

On the Axion Dark Matter Detection with DECIGO/B-DECIGO Laser Interferometry

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ABSTRACT: Dark matter is a sort of matter suggested by the cosmological hypothesis to explain the several puzzles of the galaxies and the whole universe. It is completely invisible, emitting no light or energy and thus cannot be detected by conventional sensors and detectors. Axion dark matter is one of the strong candidates for dark matter, which is introduced as the pseudo Nambu-Goldstone boson from the spontaneous breakdown of Peccei-Quinn symmetry and is produced via the Misalignment mechanism of the initial phase of the complex singlet scalar field. In this paper, we explore how axion dark matter can be detected through gravitational wave laser interferometry. We evaluated the signal-to-noise ratio(S/N) of the DECIGO observatory and gained the sensitivity curve for axion mass and coupling constant to a pair of photons in the lasers, with the assumption that the detector is placed in the galactic halo of the BEC state non-relativistic axion-like particles(ALPs). The results showed that the most sensitive mass varied with arm length, allowing for a wider range of ALP candidates to be searched as a result of longer arms.

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1 General Introduction

Structure of this thesis

This thesis consists of 8 sections. In Sec. 1, we review the basic concepts of elementary particles and dark matter. In Sec. 2, we derive the axial anomaly of Abelian gauge theory by the Fujikawa method. In Sec. 3, we explain the Peccei-Quinn mechanism and Misalignment mechanism of cosmic axions. In Sec. 4, we investigate the Bose-Einstein Condensate (BEC) state of the axion system in a galactic halo. In Sec. 5, we explore how the Maxwell equations of electromagnetism are modified in the presence of axions. In Sec. 6, we focus on the signal detection of the gravitational waves (GW). In Sec. 7, we provide our model-setting of the axion interferometry and analysis methods with the result of the expected sensitivity curve of DECIGO and B-DECIGO. Finally, in Sec. 8, we conclude and discuss our result.

The standard model of particle physics

The smallest particles that make up the Earth, the universe, and all the matter in which we live are called elementary particles[1]. The currently known elementary particles are fermions with half-integer spin and the gauge and Higgs bosons with integer spin. Fermions are elementary particles that make up matter and are divided into quarks, which are affected

by the strong force, and leptons, which are not affected by the strong force. Gauge bosons include the spin-1 gauge boson, which mediates the force between particles, and the spin-0 Higgs boson, which gives particles mass. There are three generations of six particles in each quark and lepton. Four forces are known to act between elementary particles: the strong force, the weak force, the electromagnetic force, and gravity. The Glashow-Weinberg-Salam theory[2][3] is a unified description of the weak force and the electromagnetic force. Quantum chromodynamics (QCD), which explains the strong force, provides a very good description of elementary particle phenomena involving strong interactions, especially in the high-energy region. The Glashow-Weinberg-Salam theory plus QCD is called the Standard Model. The Standard Model is a gauge theory based on $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry and is known as a theory that explains experimental results well and describes physical phenomena extremely well. The Higgs boson, long undiscovered as a Standard Model particle, was discovered in the LHC experiment in 2012[4], confirming all Standard Model particles. Of the four forces found in nature, only the gravitational interaction is outside the framework of the Standard Model and has not yet reached the stage where it can be treated in a unified manner.

Dark matter

Dark matter(DM) is a sort of matter suggested by the cosmological hypothesis to explain the several puzzles of the galaxies and the whole universe. It is entirely invisible, emitting no light or energy and thus cannot be detected by conventional sensors and detectors. It is called "dark" because it rarely interacts with the electromagnetic force, meaning it does not absorb, reflect or emit light. Its presence is implied by its gravitational attraction rather than its luminosity. DM is essential for understanding the universe's size, shape and future, as well as for explaining the gravitational effects of galaxies and galaxy clusters. It also plays a role in forming galaxies and is necessary for life to exist. Finally, its gravitational influence on the motions of stars and galaxies helps us understand how galaxies are organized on grand scales. Observations of the orbital velocities of stars moving in galaxies have revealed the existence of unknown sources of mass that do not emit light, in addition to ordinary matter that emits light[5][6]. In the "Bullet Cluster," where galaxy clusters collide with each other at 4500 km/s, a comparison of the mass distribution estimated from observations of X-rays emitted by collisions between interstellar gas and the actual mass distribution known from observations of the gravitational lensing effect, an effect due to Einstein's theory of general relativity, shows that the mass of the interstellar gas is about 1/6 of the entire cluster and that most of the gas is stars and DM, which do not emit X-rays [7][8]. In particular, recent observations of the cosmic microwave background radiation by the Planck satellite have precisely measured the fraction of DM in the cosmic constituents to be 25%, more than five times the amount of already known matter [9]. Although the existence of DM is certain, its true nature and properties are still unknown.

