{5.8.3}$$

The enthalpy represents the energy that goes into creating a system *excluding work done against the external constant pressure on the system.* The first law of thermodynamics states that:

$$U = Q - P\Delta V + W_{other} \tag{5.8.4}$$

Where the PV term is work done against the atmosphere (pushing a piston, etc.), and W_{other} is other work. For the enthalpy, then:

$$H = Q + W_{other} \tag{5.8.5}$$

Therefore the enthalpy only changes due to added heat or other forms of work, excluding the expansion of the system. This makes enthalpy a natural quantity to use for recording heat capacities etc. that are independent of the value of the external pressure.

For example, cooking directions often implicitly depend on the amount of energy required to boil water. This energy includes both the energy to boil the water, and some extra work to account for the increased volume occupied by the water vapor. This means that the amount of energy required varies based on atmospheric pressure: hence why cooking times must be adjusted at higher altitudes.

If instead cooking times could somehow be expressed in enthalpy, the same directions would work at any altitude (pressure).

5.8.3 The Helmholtz Potential (or Helmholtz Free Energy)

$$F = U - TS \tag{5.8.6}$$

Applying the fundamental thermodynamic relation for dU:

$$dF = -SdT - pdV + \sum_i \mu_i dN_i \tag{5.8.7}$$

The Helmholtz free energy represents the amount of energy necessary to create a system from nothing *excluding energy that can be leeched for free from the environment.* For example, somehow conjuring an object at some non-zero temperature would require some energy. However, if the environment is also at temperature T, this energy can be had “for free" by conduction.

5.8.4 The Gibbs Free Energy

$$G = U - TS + PV \tag{5.8.8}$$

Applying the fundamental thermodynamic relation for dU:

$$dG = -SdT + VdP + \sum_i \mu_i dN_i \tag{5.8.9}$$

The Gibbs free energy is truly the energy required to create a system out of nothing, since it includes BOTH the energy necessary to push the atmosphere out of the way and the heat that can be leeched from the environment by conduction.

5.9 Maxwell Relations

Maxwell relations are useful thermodynamic identities produced by twice differentiating a thermodynamic potential. In general, for a potential $\Phi(x_1, x_2, ...)$, a Maxwell relation is:

$$\frac{\partial}{\partial x_j}\frac{\partial \Phi}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial \Phi}{\partial x_j} \tag{5.9.1}$$

For example, the fundamental thermodynamic relation tells us that

$$T = \left.\frac{\partial U}{\partial S}\right|_V, \qquad P = -\left.\frac{\partial U}{\partial V}\right|_S \tag{5.9.2}$$

We will now take the V partial of left relation and the S partial of the right relation. Since it is true that:

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x} \tag{5.9.3}$$

We can then write:

$$\boxed{\left.\frac{\partial T}{\partial V}\right|_S = -\left.\frac{\partial P}{\partial S}\right|_V} \tag{5.9.4}$$

Applying the same process to each of the other thermodynamic potentials produces the following other potentials.

$$\boxed{\left.\frac{\partial T}{\partial P}\right|_S = \left.\frac{\partial V}{\partial S}\right|_P} \tag{5.9.5}$$

$$\boxed{\left.\frac{\partial S}{\partial V}\right|_T = \left.\frac{\partial P}{\partial T}\right|_V} \tag{5.9.6}$$

$$\boxed{-\left.\frac{\partial S}{\partial P}\right|_T = \left.\frac{\partial V}{\partial T}\right|_P} \tag{5.9.7}$$

5.10 The Thermodynamic Square

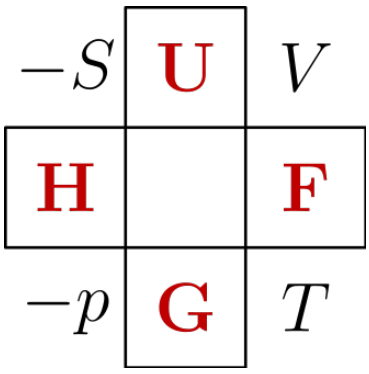


Figure 5.10.1: The thermodynamic square. Source: Wikipedia.

The thermodynamic square is a visual mnemonic for remembering both the thermodynamic

potentials and the Maxwell relations. The order of the letters on the square can be easily remembered as: "**G**ood **P**hysicists **H**ave **S**udied **U**nder **V**ery **F**ine **T**eachers "

To construct each of the potentials, follow the following process:

1. Start on the letter of the potential you wish to write the differential identity for. For example, to find dU, we start on U.
2. The two variables on the near side of the square will be differentials, while the ones on the far side will not. Variables on the opposite diagonals go with one another. If one of the NON-differential variables has a minus sign, the term picks up a minus sign (ignore minus signs for differential elements). For example, for dU, the terms will be $(-p)(dV)$ and $(T)(dS)$.
3. The final function is the sum of these terms: $dU = TdS - pdV$.

The square can also be used to generate the four Maxwell relations:

1. We will form a "U" shape, where the ends of the U are the numerators of the differentials and the bends are the denominators. For example,

$$\frac{\partial S}{\partial p} = \frac{\partial V}{\partial T} \tag{5.10.1}$$

2. If BOTH sides have a minus variable, that side gets a minus sign. If EACH side has one, we ignore the minus sign. In this case:

$$-\frac{\partial S}{\partial p} = \frac{\partial V}{\partial T} \tag{5.10.2}$$

3. Finally, each derivative is held constant with respect to the denominator of the OTHER derivative:

$$-\frac{\partial S}{\partial p}\bigg|_T = \frac{\partial V}{\partial T}\bigg|_p \tag{5.10.3}$$

4. The other three Maxwell relations can be generated by rotating the U by 90, 180, and 270 degrees.

There is a second thermodynamic square used for remembering the non-differential formulas of the potentials:

In this case the interpretation is very simple: $H = U + PV$, $G = U + PV - TS$, etc.

5.11 Heat Capacity

The heat capacity of an object is the amount of heat required to raise its temperature by some amount:

$$C = \frac{Q}{\Delta T} \tag{5.11.1}$$

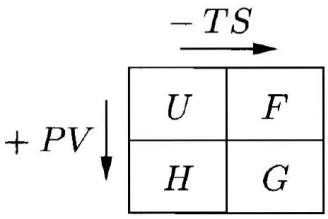


Figure 5.10.2: The second thermodynamic square. Source: Schroeder pg. 151.

We will derive two expressions for the heat capacity: one for heating at constant volume and the other for heating at constant pressure.

5.11.1 Specific Heat

The specific heat of an object is a measure of its heat capacity per unit mass: c has units of $\frac{J}{K \cdot kg}$. Just like the heat capacity, there are separately capacities for constant volume or pressure: c_V and c_P respectively.

The definition of heat capacity is often rearranged and written in the following way with the specific heat:

$$\Delta Q = mc\Delta T \tag{5.11.2}$$

A potentially useful number to know is the specific heat of water: $4.186 \frac{\text{J}}{\text{C} \cdot \text{g}}$, or $1 \frac{\text{cal}}{\text{C} \cdot \text{g}}$.

5.11.2 Constant Volume

The first law of thermodynamics states that $U = Q - W$ or, taking $dW = pdV$ for work done against the environmental pressure as the system expands: $dQ = dU + dW$.

Now, assume that $U(T, V)$ (remember, we are free to do this since the state of the system is entirely determined by the two intensive parameters T and P, which is related to V). Then:

$$dU = \left.\frac{\partial U}{\partial T}\right|_V dT + \left.\frac{\partial U}{\partial V}\right|_T dV \tag{5.11.3}$$

Substituting this into the previous expression for dQ:

$$dQ = \left.\frac{\partial U}{\partial T}\right|_V dT + \left(p + \left.\frac{\partial U}{\partial V}\right|_T\right) dV \tag{5.11.4}$$

Or, **at constant volume**:

$$\boxed{C_V = \frac{dQ}{dT} = \left.\frac{\partial U}{\partial T}\right|_V} \tag{5.11.5}$$

5.11.3 Constant Pressure

Similarly to the derivation above, if we take $U(T, P)$, then:

$$dU = \left.\frac{\partial U}{\partial T}\right|_P dT + \left.\frac{\partial U}{\partial P}\right|_T dP \tag{5.11.6}$$

Likewise, if $V(T, P)$ (as is clearly the case with the ideal gas, and is true in general for simple systems):

$$dV = \left.\frac{\partial V}{\partial T}\right|_P dT + \left.\frac{\partial V}{\partial P}\right|_T dP \tag{5.11.7}$$

Combining these equations with the expression $dQ = dU + pdV$ yields:

$$dQ = \left(\left.\frac{\partial U}{\partial P}\right|_T + P\left.\frac{\partial V}{\partial P}\right|_T\right)dP + \left(\left.\frac{\partial U}{\partial T}\right|_P + P\left.\frac{\partial V}{\partial T}\right|_P\right)dT \tag{5.11.8}$$

Or, at constant pressure:

$$\boxed{C_P = \frac{dQ}{dT} = \left(\left.\frac{\partial U}{\partial T}\right|_P + P\left.\frac{\partial V}{\partial T}\right|_P\right)} \tag{5.11.9}$$

The second term here represents extra heat that is necessary to do work against the constant pressure environment as the system expands.

5.11.4 Relationship between Heat Capacities

There is an important relationship between the heat capacities at constant pressure and volume[5.4]:

$$C_P - C_V = T\left(\left.\frac{\partial P}{\partial T}\right|_{V,n}\right)\left(\left.\frac{\partial V}{\partial T}\right|_{P,n}\right) \tag{5.11.10}$$

In particular, for an ideal gas,

$$C_P - C_V = T\left(\frac{Nk}{V}\right)\left(\frac{Nk}{P}\right) = Nk \tag{5.11.11}$$

6 Statistical Mechanics

6.1 Multiplicity, Temperature, and Entropy

The **multiplicity** of a state is defined to be the number of distinguishable states that share the same energy, and is symbolized $\Omega(E)$.

The **entropy** is a measure of the amount of information "stored" in a state, or equivalently the amount of "disorder". The entropy increases as the multiplicity increases because a greater number of accessible states affords more combinations (disorder), and therefore requires more information to fully describe the state of the system. Some (more mathematical) books define a

[5.4] The derivation seems to be too long and annoying to include here.

dimensionless entropy:

$$\sigma = \log \Omega \tag{6.1.1}$$

In general, though, the entropy is defined in terms of the Boltzmann constant k:

$$S = k \log \Omega \tag{6.1.2}$$

Although a hand-waving link is made between this thermodynamic entropy and the Shannon entropy of information theory, the mathematical link between these two concepts is more tenuous[6.1].

The *temperature* of a system is defined by the derivative of the entropy to ensures that the thermal equilibrium of two systems occurs when their temperatures are equal *and* the combined entropy of the systems is maximized:

$$\frac{1}{T} = \frac{\partial S}{\partial E} \tag{6.1.3}$$

To save notation, we also introduce a convenient expression:

$$\beta = \frac{\partial \log \Omega}{\partial E} = \frac{1}{kT} \tag{6.1.4}$$

6.2 Density of States

We will almost always discuss systems in terms of their energy states. However, many systems have multiple states with the same energy (degeneracy). This fact becomes important when applying statistics that rely on the total number of states. We therefore define the **density of states** to be the number of degenerate energy states of an energy E. In terms of the number of states, then:

$$g(E) = \frac{d\Omega}{dE} \tag{6.2.1}$$

6.2.1 Density of Quantum States in Phase Space

In classical mechanics, a particle in a volume with position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ can occupy an infinite number of degenerate states. However, quantum mechanics insists that only states of quantized momentum and position can actually be occupied. The uncertainty principle[6.2] (approximately[6.3]) states that:

$$dpdx \approx h \tag{6.2.2}$$

[6.1] For an interesting book-length analysis of this topic, see "A Farewell to Entropy: Statistical Thermodynamics Based on Information" by Arieh Ben-Naim.

[6.2] I've heard that this isn't the most rigorous derivation of the density of states, but I'm not sure why.

[6.3] I'm not sure why a factor of π is traditionally neglected here, but it seems to always be left off.

So therefore the smallest possible volume of a cell in six-dimensional[6.4] x-p phase space is:

$$dp^3dx^3 \approx h^3 \tag{6.2.3}$$

The density of states can then be written:

$$\boxed{g(E)dE = \frac{dp^3dx^3}{h^3}} \tag{6.2.4}$$

For systems with spin-based degeneracy (such as fermions in a Fermi gas, like electrons in a metal), an additional factor of 2 should also be included.

It might be easy to confuse $g(E)$ and n_s (the occupancy of a state s). n_s tells you how many particles there are IN a state s (with energy E_s), while $g(E)$ tells you how many states there ARE with energy less than E.

6.2.2 Example: Density of States for a 3D Square Well

This example illustrates a different way of calculating the density of states.

The density of states per unit volume can be expressed:

$$g(E) = \frac{dN}{dk} = \frac{dN}{dk}\frac{dk}{dE} \tag{6.2.5}$$

$N(k)$ can be found by imagining a 3D k phase space. For a square well, the volume taken up by each cell in such a phase space is $\left(\frac{\pi}{L}\right)^3$. The number of possible states for a given $|\underline{k}|$ is then:

$$N(k) = 2 \times \left(\frac{L}{\pi}\right)^3 \times \frac{1}{8} \times \frac{4}{3}\pi k^3 \tag{6.2.6}$$

Where the factor of 2 accounts for the spin of the states, and $\frac{1}{8} \times \frac{4}{3}\pi k^3$ is the volume of one quadrant of the sphere. Thus:

$$\frac{dN}{dk} = \left(\frac{L}{\pi}\right)^3 \pi k^2 \tag{6.2.7}$$

Now, in order to find a relationship between E and k, we can use the fact that $E = \frac{\hbar^2 k^2}{2m}$:

$$\frac{dk}{dE} = \frac{m}{\hbar^2 k} \tag{6.2.8}$$

Finally, since the above relationship also gives us that $k = \frac{\sqrt{2mE}}{\hbar}$, we can substitute all of these together to get:

$$g(E) = \frac{\sqrt{2}8\pi}{h^3} m^{\frac{3}{2}} \sqrt{E} \tag{6.2.9}$$

Notice that the only point in this derivation where the square well came into play was the size

[6.4] This is obviously for three dimensional space, but the same process is easy to follow in any dimensionality.

of the phase space cell, so more generally we could write:

$$g(E) = \frac{1}{\delta V}\frac{\sqrt{2}m^{\frac{3}{2}}\sqrt{E}}{\hbar^3} \tag{6.2.10}$$

Where δV is the volume of a phase space cell for the system in question.

Good source: `http://ecee.colorado.edu/~bart/book/book/chapter2/ch2_4.htm`

6.3 Energy Distribution at Equilibrium

Consider two macroscopic systems A and A' (all primed functions will refer to this second system) where $A' >> A$. Suppose that the *total* energy of the system is fixed ($E + E' = E_{tot}$), and that the two systems only weakly interact (so that they may exchange energy, but have negligible interaction energy).

If we additionally assume that the two states are in equilibrium, then system A is equally likely to be in any state that is accessible to it, given its energy. The probability of A having an energy E is therefore:

$$P(E) = C\Omega(E) = \frac{\Omega(E)}{\sum_E \Omega(E)} \tag{6.3.1}$$

Likewise, the probability of A and A' having energies E and E' respectively is:

$$P(E, E') = C\Omega(E)\Omega'(E') \tag{6.3.2}$$

However, since the total energy of the system is fixed, this can be expressed in terms of the (constant) total energy E_{tot}:

$$P(E) = C\Omega(E)\Omega'(E_{tot} - E) \tag{6.3.3}$$

Notice that, as E increases, $\Omega(E)$ rapidly increases, while $\Omega'(E_{tot} - E)$ just as rapidly decreases (since the number of states per energy level usually increases exponentially or even as a factorial). It therefore follows that the product, $P(E)$ is very sharply peaked[6.5]. The implication of this is that, for a given E, the system is practically certain to reside in a well defined **equilibrium state**.

We now want to *find* the energy value E that corresponds to this equilibrium point. Of course, this can be found by considering $\frac{\partial P}{\partial E} = 0$. However, for mathematical convenience, we will instead (equivalently) maximize the logarithm of $P(E)$:

$$\frac{\partial \log P}{\partial E} = \frac{1}{P}\frac{\partial P}{\partial E} = 0 \tag{6.3.4}$$

Since $\log P(E) = \log C + \log \Omega(E) + \log \Omega'(E_{tot} - E)$, this becomes

$$\frac{\partial \log \Omega(E)}{\partial E} + (-1)\frac{\partial \log \Omega'(E_{tot} - E)}{\partial E} = 0 \tag{6.3.5}$$

[6.5] See proof in Kittel and Kroemer, pg. 18-19.

Then in terms of β **at equilibrium**:

$$\beta(E) = \beta(E') \tag{6.3.6}$$

Since $\beta = \frac{\partial S}{\partial E}$, this implies that, at equilibrium, $S + S'$ is maximized. Equivalently, then, when two systems are at equilibrium,

$$\frac{\partial}{\partial E}(S + S') = 0 \tag{6.3.7}$$

And therefore $T = T'$.

6.4 The Microcanonical Ensemble

The microcannonical ensemble describes a system with a fixed total energy, number of particles, and volume. It is assumed that **all of the states accessible to the system at this fixed energy are equally probable**. Therefore, if there are N such accessible states,

$$P = \frac{1}{N} \tag{6.4.1}$$

6.5 The Canonical Ensemble

The Canonical Ensemble represents a system in thermal equilibrium with a much larger system that acts as a heat reservoir. The system can exchange energy, but *not* particles with the heat reservoir.

6.5.1 Probability of States

Suppose that the small system is in a definite state, E_s. We want to find the probability of this occurring, which is given (eq. 6.3.3) as:

$$P(E_s) = C\Omega(E_s)\Omega'(E_{tot} - E_s) = C\Omega'(E_{tot} - E_s) \tag{6.5.1}$$

$\Omega(E_s) = 1$ since the state of the small system is defined. In the limit where $E_s << E_{tot}$, which is satisfied in the canonical ensemble;

$$\log \Omega'(E_{tot} - E_s) \approx \log \Omega'(E_{tot}) - \left(\frac{\partial \log \Omega'}{\partial E'}\right)\Big|_{E_s=0} E_s \tag{6.5.2}$$

Since $\left(\frac{\partial \log \Omega'}{\partial E'}\right)\Big|_{E_s=0} = \beta'|_{E_s=0} = \beta'$, and at equilibrium $\beta' = \beta$ since $T' = T$:

$$\log \Omega'(E_{tot} - E_s) \approx \log \Omega'(E_{tot}) - \beta E_s \tag{6.5.3}$$

So therefore, by taking the exponential:

$$\Omega'(E_{tot} - E_s) = \Omega'(E_{tot})e^{-\beta E_s} \tag{6.5.4}$$

Since $\Omega'(E_{tot})$ is constant, this allows us to write the probability $P(E_s)$ as:

$$P(E_s) = Ce^{-\beta E_s} \tag{6.5.5}$$

Where C is an constant determined by the fact that $\int C ds = 1$.

This probability distribution is the origin of the **Boltzmann Factor**, which is:

$$e^{-\beta E_s} \tag{6.5.6}$$

6.6 The Grand Canonical Ensemble

The Grand Canonical Ensemble[6.6] represents a system that is able to exchange *both* heat and particles with a heat and particle reservoir.

If A is a small system, and A' a heat reservoir:

$$\begin{aligned} E + E' &= E_{tot} \\ N + N' &= N_{tot} \end{aligned} \tag{6.6.1}$$

By analogous arguments to those for the canonical ensemble, the probability of observing A in a particular state characterized by E_r and N_r is:

$$P(E_r, N_r) = C\Omega'(E_{tot} - E_r, N_{tot} - N_r) \tag{6.6.2}$$

Which leads, under the approximation that $E' \approx E_{tot}$ and $N' \approx N_{tot}$, to:

$$\log \Omega'(E_{tot} - E_r, N_{tot} - N_r) = \log \Omega'(E_{tot}, N_{tot}) - \left.\frac{\partial \log \Omega}{\partial E'}\right|_{E_r=0} - \left.\frac{\partial \log \Omega}{\partial N'}\right|_{N_r=0} \tag{6.6.3}$$

As before, we have

$$\beta = \left.\frac{\partial \log \Omega}{\partial E'}\right|_{E_r=0} \tag{6.6.4}$$

But now, analogously:

$$\alpha = \left.\frac{\partial \log \Omega}{\partial N'}\right|_{N_r=0} \tag{6.6.5}$$

We will then define the **chemical potential** to be

$$\mu = -kT\alpha = -\frac{\alpha}{\beta} \tag{6.6.6}$$

So

$$\log \Omega'(E_{tot} - E_r, N_{tot} - N_r) = \log \Omega'(E_{tot}, N_{tot}) e^{-\beta(E_r - \mu N_r)} \tag{6.6.7}$$

[6.6] This is possibly the most pretentious name for anything in all of physics. The only difference between this and the Canonical Ensemble is that it accounts for changes in particle number. Does that really deserve the title "grand"?

Which, by taking the exponential, allows us to write the probability neatly as:

$$\boxed{P(E_r, N_r) = Ce^{-\beta(E_r - \mu N_r)}} \tag{6.6.8}$$

Where, as usual, the constant C is set by $C^{-1} = \int P(E_r, N_r) dE_r dN_r = 1$.

6.7 The Boltzmann Factor and Boltzmann Equation

The Boltzmann factor is the name given to the exponential part of the probability distribution in the canonical ensemble:

$$e^{-\beta E_s} \tag{6.7.1}$$

This factor is only meaningful as a ratio: it gives information about the relative but not absolute probabilities of certain states[6.7]. However, this means the factor by itself can be used to find the relative probabilities of two states in thermodynamic equilibrium.

Suppose a system of many particles in which two possible energies for an individual particle are E_A and E_B. If the density of states for a particle is $D(E)$, then the ratio of Boltzmann factors tells us that the ratio between the number of particles in each state ($N_i = N(E_i)$) is:

$$\frac{N_A}{N_B} = \frac{D(E_A)}{D(E_B)} e^{-\beta(E_A - E_B)} \tag{6.7.2}$$

This result is known as the **Boltzmann Equation**.

6.8 The Partition Function

The partition function describes the probability distribution of states of different energies. The partition function takes different forms in different ensembles.

6.8.1 The Canonical Partition Function

The canonical partition function applies when describing a system within the canonical ensemble. For classical discrete systems, we define:

$$Z = \sum_s e^{-\beta E_s} \tag{6.8.1}$$

Where $\beta = \frac{1}{kT}$ and E_s is the energy of the sth state. If the spectrum of states is continuous, the sum approaches an integral as long as we include the density of states to weight it:

$$\boxed{Z = \int g(E) dE e^{-\beta E} = \frac{1}{h^3} \int \int e^{-\beta H(p,q)} d^3q d^3p} \tag{6.8.2}$$

Where in the last step we have substituted in an expression for the density of states. The factors of h^3 depends on the dimensionality (h^3 in 3D).

[6.7] This is why the absolute magnitude of the partition function is meaningless: physical quantities expressed in terms of Z must be somehow normalized, usually by a factor of $\frac{1}{Z}$.

For the quantum mechanical discrete case, the energy is replaced by the Hamiltonian:

$$Z = \mathrm{Tr}(e^{-\beta \hat{H}}) \tag{6.8.3}$$

For continuous states (again substituting in the density of states explicitly)

$$Z = \frac{1}{h^3} \int \langle q, p | e^{-\beta \hat{H}} | q, p \rangle d^3q d^3p \tag{6.8.4}$$

6.8.2 The Grand Canonical Partition Function

When working in the grand canonical ensemble (allowing for particle exchange), the discrete classical partition function is:

$$\mathcal{Z} = \sum_s e^{-\beta(E_s - \mu N_s)} \tag{6.8.5}$$

6.8.3 Combining Partition Functions

Suppose two systems 1 and 2 are weakly interacting *and distinguishable*. Then the total system energy is $E_{tot} = E_1 + E_2$. Exploiting the properties of exponentials, this means we can rewrite the partition function of the system as follows:

$$\sum_{r,s} e^{-\beta(E_r + E_s)} = \sum_{r,s} e^{-\beta E_r} \sum_{r,s} e^{-\beta E_s} \tag{6.8.6}$$

Or, therefore

$$Z_{tot} = Z_1 Z_2 \tag{6.8.7}$$

Equivalently:

$$\log Z_{tot} = \log Z_1 + \log Z_2 \tag{6.8.8}$$

If N systems are identical but distinguishable, and Z_1 is the partition function for one of the systems, then:

$$Z_{tot} = Z_1^N \tag{6.8.9}$$

If the N identical systems are *indistinguishable*, we need to account for double counting, which leaves:

$$Z_{tot} = \frac{1}{N!} Z_1^N \tag{6.8.10}$$

Or, utilizing the Sterling approximation:

$$\log Z_{tot} = N \log Z_1 - N \log N \tag{6.8.11}$$

6.8.4 Ideal Gas Partition Function

The partition function for an ideal gas in three dimensions in the canonical ensemble can be found by first writing the partition function for a single particle:

$$Z_1 = \frac{1}{h^3} \int e^{-\frac{\beta p^2}{2m}} d^3p d^3x = \frac{V}{h^3} \int 4\pi p^2 e^{-\frac{\beta p^2}{2m}} dp = \frac{V}{h^3} (2m\pi kT)^{\frac{3}{2}} \tag{6.8.12}$$

Since an ideal gas is made up of identical, indistinguishable particles, the total partition function is then:

$$\log Z = N \log Z_1 - N \log N \tag{6.8.13}$$

6.8.5 The Partition Function and Thermodynamics

The partition function contains all relevant information about a system, and therefore can be used to express the variables of thermodynamics. Suppose that a canonical ensemble is characterized by a temperature (β) and a single external parameter x, such that $Z(\beta, x)$. Then[6.8]

$$d \log Z = \frac{\partial \log Z}{\partial x} dx + \frac{\partial \log Z}{\partial \beta} d\beta \tag{6.8.14}$$

As previously derived, $\frac{\partial \log Z}{\partial \beta} = -\overline{E}$. Now, all we need to do is rewrite the first term.
The macroscopic work done on the system in a state r can be expressed as:

$$\Delta_x E_r = \frac{\partial E_r}{\partial x} dx \tag{6.8.15}$$

The average work done on a system of n external parameters can be written $dW = \sum_{i=1}^{n} -\overline{\frac{\partial E_r}{\partial x_i}} dx_i$, so if we set n=1 and explicitly evaluate the time average

$$dW = -\overline{\frac{\partial E_r}{\partial x}} = \frac{\sum_r e^{-\beta E_r} \left(- \frac{\partial E_r}{\partial x} dx \right)}{\sum_r e^{-\beta E_r}} \tag{6.8.16}$$

Since $\sum_r e^{-\beta E_r} \left(- \frac{\partial E_r}{\partial x} dx \right) = -\frac{1}{\beta} \frac{\partial Z}{\partial x}$, we can pull the Boltzmann factor and sum past the derivative to obtain $\overline{E}$ allows us to use the above relation between $\overline{E}$ and Z. The denominator is just Z, so we can write:

$$\beta dW = \frac{\partial \log Z}{\partial x} dx \tag{6.8.17}$$

This is the expression for the first term in the *original* equation that we sought. Plugging it in:

$$d \log Z = \beta dW - \overline{E} d\beta \tag{6.8.18}$$

This can equally well be rewritten as:

$$d \log Z = \beta dW - d(\overline{E}\beta) + \beta d\overline{E} \tag{6.8.19}$$

Collecting derivatives and using the first law, $dQ = dW + d\overline{E}$

$$d(\log Z + \beta \overline{E}) = \beta (dW + d\overline{E}) = \beta dQ \tag{6.8.20}$$

Since $dQ = TdS$, we can then integrate both sides to produce

$$\boxed{S = kT \log Z + \beta \overline{E} = k(\log Z + \beta \overline{E})} \tag{6.8.21}$$

[6.8] The following derivation is from Reif pg.215.

So we can also rewrite F:

$$\boxed{F = \overline{E} - TS = -kT \log Z} \tag{6.8.22}$$

6.9 Using the Partition Function

6.9.1 Average Energy and Fluctuations in Energy

The partition function has the form:

$$Z = \sum e^{-\beta\epsilon} \tag{6.9.1}$$

Which differs only slightly from the equation for the average energy:

$$\langle E \rangle = \frac{\sum \epsilon e^{-\beta\epsilon}}{\sum e^{-\beta\epsilon}} \tag{6.9.2}$$

We can therefore cleverly write:

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \tag{6.9.3}$$

Or, even more cleverly:

$$\boxed{\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z} \tag{6.9.4}$$

Taking a second derivative yields:

$$\langle E^2 \rangle = \frac{\partial^2}{\partial \beta^2} \log Z \tag{6.9.5}$$

This is useful, because the 'fluctuations' in E (i.e. the standard deviation of the E distribution) are given by:

$$\langle (\Delta E)^2 \rangle = \langle E^2 - 2\langle E \rangle E + \langle E \rangle^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 \tag{6.9.6}$$

However, we can rearrange the equation for $\langle E^2 \rangle$:

$$\langle E^2 \rangle = \frac{\partial^2}{\partial \beta^2} \log Z = \frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right) + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 = -\frac{\partial \langle E \rangle}{\partial \beta} + \langle E \rangle^2 \tag{6.9.7}$$

Which means that:

$$\boxed{\langle (\Delta E)^2 \rangle = \frac{\partial^2}{\partial \beta^2} \log Z} \tag{6.9.8}$$

Or (if we already have an expression for $\langle E \rangle$):

$$\boxed{\langle (\Delta E)^2 \rangle = -\frac{\partial}{\partial \beta} \langle E \rangle} \tag{6.9.9}$$

(See Reif pg. 213 for more details)

6.9.2 Average State Occupancy and Fluctuations

We can play a similar trick to that of the previous section in order to determine the number of particles in a given energy state. Since the partition function contains a term of $e^{-\beta\epsilon_s}$ for each state with energy ϵ_s, and since:

$$\frac{\partial}{\partial\epsilon_s}e^{-\beta\epsilon_i} = \begin{cases} -\beta e^{-\beta\epsilon_i} & i = s \\ 0 & i \neq s \end{cases} \tag{6.9.10}$$

we see that we can cleverly count the number of such states:

$$\boxed{\langle n_s\rangle = \frac{-1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial\epsilon_s} = \frac{-1}{\beta}\frac{\partial}{\partial\epsilon_s}\log Z} \tag{6.9.11}$$

It is important to note that $\langle n_s\rangle = n_s$. We will often just drop the average brackets, since obviously we don't have access to the *actual* n_s function!

Similarly to the previous section, we can use this fact to also find the fluctuations in the particle number:

$$\boxed{\langle(\Delta n_s)^2\rangle = \frac{1}{\beta^2}\frac{\partial^2}{\partial\epsilon_s^2}\log Z} \tag{6.9.12}$$

Or:

$$\boxed{\langle(\Delta n_s)^2\rangle = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_s}\langle n_s\rangle} \tag{6.9.13}$$

See Reif pg. 336-337 for more details.

6.10 The Equipartition Theorem

The Equipartition theorem states that **every quadratic degree of freedom in a system's Hamiltonian, has an average energy of $\frac{1}{2}kT$ in thermal equilibrium**. This can be easily proven classically using the Boltzmann distribution.