Nature of DM

Collision-less DM distribution functions

For a halo composed of collision-less DM with mass m and the energy $E = \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{r})$, the density profile $\rho(r)$ is derived from the distribution function

$$f(\mathbf{r}, \mathbf{v}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left[-\frac{E}{m\sigma^2}\right], \quad (1.1)$$

where ρ_1 and σ are the arbitrary parameters to be decided by hand. Because of the spherically symmetric distribution, the density profile is given by

$$\begin{aligned} \rho(r) &= \int d^3\mathbf{v} f(r, \mathbf{v}) \\ &= \rho_1 \exp\left[-\frac{\Phi(r)}{\sigma^2}\right]. \end{aligned} \quad (1.2)$$

As a result, we yield the potential function in terms of the radial density

$$\Phi(r) = -\sigma^2 \ln\left(\frac{\rho(r)}{\rho_1}\right). \quad (1.3)$$

From the Poisson equation $\nabla^2\Phi(r) = 4\pi G_N\rho(r)$, the density satisfies the following relation

$$\frac{d^2\rho(r)}{dr^2} - \frac{1}{\rho(r)} \left(\frac{d\rho(r)}{dr}\right)^2 + \frac{2}{r} \left(\frac{d\rho(r)}{dr}\right) = -\frac{4\pi G_N}{\sigma^2} \rho^2(r). \quad (1.4)$$

Assuming that the density follows the power-law $\rho(r) = Cr^b$, we have

$$\begin{aligned} Cbr^{b-2} - \frac{4\pi G_N}{\sigma^2} C^2 r^{2b} &= 0 \\ \Rightarrow \rho(r) &= \frac{\sigma^2}{2\pi G_N} \frac{1}{r^2}. \end{aligned} \quad (1.5)$$

Then from Eq (1.3), the derivative of the potential is given by

$$-\frac{d}{dr}\Phi(r) = 2\sigma^2 \frac{1}{r} \quad (1.6)$$

If the DM particle shows circular motion in this spherical system with velocity $v_c = v_c(r)$, we find the physical meaning of σ

$$\begin{aligned} m \frac{v_c^2}{r} &= -m \frac{d\Phi(r)}{dr} = 2\sigma^2 \frac{m}{r} \\ \Rightarrow v_c^2 &= 2\sigma^2. \end{aligned} \quad (1.7)$$

From this relation, σ is physically interpreted as the typical circular velocity. The circular velocity of the sun is about $v_c \simeq 220\text{km/s}$. By using this information, the local dark matter density in the solar neighbourhood is

$$\begin{aligned} \rho_0 \equiv \rho(r \sim 8\text{kpc}) &= 0.945 \times 10^{-30} \text{g cm}^{-3} \\ &= 5.3 \times 10^{-7} \text{GeV cm}^{-3}. \end{aligned} \quad (1.8)$$

Comparing with the observation of $\rho_0 \simeq 0.3 \text{GeV cm}^{-3}$ [10][11][12], we see the great difference. This is because the DM is actually colliding and experiencing attenuations. Besides, the DM halo profile given by Eq. 1.5 is also inconsistent with the observed density profile of dwarf galaxies [13]. This discrepancy is the so-called *cusp-core problem*. One of the solutions to this issue is the idea of self-interacting dark matter (SIDM) model. SIDM model states that DM particles form the central core profile as soon as the interaction rate R_{SIDM} is larger than the inverse of the age of the galaxy: $R_{\text{SIDM}} > t_{\text{gal}}$. This means that for a repulsive force, the dark matter scattering actively occurs in the central region and prevents the cuspy shape of the DM halo profile. Let σ_{SIDM} be the self-interaction cross-section, ρ be the DM mass density and v be the typical DM relative velocity. Then as shown in 1, the number of interactions within time interval T leads to the form of R_{SIDM}

$$\begin{aligned} N &= R_{\text{SIDM}} T = \frac{\sigma_{\text{SIDM}} \times (vT) \times \rho}{m_{\text{DM}}} \\ \Rightarrow R_{\text{SIDM}} &= \frac{\sigma_{\text{SIDM}} \rho v}{m_{\text{DM}}}. \end{aligned} \quad (1.9)$$