Consider a single quadratic degree of freedom, $H = Ax_i^2$. The partition function for this degree of freedom is:

$$Z_i = \int_{-\infty}^{\infty} e^{-\beta A x_i^2}dx_i = \sqrt{\frac{\pi}{\beta A}} \tag{6.10.1}$$

The *total* partition function of a system of f such independent degrees of freedom is

$$Z = \Pi_i Z_i = \left(\frac{\pi}{\beta A}\right)^{\frac{f}{2}} \tag{6.10.2}$$

The average energy of this partition function is therefore

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\log Z = \frac{f}{2}\frac{\partial}{\partial\beta}(\log(\beta) - \log(\frac{\pi}{A})) = \frac{f}{2}\frac{1}{\beta} \tag{6.10.3}$$

And thus

$$\boxed{\langle E \rangle = \frac{1}{2}kT} \tag{6.10.4}$$

6.11 Quantum Statistics: Bose, Fermi, and Boltzmann Distributions

In classical mechanics, particles are considered to be distinguishable from one another, while in quantum mechanics this is not the case. This has profound consequences on statistical mechanics when counting the number of states. Consider the following two states, where each quantum number represents one particle:

$$\Psi_1 = \langle ...Q, ...P, ... \rangle, \qquad \Psi_2 = \langle ...P, ...Q, ... \rangle \tag{6.11.1}$$

In quantum mechanics, these two states are identical (since the particles are indistinguishable), while in classical mechanics they are distinct.

$$(\Psi_1 = \Psi_2)_{quantum}, \qquad (\Psi_1 \neq \Psi_2)_{classical} \tag{6.11.2}$$

At the same time, quantum mechanics also differentiates between two types of particles; Bosons (integer spin) and Fermions (half-integer spin). For bosons, the wavefunction is symmetric under the interchange of two particles. This reiterates the requirment that:

$$\Psi_1 = \Psi_2 \tag{6.11.3}$$

For fermions, however, the wavefunction is *antisymmetric* under interchange of two particles. Therefore, we additionally require that:

$$\Psi_1 = -\Psi_2 \tag{6.11.4}$$

Therefore, we see that for a Ψ where *both* fermions occupy the same state Q:

$$\Psi_3 = (..., Q, ...Q, ...) = 0 \tag{6.11.5}$$

This is the Pauli Exclusion Principle: **no two fermions may occupy exactly the same quantum state**.

6.11.1 The Average Number of Particles

The average number of particles in a given state s can be calculated using the partition function for the system. Consider the energy of the system in some configuration R, where $N = \sum_r n_r$ particles can be in one of r energy states ϵ_r. The energy of a given configuration is:

$$E_R = n_1\epsilon_1 + n_2\epsilon_2 + ... = \sum_r n_r\epsilon_r \tag{6.11.6}$$

In the canonical ensemble, the probability of finding the system in this state is:

$$P_R = \frac{1}{Z} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)} \tag{6.11.7}$$

The partition function for the *entire* system is then:

$$Z = \sum_R e^{-\beta E_R} = \sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)} \tag{6.11.8}$$

Therefore, the average number of particles in any given energy state s can be given by averaging over the occupancy of that state for every possible configuration of the system:

$$\overline{n_s} = \frac{\sum_R n_s e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}}{\sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}} \tag{6.11.9}$$

Some clever rearrangement yields:

$$\overline{n_s} = -\frac{1}{\beta Z}\frac{\partial Z}{\partial \epsilon_s} \tag{6.11.10}$$

or

$$\boxed{\overline{n_s} = -\frac{1}{\beta}\frac{\partial \log(Z)}{\partial \epsilon_s}} \tag{6.11.11}$$

This equation allows us to find the distribution of particles given the partition function.

Another quantity often defined is the **dispersion** of the number of particles in s:

$$\overline{(\Delta n_s)^2} = \overline{(n_s - \overline{n_s})^2} = \overline{n_s^2} - \overline{n_s}^2 \tag{6.11.12}$$

This can be nicely recast using the partition function[6.9]:

$$\overline{(\Delta n_s)^2} = \frac{1}{\beta^2}\frac{\partial^2 \log(Z)}{\partial \epsilon_s^2} = -\frac{1}{\beta}\frac{\partial \overline{n_s}}{\partial \epsilon_s} \tag{6.11.13}$$

Further statistical arguments allow us to write $\overline{n_s}$ more simply for systems that obey particular quantum statistics. For systems with indistinguishable particles, the state R is completely determined by the numbers $(n_1, n_2, ...)$, so the sums over R in the general equation can be replaced by sums over all possible combinations of n_r, which we will symbolize as the set $\sum_{n1,n2,...}$:

$$\overline{n_s} = \frac{\sum_{n1,n2,...} n_s e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}}{\sum_{n1,n2,...} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}} \tag{6.11.14}$$

We can rearrange this equation (using properties of exponentials) to sum over one particular n_s first. We will symbolize the set of all possible combinations of n_r excluding n_s as $n_1, n_2, ...\neg n_s$.

$$\overline{n_s} = \frac{\sum_{n_s}\left[n_s e^{-\beta n_s \epsilon_s} \sum_{n_1,n_2,...\neg n_s} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}\right]}{\sum_{n_s}\left[e^{-\beta n_s \epsilon_s} \sum_{n_1,n_2,...\neg n_s} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}\right]} \tag{6.11.15}$$

[6.9] Rief pg. 336.

Notice that the second sums in the numerator and denominator of eq. 6.11.15 do *not* in general cancel. Fixing the total number of particles,

$$\sum_r = n_r = N \tag{6.11.16}$$

Given this requirement, the acceptable values of n_r for the sum $\sum_{n_1,n_2,...\neg n_s}$ *are dependent on the current value of* n_s. Therefore they cannot be pulled out of the sum to cancel. The exception to this rule is photon statistics, where the particle number is allowed to fluctuate.

Equation 6.11.15 can now be further reduced to an expression without sums by making further assumptions about the statistics of the system.

6.11.2 The Fermi-Dirac Distribution

For Fermions, we require that

$$n_r = 0, 1 \quad \forall r \tag{6.11.17}$$

This obviously holds for $r = s$ as well. Therefore, we can explicitly carry out the n_s sum in eq. 6.11.15 over $n_s = 0, 1$:

$$\overline{n_s} = \frac{0 + e^{-\beta\epsilon_s} Z_s(N-1)}{Z_s(N) + e^{-\beta\epsilon_s} Z_s(N-1)} = \frac{1}{\frac{Z_s(N)}{Z_s(N-1)} e^{\beta\epsilon_s} + 1} \tag{6.11.18}$$

Where, for compactness, we have introduced the following notation for the partition function of $N - n_s$ remaining particles, excluding n_s:

$$Z_s(N - n_s) \equiv \sum_{n_1,n_2,...\neg n_s} e^{\beta(n_1\epsilon_1 + n_2\epsilon_2 + ...)}, \quad \text{s.t.} \quad \sum_{r\neq s} n_r = N - n_s \tag{6.11.19}$$

If $1 << N$ (or, more generally, $\Delta N << N$), a Taylor expansion of $\log Z_s(N - \Delta N)$ becomes:

$$\log Z_s(N - \delta N) = \log Z_s(N) - \alpha_s \delta N \tag{6.11.20}$$

Where $\alpha_s = \frac{\partial \log Z_s}{\partial N}$. For large N, we approximate Z_s to be slowly varying with s, such that $\alpha_s \approx \alpha$. Combining these approximations allows us to write:

$$\frac{Z_s(N)}{Z_s(N-1)} = e^{\alpha} \tag{6.11.21}$$

So that

$$\overline{n_s} = \frac{1}{e^{\alpha + \beta\epsilon_s} + 1} \tag{6.11.22}$$

Customarily, this is rewritten by using the fact that $F = -kT \log Z$ to rewrite α as $\alpha = -\frac{1}{kT}\frac{\partial F}{\partial N} = -\frac{\mu}{kT} = -\beta\mu$. The result is the **Fermi-Dirac Distribution**:

$$\boxed{\overline{n_s} = \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1}} \tag{6.11.23}$$

Notice that $0 \geq \overline{n_s} \geq 1$, which is required by our assumption that $n_r = 0, 1$.

6.11.3 The Bose-Einstein Distribution

For bosons, the total number of particles is conserved and the particles are still indistinguishable. However, there is now no restriction on the maximum value of n_r:

$$n_r = 0, 1, 2, ... \tag{6.11.24}$$

Coorespondingly, the general $\overline{n_s}$ equation (eq. 6.11.15) is now (utilizing the definition of Z_s from the Fermi-Dirac section):

$$\overline{n_s} = \frac{0 + e^{-\beta\epsilon_s} Z_s(N-1) + 2e^{-2\beta\epsilon_s} Z_s(N-2)} {Z_s(N) + e^{-\beta\epsilon_s} Z_s(N-1) + e^{-2\beta\epsilon_s} Z_s(N-2)} \tag{6.11.25}$$

Applying the $N >> \Delta N$ approximations from the Fermi-Dirac section, this becomes

$$\overline{n_s} = \frac{Z_s(N)[0 + e^{-\beta\epsilon_s} e^{-\alpha} + 2e^{-2\beta\epsilon_s} e^{-2\alpha} + ...]}{Z_s(N)[1 + e^{-\beta\epsilon_s} e^{-\alpha} + 2e^{-2\beta\epsilon_s} e^{-2\alpha} + ...]} = \frac{\sum_{n_s} n_s e^{n_s(\alpha+\beta\epsilon_s)}}{\sum_{n_s} e^{n_s(\alpha+\beta\epsilon_s)}} \tag{6.11.26}$$

For simplicity, let $z_s = \alpha + \beta\epsilon_s$. Then

$$\overline{n_s} = \frac{\sum_{n_s} n_s e^{n_s z_s}}{\sum_{n_s} e^{n_s z_s}} = \frac{\frac{\partial}{\partial z_s} \sum_{n_s} e^{n_s z_s}}{\sum_{n_s} e^{n_s z_s}} = \frac{\partial}{\partial z_s} \log\left(\sum_{n_s} e^{n_s z_s}\right) \tag{6.11.27}$$

But the sum inside the log is a simple series (allowing us to evaluate the sum):

$$\sum_{n_s} e^{n_s z_s} = \frac{1}{1 - e^{-\beta\epsilon_s}} \tag{6.11.28}$$

So

$$\boxed{\overline{n_s} = \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}} \tag{6.11.29}$$

6.11.4 The Planck Distribution (Photons)

For the special case of photons, the number of particles is allowed to fluctuate. A similar derivation to those for the Fermi and Bose distributions can be made using this fact (Reif pg. 339). However, we can also simply observe that, in this case, the partition function for the remaining states when n_s is specified, Z_s, is independent of the total N, so:

$$\alpha_s = \frac{\partial \log Z_s}{\partial N} = 0 \tag{6.11.30}$$

and therefore

$$\boxed{\overline{n_s} = \frac{1}{e^{\beta\epsilon_s} - 1}} \tag{6.11.31}$$

6.11.5 The Maxwell-Boltzmann Distribution

In the classical case of distinguishable particles, any set of $(n_1, n_2, ...)$ can be rearranged in $\frac{N!}{n_1!n_2!...}$ combinations (instead of just one in the quantum case). This means that the partition function is:

$$Z = \sum_{n_1,n_2,...} \frac{N!}{n_1!n_2!...} e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+...)} \tag{6.11.32}$$

Utilizing

$$\overline{n_s} = -\frac{1}{\beta}\frac{\partial \log(Z)}{\partial \epsilon_s} \tag{6.11.33}$$

The **Maxwell-Boltzmann Distribution** is:

$$\boxed{\overline{n_s} = N\frac{e^{-\beta\epsilon_s}}{\sum_r e^{-\beta\epsilon_r}}} \tag{6.11.34}$$

6.12 Black Body Radiation

All bodies spontaneously emit radiation. Near thermodynamic equilibrium, the spectrum of which is described by the Bose-Einstein distribution, since photons are bosons.

A black body is a perfect absorber of radiation. For example, an enclosure with a small pinhole will absorb virtually all light that enters (since light would have to reflect many, many times to find its way back out through the pinhole). In this idealized situation, the properties of the photon gas (radiation) emitted by the black body are described *exactly* by the Bose-Einstein distribution with the energy $E = \hbar\omega$. From this fact, we can derive the characteristic energy spectrum for black body radiation (an important discovery in the early history of quantum mechanics).[6.10]

Throughout this section, it is useful to keep in mind the picture of a black body as a box with a pinhole, and to draw an analogy between photons escaping the blackbody box through the pinhole, and a heated molecular gas escaping a box through a pinhole.

6.12.1 Planck's Law

We will begin by writing the average energy of a photon gas in three dimensions[6.11]:

$$U(T) = \frac{1}{h^3}\int \frac{\epsilon d^3p d^3x}{e^{\beta\epsilon} - 1} \tag{6.12.1}$$

The x integral yields a volume, V. Since the momenta of the photons are randomly oriented, we can simplify the integral by substituting $d^3p = (2)4\pi p^2$. The additional factor of 2 must be included, since one photon of each of two polarization could inhabit a single k-state.

$$U(T) = \frac{V}{h^3}\int \frac{\epsilon(8\pi p^2)dp}{e^{\beta\epsilon} - 1} \tag{6.12.2}$$

[6.10] This section mostly follows Wikipedia, but another useful resource is: `http://disciplinas.stoa.usp.br/pluginfile.php/48089/course/section/16461/qsp_chapter10-plank.pdf`

[6.11] Three dimensions is important, since it sets the dimensionality of the integrals and the factor of h out front.

Since ϵ is a function of p, we need to change to a common variable. Since the final result is traditionally expressed in terms of frequency, we will chose that. Since $p = \hbar k$, $k = \frac{\omega}{c}$, and $\omega = 2\pi f$:

$$p = \frac{2\hbar\pi f}{c} = \frac{hf}{c} \tag{6.12.3}$$

Also,

$$\epsilon = \hbar\omega = 2\pi\hbar f = hf \tag{6.12.4}$$

So, changing variables and canceling some h's:

$$U(T) = \frac{8\pi V}{c^3}\int \frac{f^3 df}{e^{\beta hf} - 1} \tag{6.12.5}$$

We will use the integrated later, so we'll give it its own symbol:

$$u(f,T) = \frac{8\pi V}{c^3}\frac{f^3 df}{e^{\beta hf} - 1} \tag{6.12.6}$$

The spectral radiance[6.12] (power per unit surface area per solid angle per hertz, or joules per square meter) of a pinhole on the black body is[6.13]:

$$I(f,T) = \frac{u(T)c}{4\pi} \tag{6.12.7}$$

Which leaves us with **Planck's Law**:

$$\boxed{I(f,T) = \frac{2hf^3}{c^2}\frac{1}{e^{\beta hf} - 1}} \tag{6.12.8}$$

Remember that $I(T)$ has units of watts per m^2 per steradian per hertz, or joules per m^2, making it a spectral radiance.

6.12.2 The Stefan-Boltzmann Law

Suppose we now want to know the total power radiated by a section of black body surface area, A. We will include all frequencies, but only integrate over half of a solid angle, since we are interested in radiation coming *out* of the black body (picture the black body as a flat sheet).

$$P = A\int_0^\infty I(f,T)df \int_{\text{half sphere}} d\Omega \tag{6.12.9}$$

However, there is a subtle point here. Black bodies are *Lambertian*[6.14], which means that the intensity they radiate in a given direction is proportional to the cosine of the angle between the given direction and the surface normal. If we imagine our black body sheet in the XY plane,

[6.12] Warning: be wary of terms like intensity, radiance, etc. All are used loosely, but each has a separate meaning (representing different dimensions). A list (I don't know about a GOOD one) of the distinctions can be found at `https://en.wikipedia.org/wiki/Radiance#SI_radiometry_units`

[6.13] This equation is the same as for a gas escaping a heated container, except all photons move with speed c.

[6.14] `https://en.wikipedia.org/wiki/Lambert%27s_cosine_law`

then:

$$I(f,T,\theta) = I(f,T)\cos(\theta) = \frac{2hf^3}{c^2}\frac{\cos(\theta)}{e^{\beta hf}-1} \tag{6.12.10}$$

Accounting for this fact:

$$P = \frac{2hA}{c^2}\int_0^\infty \frac{f^3}{e^{\beta hf}-1}df \int_{\text{half sphere}} \cos(\theta)d\Omega \tag{6.12.11}$$

In order to carry out the integration, we'll first change variables to make the frequency integral dimensionless. Making the substitution $x = \beta hf$:

$$P = \frac{2A(kT)^4}{c^2h^3}\int_0^\infty \frac{x^3}{e^x-1}dx \int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)d\theta \int_0^{2\pi} d\phi \tag{6.12.12}$$

The x integral term is complicated[6.15], but evaluates to:

$$\int \frac{x^3dx}{e^x-1} = \frac{\pi^4}{15} \tag{6.12.13}$$

Carrying out the integral produces the **Stefan-Boltzmann Law**:

$$P = A\left(\frac{2\pi^5k^4}{15c^2h^3}\right)T^4 \tag{6.12.14}$$

The law is greatly simplified by labeling the constant in parentheses as the **Stefan-Boltzmann Constant**:

$$\sigma = \frac{2\pi^5k^4}{15c^2h^3} = \frac{\pi^2k^4}{60\hbar^3c^2} \tag{6.12.15}$$

So that the law becomes simply:

$$\boxed{P = \sigma AT^4} \tag{6.12.16}$$

This equation can be extended to more general circumstances beyond black body radiation. For an imperfect radiator with emissivity $e, 0 < e < 1$ ($e = 1$ for a black body) radiating into surroundings with a temperature $T_c, T_c < T$:

$$\boxed{P = e\sigma A(T^4 - {T_c}^4)} \tag{6.12.17}$$

6.12.3 Energy Density of a Black Body Photon Gas

The average energy density in the photon gas is given by integrating the energy distribution function, $u(f,T)$, over all frequencies to find the total energy:

$$U(T) = \int_0^\infty u(f,T)df = \frac{8\pi V}{c^3}\int_0^\infty \frac{f^3df}{e^{\beta hf}-1} \tag{6.12.18}$$

[6.15] This is a Bose-Einstein integral, but I'd recommend not worrying about integrals that are hard enough to have names.

Solving the frequency integral the same way as in the previous section, we see that the average energy density in the photon gas is:

$$\frac{U(T)}{V} = \frac{8\pi(kT)^4}{15(hc)^3)} \tag{6.12.19}$$

This expression can be simplified by using the Stefan-Boltzmann constant. However, note that this is NOT equivalent to the Stefan-Boltzmann Law, which applies to *radiated power*, rather than *internal energy*:

$$\boxed{\frac{U(T)}{V} = \frac{4\sigma T^4}{c}} \tag{6.12.20}$$

6.13 Fermi-Gases

6.13.1 The Fermi Energy

When a boson gas is in its ground state (T=0), all particles are in the same energy level, and the total energy of the gass is zero. The Pauli exclusion principle prevents fermions from occupying this same configuration. Instead, at T=0 a fermion gas will have a non-zero energy, called the **Fermi Energy**.

The following discussion holds in general for any spin-ful fermions. However, the most important application of this material is the description of electrons within a conducting material: the so called "electron gas".

The Fermi distribution gives the mean occupancy of a state as a function of energy:

$$\overline{n_s} = \frac{1}{e^{\alpha+\beta\epsilon_s}+1} \tag{6.13.1}$$

For convenience, lets define a constant: $\mu = \frac{-\alpha}{\beta}$ [6.16].

$$\overline{n_s} = \frac{1}{e^{\beta(\epsilon_s-\mu)}+1} \tag{6.13.2}$$

At some non-zero temperature T, energy levels far below μ are always occupied, while those far above μ are *never* occupied. When $\epsilon \approx \mu$, the states have some probability of being occupied (see Figure 6.13.1).

However, when $T = 0$, this middle region disappears. The Fermi distribution becomes:

$$\overline{n_s} = \frac{1}{e^{\infty(\epsilon_s-\mu)}+1} \tag{6.13.3}$$

This function switches abruptly from 0 to 1 at $\epsilon = \mu$. This is exactly what we expect to happen due to the Pauli exclusion principle: in the ground state, fermions will fill up each quantum level, starting at the bottom, until reaching some maximum. This maximum, μ is called the **Fermi energy**.

The Fermi energy corresponds to a particular momentum, and therefore a particular wave

[6.16] Some books use $\mu = \epsilon_F$ instead.

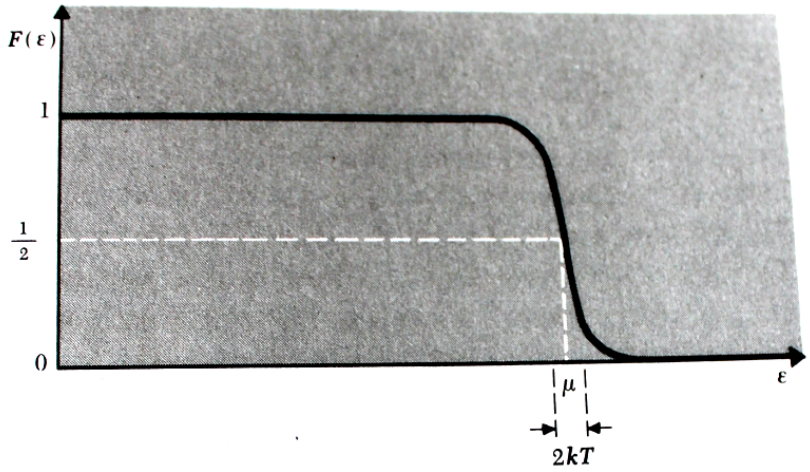


Figure 6.13.1: Graph of the Fermi distribution at temperature $T \neq 0$. $F(\epsilon) = \overline{n_s}$. Source: Reif pg. 389.

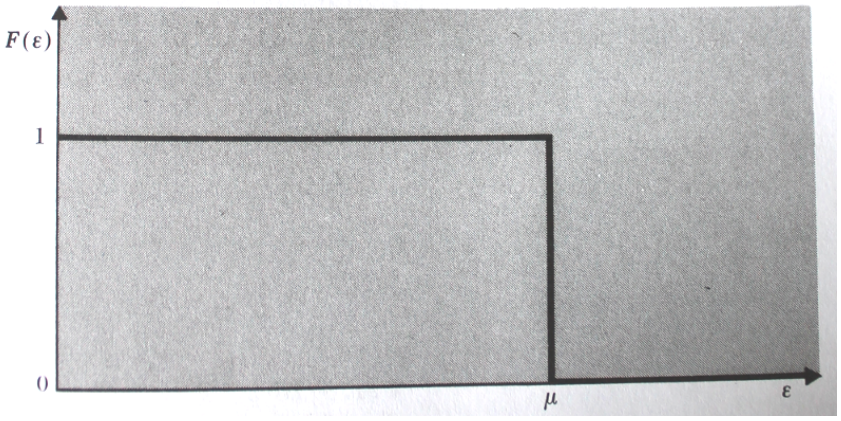


Figure 6.13.2: Graph of the Fermi distribution at temperature $T = 0$. $F(\epsilon) = \overline{n_s}$. Source: Reif pg. 389.

number k_F:

$$\mu = \frac{\hbar^2 k_F^2}{2m} \tag{6.13.4}$$

The total number of states is then the volume of a sphere in k-space: $\frac{4}{3}\pi k_F^3$. Since the volume of a single cell in k-space is $(2\pi)^3$, and each state can hold 2 fermions (of opposite spin):

$$N = 2\frac{V}{(2\pi)^3}\left(\frac{4}{3}\pi k_F^3\right) \tag{6.13.5}$$

Using these two equations, we can rearrange for an expression for the Fermi energy of a given system *at T=0*:

$$\mu_{(T=0)} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{\frac{2}{3}} \tag{6.13.6}$$

6.14 Velocity Distributions

6.14.1 The Maxwell Velocity Distribution: Rigorous Derivation

Consider a single molecule in a gas with a mass m, position $\boldsymbol{r}$, and momentum $\boldsymbol{p}$. The energy of the particle can be subdivided into kinetic energy, ($\frac{p^2}{2m}$) and internal energy, E_{int}. In general $E_{int}(\boldsymbol{r})$, because of interactions with other molecules. However, if we assume that the gas is sufficiently dilute that these are negligible, we can write:

$$E_{tot} = \frac{p^2}{2m} + E_{int} \tag{6.14.1}$$

Where E_{int} is constant. Since the molecule is in thermal contact with the rest of the gas, we can describe its states as a canonical ensemble. The probability of occupying a particular state, with a particular $\boldsymbol{r}$ and $\boldsymbol{p}$ is then:

$$P(\boldsymbol{r},\boldsymbol{p}) \propto e^{-\beta(\frac{\boldsymbol{p}^2}{2m}+E_{int})} \propto e^{-\beta(\frac{\boldsymbol{p}^2}{2m})} \tag{6.14.2}$$

Where we have absorbed $e^{-\beta E_{int}}$ into the proportionality constant. In this system, $\boldsymbol{r}$ and $\boldsymbol{p}$ are continuous variables, so we can ask about the probability of finding the molecule within $\boldsymbol{r} \pm d\boldsymbol{r}$, and $\boldsymbol{p} \pm d\boldsymbol{p}$. In three dimensions, this is represented by the differential:

$$P(\boldsymbol{r},\boldsymbol{p})d^3\boldsymbol{r}d^3\boldsymbol{p} = Ce^{-\beta(\frac{\boldsymbol{p}^2}{2m})}d^3\boldsymbol{r}d^3\boldsymbol{p} \tag{6.14.3}$$

Conventionally, we will chose to change variables from $\boldsymbol{p}$ to $\boldsymbol{v}$. :

$$P(\boldsymbol{r},\boldsymbol{v})d^3\boldsymbol{r}d^3\boldsymbol{v} = Ce^{-\beta(\frac{\boldsymbol{m}\boldsymbol{v}^2}{2})}d^3\boldsymbol{r}d^3\boldsymbol{v} \tag{6.14.4}$$

So far, this probability distribution describes just a single particle. However, multiplying both sides of this equation by the total number of particles, N, turns the *probability* $P(\boldsymbol{r},\boldsymbol{v})$ into the average number of particles that occupy this state (On the RHS, N is absorbed into the constant of proportionality C). We will symbolize average number function as $f(\boldsymbol{r},\boldsymbol{v})$:

$$f(\boldsymbol{r},\boldsymbol{v})d^3\boldsymbol{r}d^3\boldsymbol{v} = Ce^{-\beta(\frac{\boldsymbol{m}\boldsymbol{v}^2}{2})}d^3\boldsymbol{r}d^3\boldsymbol{v} \tag{6.14.5}$$

Now, the integral of this expression must equal the total number of particles in the gas:

$$\begin{aligned} \int\int f(\boldsymbol{r},\boldsymbol{v})d^3\boldsymbol{r}d^3\boldsymbol{v} &= N \\ \rightarrow \quad C\int\int e^{-\beta(\frac{\boldsymbol{m}\boldsymbol{v}^2}{2})}d^3\boldsymbol{r}d^3\boldsymbol{v} &= N \end{aligned} \tag{6.14.6}$$

The integral is independent of $\boldsymbol{r}$, so $\int d^3\boldsymbol{r} = V$, where V is the volume. The remaining integral yields[6.17] $C = \frac{N}{V}\left(\frac{\beta m}{2\pi}\right)^{\frac{3}{2}}$.

If we define the **number density** $n = \frac{N}{V}$, we can then write the **Maxwell velocity**

[6.17] The integral can be evaluated by calculating $\int_{-\infty}^{\infty} e^{-\beta\frac{mv_x^2}{2}}dv_x$ and then cubing the result.

distribution as:

$$\boxed{f(v)d^3\boldsymbol{r}d^3\boldsymbol{v} = n\left(\frac{\beta m}{2\pi}\right)^{\frac{3}{2}} e^{-\beta(\frac{\boldsymbol{m}v^2}{2})} d^3\boldsymbol{r}d^3\boldsymbol{v}} \tag{6.14.7}$$

Note that $f(v)$: the distribution is independent of $\boldsymbol{r}$, and only depends on the *magnitude* of $\boldsymbol{v}$.

6.14.2 The Maxwell Velocity Distribution: Fast Derivation

We want to calculate the number of molecules in an ideal, dilute gas (N particles in a volume V) within some velocity and position range, $f(v)d^3\boldsymbol{r}d^3\boldsymbol{v}$. The probability of a single molecule having that speed is given by the Boltzmann factor:

$$f(v)d^3\boldsymbol{r}d^3\boldsymbol{v} \propto e^{-\frac{\beta v^2}{2m}} \tag{6.14.8}$$

We are integrating over vectors ($\boldsymbol{v}$), but we want our final function to be of a scalar, $f(v)$. Therefore, we need to think about the surface area of a sphere in velocity space, which means that:

$$f(v)d^3\boldsymbol{r}d^3\boldsymbol{v} \propto 4\pi v^2 \tag{6.14.9}$$

Lastly, we know that, since the function we are looking for is a fraction of the total number of particles:

$$\int f(v)d^3\boldsymbol{r}d^3\boldsymbol{v} = CN \int 4\pi v^2 e^{-\frac{\beta v^2}{2m}} d^3\boldsymbol{r}d^3\boldsymbol{v} = N \tag{6.14.10}$$

Where C, the normalization coefficient, can now be determined by evaluating the integral. If we define the number density $n = \frac{N}{V}$, we recover:

$$\boxed{f(v)d^3\boldsymbol{r}d^3\boldsymbol{v} = n\left(\frac{\beta m}{2\pi}\right)^{\frac{3}{2}} e^{-\beta(\frac{\boldsymbol{m}v^2}{2})} d^3\boldsymbol{r}d^3\boldsymbol{v}} \tag{6.14.11}$$

6.14.3 Maxwell Distribution of a Single Velocity Component

Given the Maxwell distribution, we may wish to find the number of molecules in a gas with a particular velocity component, say v_x:

$$g(v_x) = dv_x = \int_{v_y}\int_{v_z} f(\boldsymbol{v})d^3\boldsymbol{v} \tag{6.14.12}$$

Or, evaluating the integral:

$$g(v_x)dv_x = n\left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} e^{-\frac{mv_x^2}{2kT}} dv_x \tag{6.14.13}$$

6.14.4 Maxwell Speed Distribution

Given the Maxwell distribution, we might want to find the number of molecules with a given speed, regardless of direction. If we visualize the Maxwell distribution as existing in a 3D

spherical coordinate velocity space, this function is just the integral over the surface area of a sphere:

$$F(v)dv = 4\pi v^2 f(v)dv \tag{6.14.14}$$

Or, plugging in the Maxwell distribution:

$$F(v)dv = 4\pi n v^2 \left(\frac{\beta m}{2\pi}\right)^{\frac{3}{2}} e^{-\beta(\frac{mv^2}{2})} d^3\boldsymbol{r} d^3\boldsymbol{v} \tag{6.14.15}$$

6.15 Kinetic Theory

Kinetic theory applies statistical mechanics (particularly the Maxwell velocity distribution) to analyze the motion of individual molecules in an ideal gas to make predictions about the behavior of the whole.

6.15.1 Number of Particles Hitting a Surface

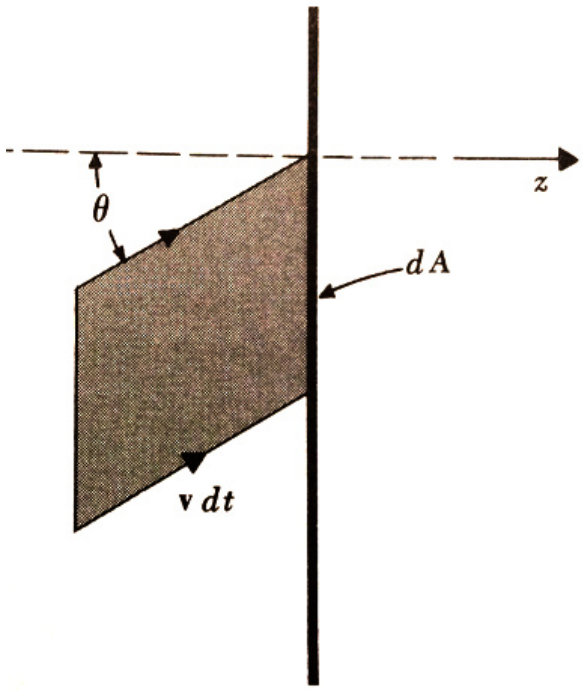


Figure 6.15.1: Cylinder of molecules that will strike an area dA. Source: Reif pg. 271.

Suppose we have an ideal gas in an enclosure. Consider a particular velocity class of molecules with velocity v at an angle θ to the surface normal of the wall. The number of those molecules that will hit a surface area dA in a time dt is the volume of a cylinder (shown in Fig. 6.15.1) with sides $|\boldsymbol{v}|dt$ long and slanted at an angle θ.

The volume of a slanted cylinder is $V =$ (base)(height), just like a parallelogram in 2D. The base here is dA, while the height is $|\boldsymbol{v}|dt\cos(\theta)$. A similar (but obviously different) cylinder could be drawn for every other $|\boldsymbol{v}|$ and θ class. The Maxwell velocity distribution $f(v)$ tells us the number of molecules in each $|\boldsymbol{v}|$ state.Therefore, the number of molecules impacting dA in

dt is the :

$$n(\boldsymbol{v})d^3\boldsymbol{v} = (vdtf(v)\cos(\theta)dAd^3\boldsymbol{v} \tag{6.15.1}$$

We will then define $\Phi(\underline{v})$ (which we could call the 'number intensity' in analogy to light intensity) to be the number of molecules hitting the surface, per dA, per dt:

$$\Phi(\boldsymbol{v})d^3\boldsymbol{v} = (vf(v)\cos(\theta)d^3\boldsymbol{v} \tag{6.15.2}$$

In order to calculate the total number of particles hitting dA per dt, we need to integrate $\Phi(\boldsymbol{v})$. However, we should not include particles whose velocities are going *away* from the wall! We avoid this by requiring that $v_z > 0$ on our integration bounds. In spherical velocity coordinates[6.18], then, $0 < \theta < \frac{\pi}{2}$.

Notice that we do *not* need to change the bounds of the v integral: the v we added determines which speed classes should be counted. All we have to do is not count particles that won't hit the wall at all.

The resulting integrals then become:

$$\Phi_{tot}(\boldsymbol{v}) = \int_0^\infty (vf(v))v^2dv \int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)d\theta \int_0^{2\pi} d\phi \tag{6.15.3}$$

This equation is easy to remember if you see that it is the same as the average velocity, except with the added $\cos(\theta)$ and some changes to the integration bounds. In fact, explicitly applying the Maxwell velocity distribution and integrating now yields a simple expression in terms of the average velocity of the gas molecules:

$$\boxed{\Phi_{tot}(\boldsymbol{v}) = \frac{1}{4}n\bar{v}} \tag{6.15.4}$$

Where n is the number density. Since we know $\bar{v} = \sqrt{\frac{8kT}{\pi m}}$ for an ideal gas, for that case we can write:

$$\Phi_{tot}(n, m, T) = \frac{1}{4}n\sqrt{\frac{8kT}{\pi m}} \tag{6.15.5}$$

6.15.2 Effusion Through an Aperture

The discussion in the previous section can be easily applied to determine the number of particles passing through an aperture of area A in a time interval dt:

$$\frac{\text{Number of Particles}}{dt}dv = A(f(v)v\cos(\theta))(v^2)d\Omega dv \tag{6.15.6}$$

The rate of effusion (total number of particles per dt), R, can then be found by integrating as above over bounds restricted such that the velocities are all *pointing* in the right direction:

$$R = A\int_0^\infty f(v)v^3dv \int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)d\theta) \int_0^{2\pi} d\phi \tag{6.15.7}$$

[6.18] Don't just blindly add a $4\pi v^2$! The extra $\cos(\theta)$ means we need to actually do the θ integral, rather than just jump to 4π.

Clearly, then:

$$R(n,m,T) = A\Phi_{tot} = \frac{nA}{4}\bar{v} = \frac{nA}{4}\sqrt{\frac{8kT}{\pi m}} \tag{6.15.8}$$

Consider two boxes (A and B), filled with different gases, with a small pinhole of area A in wall that separates them. The aperture is clearly the same size for each box. The ratio between these effusion rates of the two gases is known as **Graham's Law**:

$$\frac{R_A}{R_B} = \frac{n_A\sqrt{\frac{T_A}{m_A}}}{n_B\sqrt{\frac{T_B}{m_B}}} \tag{6.15.9}$$

6.16 The Saha Equation

The Saha equation[6.19] is an expression for the percentage of ionization due to thermal collisions for a gas at equilibrium (so that the canonical ensemble applies). The gas is assumed to be weakly ionized (short Debye length), so that there is negligible ion shielding.

Consider two ionization states: an effective "ground state"[6.20] (n) and an ionized state $(n+1)+e^-$ that includes a free electron.

For two subsequent energy levels, the Boltzmann equation(eq. 6.7.2) tells us that:

$$\frac{N_{n+1}}{N_n} = \frac{D(E_{n+1}+e^-)}{D(E_n)}e^{-\beta(E_{n+1}-E_n)} \tag{6.16.1}$$

We can explicitly include the density of state for the free electron (eq. **??**):

$$D(E_{n+1}+e^-) = D(E_{n+1})\frac{4\pi p^2 dp}{\rho_{e^-}h^3} \tag{6.16.2}$$

Where ρ_{e^-} is the density of free electrons within the gas. If the transition $n \to n+1$ takes an energy χ, then the energy difference between the states is:

$$E_{n+1} - E_n = \frac{p^2}{2m} + \chi \tag{6.16.3}$$

Where $\frac{p^2}{2m}$ is the energy of the free electron.
Now, in terms of experimentally important variables, we don't really *care* about the state of the free electron. We can calculate ratio between the states independent of $\boldsymbol{p}_{e^-}$ by integrating over all $\boldsymbol{p}$:

$$\frac{N_{n+1}}{N_n} = \frac{D(E_{n+1})}{D(E_n)}\frac{2}{\rho_{e^-}h^3}\int_0^\infty 4\pi p^2 dp e^{-\beta(\frac{p^2}{2m}+\chi)} \tag{6.16.4}$$

Evaluating the integral leaves us with the **Saha equation**:

$$\frac{N_{i+1}\rho_{e^-}}{N_i} = \frac{(2\pi mkT)^{\frac{1}{2}}}{h^3}\frac{2D(E_{i+1})}{D(E_i)}e^{-\beta\chi} \tag{6.16.5}$$

[6.19] Good PDF: `http://www.astro.princeton.edu/~gk/A403/ioniz.pdf`
[6.20] Effective because this could not be the *actual* ground state.

7 Quantum Mechanics

7.1 The Bra-ket Notation

Bra-ket notation is a shorthand used in the matrix representation of quantum mechanics. Vectors, or "kets" are column vectors that live in the Hilbert space $\mathcal{H}$, and are written $|\alpha\rangle$.

Bras, or row vectors, are written $\langle\alpha|$. Bras do not live in Hilbert space, but rather in the "dual" space to $\mathcal{H}$. This dual space is defined by the inner (or "dot") product. The dual space contains all elements whose inner product with an element of $\mathcal{H}$ is in $\mathbb{C}$. If $\mathcal{H}$ is the space of column vectors, the dual space is the corresponding space of row vectors.

7.1.1 Tips for Working with Bra-kets

- Remember that $\langle A|B\rangle$ is just a number! Therefore, $|C\rangle\langle A|B\rangle = \Big(\langle A|B\rangle\Big)|C\rangle$.

7.1.2 Inner and Outer Products

The inner product (or "dot product") of α and β is written $\langle\alpha|\beta\rangle$. By definition, this operation produces a scalar in $\mathbb{C}$.

The outer product (or "tensor product") α and β is written $|\alpha\rangle\langle\beta|$. The outer product takes two vectors and returns a matrix

7.1.3 Basis

Suppose that $|\alpha_i\rangle$ is a basis for $\mathcal{H}$. Then the following relation, known as the **closure relationship**, holds:

$$\sum_{a'} |a'\rangle\langle a'| = \mathbb{I} \tag{7.1.1}$$

Where $\mathbb{I}$ is the identity matrix.

This closure relationship can be used to express an arbitrary state vector $|\alpha\rangle$ in terms of the a' basis:

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \tag{7.1.2}$$

So that $|\alpha\rangle$ can be expressed as a linear combination of the basis kets with coefficients $\langle a'|\alpha\rangle$. We can also apply the same relationship to the self-inner product of $|\alpha\rangle$:

$$1 = \langle\alpha|\alpha\rangle = \langle\alpha|\Big(\sum_{a'} |a'\rangle\langle a'|\Big)|\alpha\rangle = \sum_{a'} |\langle a'|a'\rangle|^2 \tag{7.1.3}$$

Therefore the expansion coefficients of $|\alpha\rangle$ in the a' basis must satisfy this normalization condition.

The single outer product of two basis vectors is known as the **projection operator**: $\Lambda_{a'} = |a'\rangle\langle a'|$. Notice that, when applied to an arbitrary ket $|\alpha\rangle$, returns the component of $|\alpha\rangle$ along a': $|a'\rangle\langle a'|\alpha\rangle$.

An operator X can be expressed as a matrix by applying the closure relationship twice:

$$X = \sum_{a''} \sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'| \tag{7.1.4}$$

The center term, $\langle a''|X|a'\rangle$, is called the **matrix element** of X. Since it is just a number, it can be pulled forward, leaving:

$$X = \sum_{a''} \sum_{a'} \langle a''|X|a'\rangle|a''\rangle\langle a'| \tag{7.1.5}$$

This can be visualized as a matrix, indexed by a' and a'', with the associated matrix elements as the values at each position in the matrix.

7.2 Complete Sets of Compatible Observables (CSCO)

Most quantum mechanical systems have enough degrees of freedom that any single observable (i.e. operator) will be degenerate. In order to specify a unique state, a series of operators must be applied, together removing the degeneracy. This process can be thought of as choosing basis vectors for a vector space. If too few operators are included, then the system will be degenerate. If too many are chosen, the system will not be self-consistent (the state will be over-determined).

In order to be simultaneously measurable, two operators must share the same eigenvectors. We say that two operators with this property are *compatible observables*. A set of such operators that is sufficient to eliminate any degeneracy in a system is referred to as a complete set of compatible observables (CSCO).

If two operators A and B have the same eigenvectors $|a', b'\rangle$, then we see that

$$AB|a', b'\rangle = a'b'|a', b'\rangle \tag{7.2.1}$$

But also that

$$BA|a', b'\rangle = b'a'|a', b'\rangle = a'b'|a', b'\rangle \tag{7.2.2}$$

Since a' and b' are just numbers. Therefore $AB = BA$ or, equivalently:

$$[A, B] = 0 \tag{7.2.3}$$

If this is the case, then A and B must be compatible. If on the other hand A and B are incompatible, then $[A, B] \neq 0$.

Another exercise illustrates this principle well. Consider the matrix element of the commutator $[A, B]$ in the basis of A's eigenvectors:

$$\langle a''|[A, B]|a'\rangle = (a'' - a')\langle a''|B|a'\rangle = 0 \tag{7.2.4}$$

Where the $|a\rangle$ vectors are the eigenvectors of the A operator. This equation shows that the matrix elements of B expressed in the $|a\rangle$ basis must be zero, unless they are along the diagonal. Thus the $|a\rangle$ vectors must also be eigenvectors of B.

With this knowledge, the task of finding a CSCO is reduced to finding a set of operators $\{A, B, C, ...\}$ that all commute with one another, i.e. $[A, B] = [B, C] = [A, C] = 0$ etc.

7.3 Time Evolution, Translation, and Rotation Operators

A number of operators exist that allow us to turn one ket into another. These operators have several important properties in common. For a given operator $\mathcal{A}$:

- $\mathcal{A}^\dagger\mathcal{A} = 1$ ($\mathcal{A}$ is unitary). This property is necessary for probability conservation to be preserved under the operator[7.1].
- $\mathcal{A}(x_2, x_1)\mathcal{A}(x_1, x_0) = \mathcal{A}(x_2, x_0)$. (The operator $\mathcal{A}$ is transitive.

When using these operators, it is often useful to remember the Taylor series of an exponential e^x when $x << 0$:

$$e^x \approx 1 + x + \frac{x^2}{2} + ... + \frac{x^n}{n!} \tag{7.3.1}$$

7.3.1 The Translation Operator

The translation operator takes a ket and translates it through space. The infinitesimal form of the operator is

$$\mathcal{J}_x(d\boldsymbol{x'}) = 1 - \frac{ip_x d\boldsymbol{x'}}{\hbar} \tag{7.3.2}$$

For finite translations (in the x direction), this operator becomes

$$\mathcal{J}_x(x', x'_0) = exp\left(\frac{-ip_x(x' - x'_0)}{\hbar}\right) \tag{7.3.3}$$

7.3.2 The Time Evolution Operator

The job of the time evolution operator, written as $\mathcal{U}(t, t_0)$, is to take a ket and move it forwards or backwards in time.

The infinitesimal form of the time evolution operator is given by

$$\mathcal{U}(t_0 + dt, t_0) = 1 - \frac{iHdt}{\hbar} \tag{7.3.4}$$

For finite time translations, this operator can be more easily expressed as an exponential. However, its exact form depends on the time dependence of the Hamiltonian.

- **Case 1:** The Hamiltonian is time-independent. The time evolution operator can then be written:

$$\mathcal{U}(t, t_0) = exp\left(\frac{-iH(t - t_0)}{\hbar}\right) \tag{7.3.5}$$

This is by far the most commonly used form of the operator.

[7.1] Sakurai 68.

- **Case 2:** The Hamiltonian is time dependent, but the H's at each time commute with one another. The time evolution operator can then be written:

$$\mathcal{U}(t,t_0) = exp\Big(\frac{-i}{\hbar}\int_0^t H(t')dt'\Big) \tag{7.3.6}$$

  Notice that all we have done is replace $H(t-t_0)$ with the integral $\int_0^t H(t')dt'$!

- **Case 3:** The Hamiltonian is time dependent, and the H's at different times do NOT commute. The time evolution operator can then be written:

$$\mathcal{U}(t,t_0) = 1 + \sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n \int_{t_0}^{t}\int_{t_0}^{t_1}...\int_{t_0}^{t_{n-1}} H(t_1)H(t_2)...H(t_n)dt_1dt_2..dt_n \tag{7.3.7}$$

  This horrendous operator can apparently be made more tractable using the *time ordering operator*. Either way, it's a horrible mess.

7.3.3 The Rotation Operator

The rotation operator takes a ket and rotates it about the origin.

The infinitesimal form of the operator is

$$\mathcal{D}(\hat{n},d\phi) = 1 - i\Big(\frac{\boldsymbol{J}\cdot\hat{n}}{\hbar}\Big)d\phi \tag{7.3.8}$$

For finite rotations, the operator can be written as an exponential:

$$\mathcal{D}(\hat{n},\phi) = exp\Big(\frac{-i\boldsymbol{J}\cdot\hat{n}\phi}{\hbar}\Big) \tag{7.3.9}$$

7.4 Approaches to Quantum Mechanics (Schrodinger vs. Heisenberg)

There are two primary (equivalent) approaches to solving problems in Quantum Mechanics.

- **Schrodinger Picture**: In the Schrodinger picture, kets are considered to evolve in time, while operators remain constant. Over time, $|\alpha\rangle \to \mathcal{U}(t)|\alpha\rangle$.

- **Heisenberg Picture**: In the Heisenberg picture, kets are constant, while operators evolve in time. Over time, $X \to \mathcal{U}^\dagger X\mathcal{U}$.

Of course, these two pictures are *mathematically* identical when written in bra-ket notation, because as $|\alpha\rangle \to \mathcal{U}|\alpha\rangle = |\alpha,t\rangle$ and $|\beta\rangle \to \mathcal{U}|\beta,t\rangle$, the expectation value of some operator X becomes

$$\langle\alpha,t|X|\beta,t\rangle = \langle\alpha|\mathcal{U}^\dagger X\mathcal{U}|\beta\rangle \tag{7.4.1}$$

Where, of course, $\mathcal{U}^\dagger X\mathcal{U}$ is just the time-evolved operator $X(t)$ in the Heisenberg picture.

7.5 The Schrodinger Equation

The famous Schrodinger equation is a direct consequence of our definition of the time-evolution operator. Consider the difference

$$\mathcal{U}(t+dt,t_0)-\mathcal{U}(t,t_0) \tag{7.5.1}$$

Since $\mathcal{U}$ is transitive, we know that

$$\mathcal{U}(t+dt,t_0)=\mathcal{U}(t+dt,t)\mathcal{U}(t,t_0)=\Big(1-\frac{iHdt}{\hbar}\Big)\mathcal{U}(t,t_0) \tag{7.5.2}$$

So

$$\mathcal{U}(t+dt,t_0)-\mathcal{U}(t,t_0)=-i\frac{H}{\hbar}\mathcal{U}(t,t_0)dt \tag{7.5.3}$$

However, from the definition of the derivative, we know that

$$\frac{\partial}{\partial t}\mathcal{U}(t,t_0)=\frac{\mathcal{U}(t+dt,t_0)-\mathcal{U}(t,t_0)}{dt} \tag{7.5.4}$$

Therefore, we can rewrite this difference to obtain the Schrodinger equation:

$$\boxed{i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0)=H\mathcal{U}(t,t_0)} \tag{7.5.5}$$

In the Schrodinger picture, the wavefunction Ψ of a state $|\alpha\rangle$ is represented as $\Psi=\mathcal{U}|\alpha\rangle$. Therefore, applying each side of the Schrodinger equation to some ket, we obtain:

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi \tag{7.5.6}$$

Or, writing out the Hamiltonian and momentum operators explicitly:

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi=\Big(\frac{-\hbar^2}{2m}\nabla^2+V(x)\Big)\Psi} \tag{7.5.7}$$

7.5.1 Spherical Solutions to the Schrodinger Equation

Spherical solutions to the Schrodinger equation are found via separation of variables:

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi) \tag{7.5.8}$$

After separation of variables, the radial component yields the **Radial Schrodinger Equation**:

$$\boxed{\Big(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2}+\frac{\hbar^2 l(l+1)}{2mr^2}+V(r)\Big)u(r)=Eu(r)} \tag{7.5.9}$$

Where $u(r)=rR(r)$. Notice that this change of variables to $u(r)$ means that our boundary conditions change too: $R(\infty)=0$ is automatically enforced for oscillatory $u(r)$, and is somewhat complicated to enforce for other solutions.

We can now however require that $u(0) = 0$. Here is a sketchy proof[7.2]: the wave function must be finite and single valued at the origin, which normally translates into the boundary condition $\psi(0, \theta, \phi) =$ constant $\forall \theta, \phi$. Therefore, $u(0) = 0 \times$ constant $= 0$.

Also note that some authors prefer to preserve the form of the cartesian Schrodinger equation by creating an effective potential:

$$V_{eff}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \tag{7.5.10}$$

So that the radial equation can then be written:

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + V_{eff}(r)\right)u(r) = Eu(r) \tag{7.5.11}$$

7.6 The Heisenberg Equation

The Heisenberg equation of motion is the equation of motion in the Heisenberg picture. To derive it, we start with the relation between an operator $A^{(S)}$ in the Schrodinger picture with the corresponding operator in the Heisenberg picture, $A^{(H)}$:

$$A^{(H)}(t) = \mathcal{U}^\dagger(t) A^{(S)} \mathcal{U}(t) \tag{7.6.1}$$

Where we have assumed that $A^{(S)}$ is independent of time, as is usually the case. Now, we calculating the time derivative:

$$\frac{\partial}{\partial t}A^{(H)}(t) = \frac{\partial \mathcal{U}^\dagger(t)}{\partial t}A^{(S)}\mathcal{U}(t) + \mathcal{U}^\dagger(t)A^{(S)}\frac{\partial \mathcal{U}(t)}{\partial t} \tag{7.6.2}$$

However, assuming that $\mathcal{U}(t) = exp\left(\frac{-iHt}{\hbar}\right)$, this expression becomes

$$\frac{\partial}{\partial t}A^{(H)}(t) = \frac{1}{i\hbar}\left(\mathcal{U}^\dagger A^{(S)}\mathcal{U}\mathcal{U}^\dagger H\mathcal{U} - \mathcal{U}^\dagger H\mathcal{U}\mathcal{U}^\dagger A^{(S)}\mathcal{U}\right) \tag{7.6.3}$$

Now, noting that by our assumption of the form of $\mathcal{U}$, $[\mathcal{U}, H] = 0$, and that per definition $\mathcal{U}$ is unitary, so that $\mathcal{U}^\dagger\mathcal{U} = 1$.

The equation then reduces to a simple commutator relationship, which is the final form of the Heisenberg equation:

$$\boxed{\frac{\partial}{\partial t}A^{(H)}(t) = \frac{1}{i\hbar}[A^{(H)}, H]} \tag{7.6.4}$$

[7.2]The origin and existence of this boundary condition is non-trivial, and purportedly caused Feynman some anxiety (why should the particle have ZERO probability of existing at the origin?!). Details and discussion can be found here: `http://arxiv.org/ftp/arxiv/papers/1001/1001.3285.pdf` and here `http://arxiv.org/ftp/arxiv/papers/1302/1302.0839.pdf`

7.7 Commutator Relations

7.7.1 The Canonical Commutation Relations

The Canonical Commutation Relations are often taken as axioms of quantum mechanics (or, if you will, as definitions of the variables they include). They are:

$$[x_i, x_j] = 0 \tag{7.7.1}$$

$$[p_i, p_j] = 0 \tag{7.7.2}$$

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{7.7.3}$$

7.7.2 Classical Correspondence of Commutators

In general, the following relationship holds between quantum mechanical commutator brackets and the corresponding classical Poisson bracket:

$$\{A, B\}_{classical} \leftrightarrow \frac{1}{i\hbar}[A, B]_{quantum} \tag{7.7.4}$$

7.7.3 Ehrenfest's Theorem

Ehrenfest's theorem is a relatively simple result that follows directly from the rule that $[A, BC] = [A, B]C + B[A, C]$:

$$\boxed{[x, F(p)] = i\hbar\frac{\partial F}{\partial p}} \tag{7.7.5}$$

$$\boxed{[p, G(p)] = -i\hbar\frac{\partial G}{\partial x}} \tag{7.7.6}$$

7.8 Dispersion and Uncertainty

7.8.1 Dispersion

The dispersion of an operator (sometimes also called the mean square deviation or variance) is defined to be:

$$\Delta A = A - \langle A \rangle \tag{7.8.1}$$

The quantity we are most often interested in is the expectation value square of the dispersion:

$$\langle (\Delta A)^2 \rangle = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle \tag{7.8.2}$$

Since $\langle A \rangle$ is just a number, taking the expectation value again doesn't change anything: $\langle \langle A \rangle^2 \rangle = \langle A \rangle^2$. The middle term, however, is not constant: $\langle -2A\langle A \rangle \rangle = -2\langle A \rangle^2$, again since $\langle A \rangle$ is

constant. Therefore the expression becomes:

$$\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2 \tag{7.8.3}$$

This equation is by far the most useful for calculating uncertainties.

7.8.2 The Uncertainty Principle

For two operators A and B, the uncertainty principle states [7.3]:

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2 \tag{7.8.4}$$

For example, for the operators x and p, where $[x,p] = i\hbar$, this produces the famous uncertainty relation:

$$(\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4} \tag{7.8.5}$$

7.9 Spin 1/2 Systems

7.9.1 Constructing the state vectors

We will begin by assuming (without loss of generality) that the spin vector points in the Z direction, and will introduce two kets to describe its two possible orientations: $|+\rangle$ and $|-\rangle$, each of which we will take to be normalized.

Measuring S_x for either of these kets will yield a 50/50 split between $|S_x,+\rangle$ and $|S_x,-\rangle$, so we require that:

$$\langle\pm|S_x,\pm\rangle = \frac{1}{\sqrt{2}} \tag{7.9.1}$$

We also require that $\langle S_x,\pm|S_x,\pm\rangle = 1$. Together, these two conditions allow us to write an expression for $|S_x,\pm\rangle$, specifying its coefficients up to an arbitrary phase factor, which we will chose to collect onto the second term. We can also easily construct $|S_x,-\rangle$ by noting that it must be orthogonal to $|S_x,+\rangle$.

$$|S_x,\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}e^{i\phi_1}|-\rangle \tag{7.9.2}$$

$|S_y,\pm\rangle$ can be constructed in a similar manner. The coefficients are determined up to an arbitrary phase factor by requiring that:

$$\langle S_y,\pm|S_z,\pm\rangle = \frac{1}{\sqrt{2}} \tag{7.9.3}$$

So our expression for S_y is then:

$$|S_y,\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}e^{i\phi_2}|-\rangle \tag{7.9.4}$$

[7.3] For a derivation, see Sakurai 2nd edition, pg. 34-35

The arbitrary phase factors ϕ_1 and ϕ_2 can be eliminated by requiring that:

$$\langle S_y, \pm | S_x, - \rangle = \langle S_y, \pm | S_x, + \rangle = \frac{1}{\sqrt{2}} \tag{7.9.5}$$

From here, inserting the kets constructed above, we see that $\phi_2 - \phi_1 = \pm\frac{\pi}{2}$. Therefore, setting one ket's phase to 0, we can rewrite both kets to eliminate the phase. The final kets for each direction (in the z basis) are then:

$$|S_z, \pm\rangle = |\pm\rangle \tag{7.9.6}$$

$$|S_x, \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle \tag{7.9.7}$$

$$|S_y, \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle \tag{7.9.8}$$

7.9.2 Example: Deriving the $|\vec{S} \cdot \hat{n}, +\rangle$ State Ket

We start by defining the unit vector $\hat{n}$ in the usual way for spherical coordinates (with $\phi = \alpha$ and $\theta = \beta$):

$$\hat{n} = \sin(\beta)\cos(\alpha)\hat{n}_x + \sin(\beta)\sin(\alpha)\hat{n}_y + \cos(\beta)\hat{n}_z \tag{7.9.9}$$

Of course,

$$\boldsymbol{S} \cdot \hat{n} = n_x S_x + n_y S_y + n_z S_z \tag{7.9.10}$$

We know that the $\{|+\rangle, |-\rangle\}$ basis is complete, so:

$$|\boldsymbol{S} \cdot \hat{n}, +\rangle = a|+\rangle + b|-\rangle \tag{7.9.11}$$

Where

$$a^2 + b^2 = 1 \tag{7.9.12}$$

Now, we will take as given that

$$\boldsymbol{S} \cdot \hat{n} | \boldsymbol{S} \cdot \hat{n}, +\rangle = \frac{\hbar}{2} | \boldsymbol{S} \cdot \hat{n}, +\rangle \tag{7.9.13}$$

We will evaluate the LHS explicitly, then set $RHS = |\boldsymbol{S} \cdot \hat{n}, +\rangle = a|+\rangle + b|-\rangle$ to solve for constants a and b.

Using the expressions derived for S_x, S_y, and S_z previously, we can expand the LHS into the $\{|+\rangle, |-\rangle\}$ basis, then expand. Matching terms then produces the following equations:

$$((n_x - i n_y)b + n_z a)|+\rangle = a|+\rangle \tag{7.9.14}$$

$$((n_x + in_y)a + n_z b)|-\rangle = b|-\rangle \tag{7.9.15}$$

Combined with the normalization condition, the solution to this set of equations is:

$$\boxed{|\boldsymbol{S}\cdot\hat{n},+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle} \tag{7.9.16}$$

Similarly, you can also find the minus eigenstate:

$$\boxed{|\boldsymbol{S}\cdot\hat{n},-\rangle = -\sin\left(\frac{\beta}{2}\right)e^{-i\alpha}|+\rangle + \cos\left(\frac{\beta}{2}\right)|-\rangle} \tag{7.9.17}$$

This is a VERY useful result to have memorized!

7.9.3 Deriving the Spin Operators

Operators may be assembled through a sum over their eigenvectors (Sakurai 1.3.34):

$$\sum_{a'} a'|a'\rangle\langle a'| \tag{7.9.18}$$

For each of the spin states constructed in the previous section, the eigenvalue $a' = \pm\frac{\hbar}{2}$. We can therefore easily construct the operators for the spin in each direction:

$$S_x = \frac{\hbar}{2}\Big(|S_x,+\rangle\langle S_x,+|\Big) + \frac{-\hbar}{2}\Big(|S_x,-\rangle\langle S_x,-|\Big) \tag{7.9.19}$$

After the expressions for $|S_x,\pm\rangle$ have been substituted, expanded, and terms are allowed to cancel, the S_x operator is given by:

$$S_x = \frac{\hbar}{2}\Big((|+\rangle\langle -|) + (|-\rangle\langle +|)\Big) \tag{7.9.20}$$

The other two operators are similarly derived:

$$S_y = \frac{\hbar}{2}\Big(-i(|+\rangle\langle -|) + i(|-\rangle\langle +|)\Big) \tag{7.9.21}$$

$$S_z = \frac{\hbar}{2}\Big((|+\rangle\langle +|) + (|-\rangle\langle -|)\Big) \tag{7.9.22}$$

7.9.4 Deriving the Commutator Relationships

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k \tag{7.9.23}$$

$$\{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij} \tag{7.9.24}$$

From the anti-commutator:

$$\boldsymbol{S}^2 = {S_x}^2 + {S_y}^2 + {S_z}^2 = \frac{3}{4}\hbar^2 \tag{7.9.25}$$

7.9.5 Pauli Spin Matrices for the Spin 1/2 System

Rotations in 3 dimensions make up the 'rotation group'; SO(3), or Special Orthogonal Group 3. The elements of this group correspond to rotations about the origin, and contain only real valued entries.

Similarly, in quantum mechanics, rotations in 2D spin systems belong to the SU(2) group. The Pauli spin matrices are a basis for this group:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.9.26}$$

$$\sigma_2 = \sigma_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{7.9.27}$$

$$\sigma_3 = \sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.9.28}$$

Together, these are often written as:

$$\sigma_j = \begin{pmatrix} \delta_{j3} & \delta j1 - i\delta j2 \\ \delta j1 + i\delta j2 & -\delta_{j3} \end{pmatrix} \tag{7.9.29}$$

Conveniently, this group happens to correspond to the spin operators we have previously been using:

$$S_j = \frac{\hbar}{2}\sigma_j \tag{7.9.30}$$

Each matrix has two eigenvalues, ± 1, which are often just denoted by $\pm$. The eigenvectors of each matrix can be identified with the spin states derived earlier in this section:

- $\sigma_1 = \sigma_x$

$$\sigma_{x,+} = |S_x, +\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.9.31}$$

$$\sigma_{x,-} = |S_x, -\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7.9.32}$$

- $\sigma_2 = \sigma_y$

$$\sigma_{y,+} = |S_y, +\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{7.9.33}$$

$$\sigma_{y,-} = |S_y, -\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{7.9.34}$$

- $\sigma_3 = \sigma_z$

$$\sigma_{z,+} = |S_z, +\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.9.35}$$

$$\sigma_{z,-} = |S_z, -\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.9.36}$$

The following are facts about the Pauli matrices:

- All three matrices are Hermitian.
- The square of each matrix is identity: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$.
- $det(\sigma_i) = -1$
- $Tr(\sigma_i) = 0$
- The **Pauli vector** is defined to be $\boldsymbol{\sigma} = \sigma_x \hat{x} + \sigma_y + \hat{y} + \sigma_z \hat{z}$
- $[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c$
- $\{\sigma_a, \sigma_b\} = 2\delta_{ab} I$.

7.10 Total Angular Momentum: $\vec{J}$

7.10.1 Getting s, S, l, L, j, and J straight

On one level, these variables are all used for the *same thing*: angular momenta! However, there are some general conventions for when each variable is used, and these can occasionally be important when you get into a problem that involves one or more.

First, when dealing with atoms, **lowercase variables refer to an individual electron**, while **uppercase variables refer to the entire atom or system!**. This convention is not always followed with other systems (say, a bunch of particles in a box), but it is generally used when working with atoms.

Then:

- s/S: Spin
- l/L: Angular momentum
- j/J: TOTAL Angular momentum

7.10.2 Quantum Numbers

A state of a spin-ful particle is specified by three quantum numbers, n, l, and m. n is called the **principal quantum number** and (in the absence of any external fields, etc.) solely determines the energy of the particle. It can take on any integer value:

$$n = 0, 1, 2, ... \tag{7.10.1}$$

The second, $s/l/j$, has many names in different situations. When a particle exists on its own, it is called s, and represents the total spin of the particle. If the particle has angular momentum because it is undergoing circular motion, the quantum number is l, the **orbital angular momentum quantum number** and represents the total angular momentum of the orbit. Finally, if the system is a composite system, (say, two spin-ful particles, or one spin-ful particle also undergoing circular motion) the quantum number is j, the **total angular momentum**, and represents the total (vector sum) angular momentum of the system. In any case:

$$j = 0, 1, 2, ...n - 1 \tag{7.10.2}$$

Finally, for any of the types of systems above, the third quantum number is the projection of the angular momentum on the z axis. This is denoted m, and is called the **magnetic quantum number** because it plays an important role in the Zeeman Effect. It ranges:

$$m = -j, -|j-1|, ...0...|j-1|, j \tag{7.10.3}$$

7.10.3 A Note on Excessive j's in Notation

Discussions of total angular momentum necessarily involve multiple particles, each of which has its own angular momentum and magnetic quantum numbers, j_i and m_i respectively. To make matters messier,these must all often be written in a single ket, in order to write down the current state of a system of particles! Unfortunately, essentially many sources use conflicting notation schemes, which gets incredibly confusing.

In this book, the normal convention will be followed that states in the total angular momentum basis will be written as:

$$|j_{total}, m_{total}\rangle \tag{7.10.4}$$

While kets written in the individual angular momentum basis will be written:

$$|j_1, m_1\rangle_1 |j_2, m_2\rangle_2 ... |j_i, m_i\rangle_i \tag{7.10.5}$$

My personal opinion is that this notation is nicer than the alternative $|j_1, j_2, ...j_i; m_1, m_2...m_i\rangle$, because it emphasizes the separation between the particles in this basis. The indices after each closing karet are helpful when variables are replaces with numbers, which makes the kets easy to accidentally interchange.I will include them whenever helpful for clarity.

In many problems the j_i's of each particle will be given and equal, so these they can be dropped from the kets for simplicity's sake.

7.10.4 Addition of Angular Momenta

Imagine a system of two quantum mechanical particles, each with spin $\frac{1}{2}$ [7.4]. If these particles were classical, their angular momentum vectors could be easily added to produce a definite combined angular momentum vector. However, in quantum mechanics it isn't that easy.

[7.4] The best reference for this problem is Griffiths QM, pg. 184

The general problem we have is that a given system of particles can be written in one of two CSCOs:

- The **separate** particles CSCO, which consists of the J_1^2, J_2^2, J_{1z}, and J_{2z} operators. Notice how the J_z operators are *separate*. In this CSCO, kets are specified by: $|j_1, m_1\rangle|j_2, m_2\rangle$.
- The **combined** particles CSCO, which consists of the J_1^2, J_2^2, J^2, and J_z operators, where J and J_z are of the TOTAL system. In this CSCO, kets are specified by: $|j_1, j_2, j, m\rangle$.

Our goal is to find a way to write a combined state, say $|1, 1\rangle$, as a linear combination of separate particle states for the two spin $\frac{1}{2}$ particles. This is easy for $|1, 1\rangle$, because there is only ONE combination of two spin $\frac{1}{2}$ particles that can produce this state!:

$$|1, 1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle_1|\frac{1}{2}, \frac{1}{2}\rangle_2 \tag{7.10.6}$$

Now, recall that, *in descending order of energy*, the combined states of this system must be:

- $|1, 1\rangle$
- $|1, 0\rangle$
- $|1, -1\rangle$
- $|0, 0\rangle$

This means that we can move from one of these states to the other by applying the raising and lowering operator! Since the two particles are independent, we can write:

$$J_\pm = J_{1,\pm} + J_{2,\pm} \tag{7.10.7}$$

Now we can use the lowering operator, along with the ket we already managed to "translate" into the separate particle basis, to find the others!

$$J_-|1, 1\rangle = (J_{1,\pm} + J_{2,\pm})|\frac{1}{2}, \frac{1}{2}\rangle_1|\frac{1}{2}, \frac{1}{2}\rangle_2 \tag{7.10.8}$$

Now, remembering that:

$$J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m - 1\rangle \tag{7.10.9}$$

We can determine that:

$$\sqrt{2}|1, 0\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle_1|\frac{1}{2}, \frac{1}{2}\rangle_2 + |\frac{1}{2}, \frac{1}{2}\rangle_1|\frac{1}{2}, -\frac{1}{2}\rangle_2 \tag{7.10.10}$$

Or, more properly:

$$|1, 0\rangle = \frac{1}{\sqrt{2}}\Big(|\frac{1}{2}, -\frac{1}{2}\rangle_1|\frac{1}{2}, \frac{1}{2}\rangle_2 + |\frac{1}{2}, \frac{1}{2}\rangle_1|\frac{1}{2}, -\frac{1}{2}\rangle_2\Big) \tag{7.10.11}$$

This IS the representation of the system state $|1,0\rangle$ in the single particle basis! Carrying out the same process again (remembering that $J_-||\frac{1}{2},-\frac{1}{2}\rangle = 0$) yields:

$$|1,0\rangle = |\frac{1}{2},-\frac{1}{2}\rangle_1|\frac{1}{2},-\frac{1}{2}\rangle_2 \tag{7.10.12}$$

Which is exactly what we should have been expecting. Now, the lowering operator won't help us break $|0,0\rangle$ into the separate particle basis. However, we can easily guess this one too. We know that $|0,0\rangle$ should be a linear combination of $|\frac{1}{2},-\frac{1}{2}\rangle_1|\frac{1}{2},\frac{1}{2}\rangle_2$ and $|\frac{1}{2},\frac{1}{2}\rangle_1|\frac{1}{2},-\frac{1}{2}\rangle_2$, just like $|1,0\rangle$. However, it must also be orthogonal to $|1,0\rangle$, and still normalize to 1! The only combination that fits all these parameters is:

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle_1|\frac{1}{2},\frac{1}{2}\rangle_2 - |\frac{1}{2},\frac{1}{2}\rangle_1|\frac{1}{2},-\frac{1}{2}\rangle_2\right) \tag{7.10.13}$$

7.10.5 Clebsch-Gordan Coefficients

In the section above, the states of a system in the combined basis were represented as a linear combination of states in the *separate* particle basis. **The coefficients in this linear combination are the Clebsch-Gordan Coefficients**. The coefficients are most accurately symbolized as:

$$C^{j_1,j_2,j}_{m_1,m_2,m} \tag{7.10.14}$$

Where j_1, m_1 and j_2, m_2 are the quantum numbers of each particle in the separate basis, and j, m are the quantum numbers of the combined state. Notice that there are no Clebsch-Gordan Coefficients for more than 2 particles! That would be an *utter nightmare*. Instead, when combining or breaking apart states with many particles, you will first break the state into halves, then continue to break those apart until you have the individual basis.

Perhaps the easiest way to think about the Clebsch-Gordan Coefficients is as a function, like in a programming language. For example, the command Mathematica that generates Clebsch-Gordan Coefficients is:

$$C^{j_1,j_2,j}_{m_1,m_2,m} = \text{ClebschGordan[\{j1,m1\},\{j2,m2\},\{j,m\}]}$$

You can use this function to break down any state in the combined basis you want. Say you have a system of two particles, 1 and 2, and total spins j and m. Then we know immediately that:

$$|j,m\rangle = \sum_{j_1}\sum_{j_2}\left(\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} C^{j_1,j_2,j}_{m_1,m_2,m}|j_1,m_1\rangle|j_2,m_2\rangle\right) \tag{7.10.15}$$

The upside of this complicated mess is that you need never perform the computation to find these coefficients (which is horrible for systems higher than $\frac{1}{2}\otimes\frac{1}{2}$): you can use the computer functions instead!

Occasionally (on an exam, perhaps, or while time traveling in the 20th century) you may be required to read these coefficients off of a strange and very dense table (fig. 7.10.2). Figure 7.10.1 helps explain how to decipher these tables:

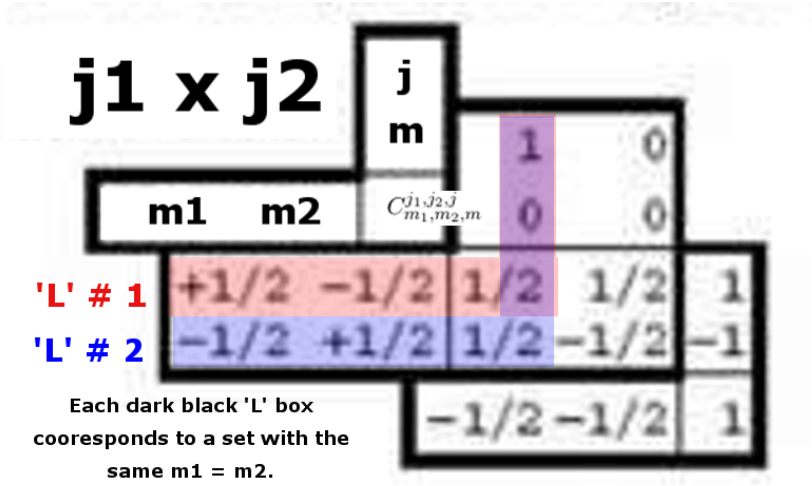


Figure 7.10.1: Detail of a CGC table (for $\frac{1}{2} \otimes \frac{1}{2}$), with labels. Source: `https://www.physicsforums.com/attachments/cg-table-jpg.32635/`

- **Important:** A square root sign is understood over every entry in the table. If the entry has a minus sign, you're meant to put that OUTSIDE of the square root. So $-\frac{1}{2} \to -\sqrt{\frac{1}{2}}$.
- The bold letters in the upper left corner (outside of the boxes) are j_1 and j_2: the j's for the individual particles. For this table $j_1 = j_2 = \frac{1}{2}$.
- The remainder of the chart is grouped into L-shaped boxes, each of which groups states that contain the same m (total m) values.
- Each column in the top segment of the 'L' contains j and m: the quantum numbers of a state in the combined representation.
- Each row in the side segment of the 'L' contains m_1 and m_2 for a state in the single particle representation.
- In the middle of the 'L' are the actual Clebsch-Gordan Coefficients.

We can read the table in two ways. Suppose we want to break down the combined state $|j = 1, m = 0\rangle$. We make two 'L's through the box with the coefficients. The first L (shaded red on the drawing)tells us that ($j = 1$, $m = 0$, $m_1 = \frac{1}{2}$, $m_2 = -\frac{1}{2}$):

$$C^{\frac{1}{2},\frac{1}{2},1}_{\frac{1}{2},-\frac{1}{2},0} = \sqrt{\frac{1}{2}} \tag{7.10.16}$$

Similarly, the second L (shaded blue on the drawing) tells us that($j = 1$, $m = 0$, $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{2}$):

$$C^{\frac{1}{2},\frac{1}{2},1}_{-\frac{1}{2},\frac{1}{2},0} = \sqrt{\frac{1}{2}} \tag{7.10.17}$$

Therefore:

$$|1,0\rangle = \sqrt{\frac{1}{2}}|\frac{1}{2},-\frac{1}{2}\rangle + \sqrt{\frac{1}{2}}|-\frac{1}{2},\frac{1}{2}\rangle \tag{7.10.18}$$

The table can also be read by making L's in the other direction to express individual basis states in terms of the combined representation. For example, for $m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}$:

$$|-\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{2}}|1,0\rangle + -\sqrt{\frac{1}{2}}|0,0\rangle \tag{7.10.19}$$

For more information: `http://www.eng.fsu.edu/~dommelen/quantum/style_a/clgrdn.html`

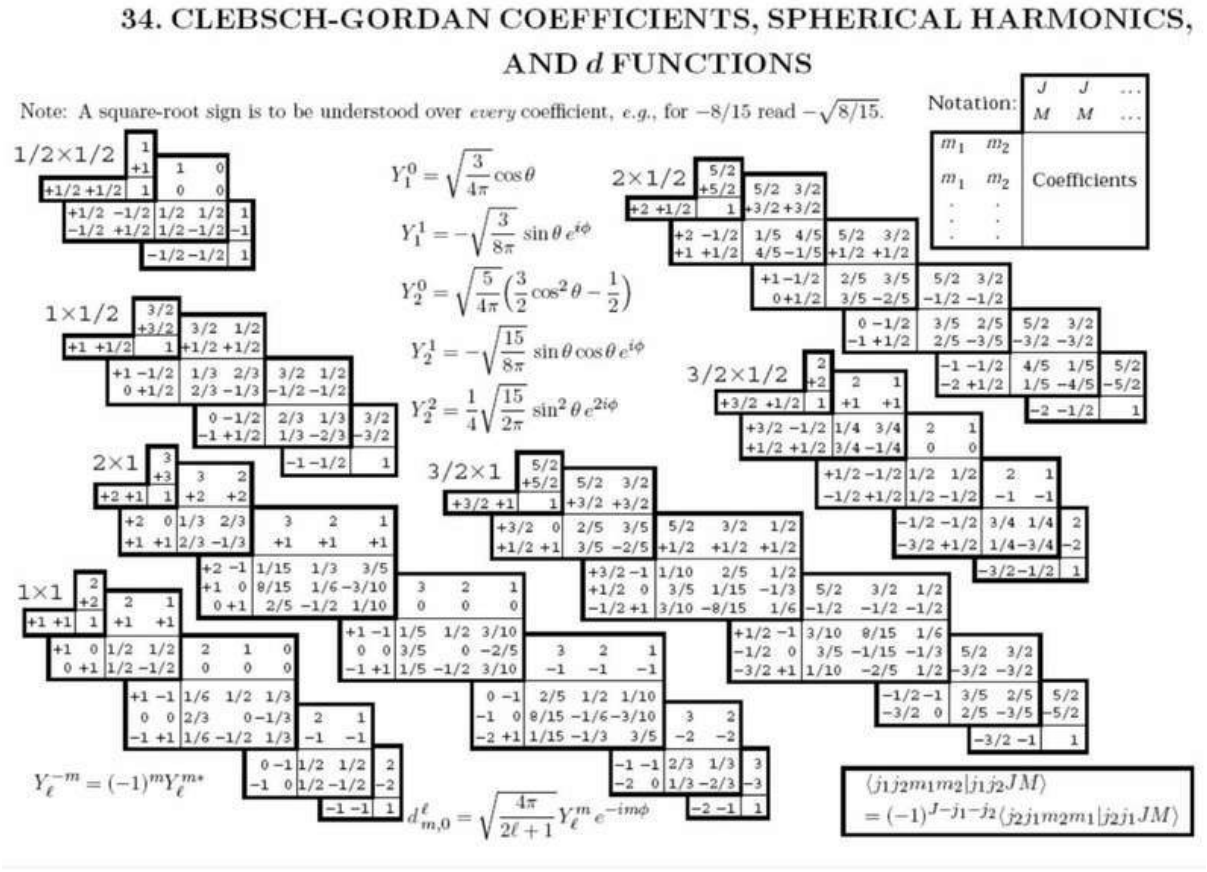


Figure 7.10.2: CGC table. Source: `https://www.physicsforums.com/attachments/cg-table-jpg.32635/`

7.11 Time Independent Perturbation Theory

Some quantum mechanical systems are fully and analytically solvable (hydrogen atom, SHO, square well, etc.), while others are much more complicated. Perturbation theory provides approximations of solutions to difficult problems in terms of the solutions to already-solved easier problems. If the Hamiltonian for a system under study is H, the goals is to write:

$$H = H_0 + H' \tag{7.11.1}$$

Where H_0 is the Hamiltonian of a solved system (i.e. we know the wave functions and their associated energies).

Simple formulas exist for systems that are non-degenerate, but these equations quickly fail when applied to degenerate systems (for example, some have denominators that can be zero for degenerate systems). Therefore, a more complicated set of equations exists for these situations.

In general, the energies of a system are just the eigenvalues of its Hamiltonian in matrix form. For non-degenerate systems, the matrix that approximates a system to some order will automatically be diagonalized, making it trivial to read off the eigenvalues. However, for degenerate systems, the matrix will NOT be diagonalized. The energies are then found (in theory) by choosing a different, special basis that diagonalizes the matrix or (usually in practice) by simply finding the eigenvalues of the matrix.

7.11.1 Non-Degenerate

To first order:

$$\boxed{E_n^{(1)} = \langle n|H'|n\rangle} \tag{7.11.2}$$

For the wavefunction:

$$\boxed{\psi_n^{(1)} = \sum_{m\neq n} \frac{\langle m|H'|n\rangle}{E_n^{(0)} - E_m^{(0)}}\psi_m^{(0)}} \tag{7.11.3}$$

To second order:

$$\boxed{E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m|H'|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}} \tag{7.11.4}$$

In general, you won't need to calculate $\psi_n^{(2)}$, and the equation is a bit nasty, so I won't reproduce it here.

7.11.2 Degenerate

To first order, the energies of a degenerate system are the eigenvalues of the matrix:

$$\boxed{W_{ij} = \langle i|H'|j\rangle} \tag{7.11.5}$$

To second order, the energies are the eigenvalues of a slightly more complicated matrix:

$$\boxed{a_{ni} = \sum_{m\neq n} \frac{W_{nm}W_{mi}}{E_n^{(0)} - E_m^{(0)}}} \tag{7.11.6}$$

Where W_{ij} is the same matrix used above for first order, n, i range **only over the degenerate subspace**, while m ranges over the whole space. For example, if your original Hamiltonian matrix (W_{ij}) is 4×4, but only the energies corresponding to 1 and 2 are degenerate, then $n, i = 1, 2$ and $m = 1, 2, 3, 4$. The resulting a_{ni} matrix would in this case then just be a 2×2.

7.12 Time Dependent Non-Degenerate Perturbation Theory

Time dependent perturbation theory allows us to analyze time dependent perturbations to a solved system. Suppose that we have the Hamiltonian:

$$H = H_0 + V(t) \tag{7.12.1}$$

Where H_0 is the Hamiltonian of our solved system, for which we will label the states $|n\rangle$ and the energy levels E_n. Assuming that the states $|n\rangle$ are complete (which is generally true), we can write any state of our *new* Hamiltonian (at any time) as a linear combination of the time evolved basis kets:

$$\boxed{|\psi(t)\rangle = \sum_n c_n(t) e^{-\frac{iE_n t}{\hbar}} |n\rangle} \tag{7.12.2}$$

The coefficients $c_n(t)$ are generally complex, and are called the **amplitudes**. The probability of finding the system in a state n is given by $|c_n(t)|^2$.The coefficients can be found via an iterative formula:

$$c_n(t) = c_n(0) + \frac{-i}{\hbar} \sum_k \int_0^t e^{-i\omega_{nk}t'} \langle n|V(t')|k\rangle c_k(t') dt' \tag{7.12.3}$$

Where:

$$\omega_{nk} = \frac{E_n - E_k}{\hbar} \tag{7.12.4}$$

Each time this formula is used recursively, it increases the order to which $c_n(t)$ is accurate by 1. In principle, then:

$$c_n(t) = c_n^{(0)} + c_n^{(1)} + c_n^{(2)} + ... \tag{7.12.5}$$

The first term is constant: $c_n^{(0)} = c_n(0)$, and represents the fact that, to 0th order, a perturbed system will just stay in the state it already occupies! To first order, then[7.5]:

$$\boxed{c_n^{(1)}(t) = \frac{-i}{\hbar} \sum_k \int_0^t e^{-i\omega_{nk}t'} \langle n|V(t')|k\rangle c_k(0) dt'} \tag{7.12.6}$$

This equation is extremely useful, since we usually know the exact values of $c_k(0)$! For example, if $|n\rangle$ are the kets for the SHO, and at $t = 0$ we have:

$$|\psi(t=0)\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle \tag{7.12.7}$$

Then $c_0(0) = \sqrt{\frac{3}{4}}$ and $c_1(0) = \sqrt{\frac{1}{4}}$.

7.12.1 Fermi's Golden Rule

Fermi's golden rule gives the transition rate R between two states f and i effected by a perturbation H':

$$R_{1\to 2} = \frac{2\pi}{\hbar} |\langle f|H'|i\rangle|^2 \rho_f \tag{7.12.8}$$

[7.5]The $c_n(0)$ from the formula above isn't included, of course, since that term represents all *lower* orders.

Where ρ_f is the density of final states, which makes sense, since we'd expect the transition probability to scale linearly with the number of possible states the system could jump into at the final energy.

7.13 Scattering

Here's a good source on scattering in general: `http://juser.fz-juelich.de/record/20885/files/A2_Bluegel.pdf`

7.13.1 Partial Wave Analysis: Theory

(Following Griffiths QM, pg. 399 to 408)

The general (separated) solution to the Schrodinger equation for a spherically symmetric potential (of which spherically symmetric scattering potentials are a subset) can be written:

$$\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi) \tag{7.13.1}$$

The bulk of the physics here takes place in the radial direction. If we make the (common) change of variables that $u(r) = rR(r)$, the radial schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu \tag{7.13.2}$$

We will search for solutions to this equation by considering three regions. In the scattering region, $V \neq 0$. In the intermediate region, $V = 0$ but we may not neglect $\frac{1}{r^2}$ (forcing us to retain the centrifugal term of the Schrodinger equation). Finally, in the radiation zone, we take $r >> 1$ and, of course, still $V = 0$.

Let's start with the radiation zone. Rewriting E in terms of the wave number k; $E = \frac{k^2\hbar^2}{2m}$, the equation then simplifies to

$$\frac{d^2u}{dr^2} \approx -k^2u \tag{7.13.3}$$

For which we know the solution:

$$R(r) \approx \frac{e^{ikr}}{r} \tag{7.13.4}$$

Now we can progress onto the intermediate zone. Here, since $V = 0$, the Schrodinger equation simplifies to:

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2} = -k^2u \tag{7.13.5}$$

The solutions to this equation are the Spherical Hankel functions, $h_l^{(1)}$ and $h_l^{(2)}$. Asymptotically, $h_l^{(1)} \approx \frac{(-i)^{l+1}}{x}e^{ix}$ and $h_l^{(2)} \approx \frac{(i)^{l+1}}{x}e^{-ix}$. Therefore, since we want our solution to represent an *outgoing* (scattered) wave, we choose only $h_l^{(1)}$, so

$$R(r)\ h_l^{(1)}(kr) \tag{7.13.6}$$

Therefore, the complete wavefunction outside of the scattering region is

$$\psi(r,\theta,\phi) = A\left[e^{ikz} + \sum_{l,m} C_{l,m} h_l^{(1)}(kr) Y_l^m(\theta,\phi)\right] \tag{7.13.7}$$

Where the first term represents the incoming plane wave in the z direction, and the second term is the scattered part of the wavefunction. Since we usually deal with azimuthally symmetric potentials, it is convenient to reduce the Y_l^m's down in terms of Legendre polynomials. We will also make the (later intelligable) substutition:

$$C_{l,0} = i^(l+1)k\sqrt{4\pi(2l+1)}a_l \tag{7.13.8}$$

Where a_l is called the **partial wave amplitude**. The solution can now be written:

$$\psi(r,\theta,\phi) = A\left[e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l h_l^{(1)}(kr)P_l(\cos\theta)\right] \tag{7.13.9}$$

In the large r limit, this equation can be written in terms of the scattering amplitude $f(\theta)$:

$$\psi(r,\theta,\phi) = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right] \tag{7.13.10}$$

Where

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l P_l(\cos\theta) \tag{7.13.11}$$

This is useful, since it can be shown that

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{7.13.12}$$

In order to actually find the partial wave amplitudes, we will need to first need to write the wavefunction entirely in spherical coordinates (instead of the current mix of spherical and linear coordinates). The problem coordinate is z, which can be rewritten in terms of spherical bessel functions via the Rayleigh formula:

$$e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \tag{7.13.13}$$

to yield:

$$\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_l h_l^{(1)}(kr)\right]P_l(\cos\theta) \tag{7.13.14}$$

From here, the general strategy for applying partial waves analysis is as follows:

1. Identify the relevant boundary conditions between the scattering region and the intermediate region.
2. Fit the general form of the wavefunction to the boundary conditions, and solve for a_l.
3. Using a_l, compute $f(\theta)$ and then $\frac{d\sigma}{d\omega}$ etc.

7.13.2 Phase Shift Analysis

When a particle scatters, the amplitude of its wavefunction must remain the same (in order to conserve probability flow into and out of the scattering region), but the outgoing wave may pick up a phase shift. The Rayleigh formula allows us to express an incoming plane wave as:

$$\psi_0^l = Ai^l(2l+1)j_l(kr)P_l(\cos\theta) \tag{7.13.15}$$

Using the asymptotic form of the spherical Hankel functions to re-write the spherical bessel function when $kr >> 1$, this becomes

$$\psi_0^l = Ai^l(2l+1)\left[\frac{1}{2}(h_l^{(1)} + h_l^{(2)})\right]P_l(\cos\theta) = A\frac{(2l+1)}{2ikr}\left[e^{ikr} - (-1)^l e^{-ikr}\right]P_l(\cos\theta) \tag{7.13.16}$$

This equation represents the wavefunction with *no* potential present. However, the wavefunction above is already clearly divided into an incoming wave (the second term) and an outgoing wave (the first term). Therefore, if there is a NON-zero potential, the effect will be to add a phase shift to the outgoing term.

$$\psi_0^l = A\frac{(2l+1)}{2ikr}\left[e^{i(kr+2\delta_l)} - (-1)^l e^{-ikr}\right]P_l(\cos\theta) \tag{7.13.17}$$

By comparing this equation to the wavefunction we calculated by partial wave analysis, we conclude that:

$$a_l = \frac{1}{k}e^{i\delta_l}\sin(\delta_l) \tag{7.13.18}$$

7.13.3 Partial Wave Analysis: Practical Application

In practice, for finding scattering cross sections, the most useful result of partial wave analysis is:

$$\boxed{\sigma = \sum_{l=0}^{\infty}\sigma_l = \sum_{l=0}^{\infty}\frac{4\pi}{k^2}(2l+1)\sin^2(\delta_l)} \tag{7.13.19}$$

Where δ_l is called the **phase shift**. For scattering at low energies of off of a central potential, we can approximate that:

$$\sigma \approx \sigma_0 = \frac{4\pi}{k^2}\sin^2(\delta_0) \tag{7.13.20}$$

Therefore, the problem of finding the cross section is reduced to that of finding the phase shift. To accomplish this, you must solve the radial Schrodinger equation[7.6]:

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right]u(r) = Eu(r) \tag{7.13.21}$$

Where $u(r) = rR(r)$, where $R(r)$ is the actual radial component of the wave function. Remember that this change of variables also changes our boundary conditions: $R(\infty) = 0$ automatically now (at least for oscillatory solutions to $u(r)$), and $u(0) = 0$ (which is NOT generally true for

[7.6] Radial, since this only applies to central potentials.

$R(r)$). Notice that, for $l = 0$, this equation reduces to a very familiar form:

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + V(r)\right)u(r) = Eu(r) \tag{7.13.22}$$

Normally, we'd seek solutions of the form $u(r) = Ae^{ikr} + Be^{-ikr}$). However, in this case, we want to instead chose the form:

$$\boxed{u(r) = A\sin(kr + \delta_0)} = \tag{7.13.23}$$

The constant δ_0 is the phase shift between the incoming wave and the outgoing wave, and is exactly the zeroth order phase shift we are looking for!. The scattering cross section can now be determined simply by plugging this result in the formula above for σ.

7.13.4 The Born Approximation

The Born Approximation is a rearrangement of the Schrodinger equation into a nicer integral equation. The laborious details are well covered in Griffths QM, pg. 408-412. The resulting equation is:

$$\psi(\boldsymbol{r}) = \phi_0(\boldsymbol{r}) - \frac{m}{2\pi\hbar^2}\int_V \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}_0|}}{|\boldsymbol{r}-\boldsymbol{r}_0|}V(\boldsymbol{r}_0)\psi(\boldsymbol{r}_0)d^3\boldsymbol{r}_0 \tag{7.13.24}$$

This looks like a direct solution for ϕ, but it is not, since the integral on the right hand side ALSO depends on ψ.

Suppose that the potential $V(\boldsymbol{r})$ is localized near $\boldsymbol{r} = 0$. If we additionally now suppose that the incoming wave is not substantially altered by the potential, then we arrive at the 1st Born approximation:

$$\boxed{f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}V(\boldsymbol{r}')d^3\boldsymbol{r}'} \tag{7.13.25}$$

Where $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$. This equation is extremely useful: most problems about scattering in the 1st Born approximation essentially amount to simply solving it. Notice that $\boldsymbol{q}(r,\theta,\phi)$, while the integration is over r', θ', and ϕ'. In other words, **$\boldsymbol{q}$ is a constant with respect to the variables of integration!**.

With the further assumption of a spherically symmetric potential, the 1st Born Approximation then becomes:

$$\boxed{f(\theta) \approx -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin(qr)dr, \qquad (spherical \quad symmetry)} \tag{7.13.26}$$

Where now we can also simplify $q = |\boldsymbol{q}| = 2k\sin\left(\frac{\theta}{2}\right)$[7.7]. Notice that f is now only a function of θ: a spherically symmetric cannot depend on ϕ!.

In the (separate) limit where the energy of the scattered particle is low, the 1st Born

[7.7] This follows from the Law of Cosines.

approximation reduces to:

$$f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int_V V(\boldsymbol{r})d^3\boldsymbol{r}, \qquad (low \quad energy) \tag{7.13.27}$$