Consider the Draco dwarf galaxy, whose age is about $t_{\text{gal}} \sim 10 \text{Gyr}$ and velocity dispersion is measured approximately $\sigma_v \simeq 20 \text{km/s}$. The maximum core radius of Draco is estimated by

$$\begin{aligned} r_{\text{max}} &< \sqrt{\frac{\sigma_{\text{SIDM}}}{m_{\text{DM}}}} \times \sqrt{10 \text{Gyrs}} \frac{(2.0 \times 10^6)^{3/2}}{2\sqrt{6.67\pi} \times 10^{-4}} \text{g}^{1/2} \text{s}^{-1/2} \\ &= 0.24 \times 10^{22} \sqrt{\frac{\sigma_{\text{SIDM}} [\text{cm}^2]}{m_{\text{DM}} [\text{g}]} \text{g}^{1/2}}. \end{aligned} \quad (1.10)$$

Since the size of the Draco core is 1kpc, we find $\sqrt{\frac{\sigma_{\text{SIDM}}}{m_{\text{DM}}}} \simeq 1.7 \text{ cm}^2 \text{g}^{-1}$. This result is almost compatible with the bound from the Bullet Cluster observation $\sqrt{\frac{\sigma_{\text{SIDM}}}{m_{\text{DM}}}} \sim 1.5 \text{ cm}^2 \text{g}^{-1}$ [14].

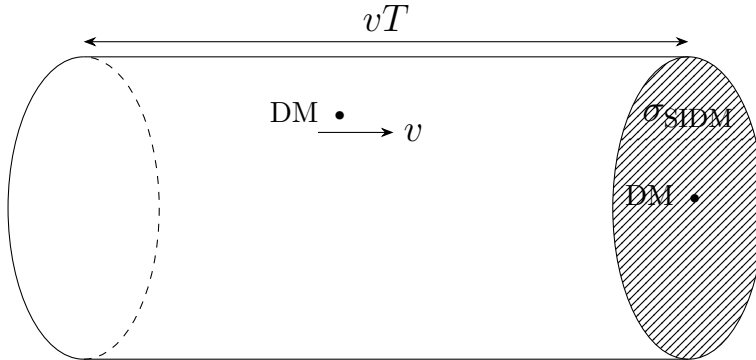


Figure 1. The relation between DM self-interaction cross-section and the reaction rate

2 Axial Anomaly

$U(1)_A$ Anomaly in the massless fermion field theory

We followed the discussion in the Fujikawa [15]. The generating functional of massless fermions coupled to $U(1)$ gauge field is given by

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(i \int d^4x \mathcal{L} + \bar{\eta}\psi + \bar{\psi}\eta\right), \quad (2.1)$$

where

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (D_\mu = \partial_\mu - ig A_\mu). \quad (2.2)$$

Spinors ψ and $\bar{\psi}$ can be expanded in terms of complete orthonormal basis functions $\{\phi_n(x)\}$ as

$$\psi(x) = \sum_n a_n \phi_n(x), \quad \bar{\psi}(x) = \sum_n \phi_n^*(x) \bar{a}_n(x), \quad (2.3)$$

where every element of $\{\phi_n(x)\}$ satisfies orthonormal conditions

$$\langle \phi_n, \phi_m \rangle = \int d^4x \phi_n^*(x) \phi_m(x) = \delta_{n,m}. \quad (2.4)$$

The path integral measures $\mathcal{D}\psi \mathcal{D}\bar{\psi}$ are related to these expansion coefficients by definition:

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = \prod_n da_n \prod_m d\bar{a}_m. \quad (2.5)$$

The classical Lagrangian $\mathcal{L}$ itself is invariant under the following global axial transformation

$$\psi \rightarrow \psi' = e^{i\alpha\gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} e^{i\alpha\gamma_5}. \quad (2.6)$$

By using the transformed spinors of the form

$$\psi(x) = \sum_n b_n \phi_n(x), \quad \bar{\psi}(x) = \sum_n \phi_n^*(x) \bar{b}_n(x), \quad (2.7)$$

we obtain

$$b_n = \int d^4x \phi_n^*(x) \psi'(x) = \sum_m \left(\int d^4x \phi_n^*(x) e^{i\alpha\gamma_5} \phi_m(x) \right) a_m =: \sum_m C_{nm} a_m, \quad (2.8)$$

$$\bar{b}_n = \int d^4x \bar{\psi}'(x) \phi_n(x) = \sum_m \left(\int d^4x \phi_n^*(x) e^{i\alpha\gamma_5} \phi_m(x) \right) \bar{a}_m = \sum_m C_{nm} \bar{a}_m. \quad (2.9)$$

The change of the path integral measure under the axial transformation is given by

$$\begin{aligned} \mathcal{D}\psi' \mathcal{D}\bar{\psi}' &= \prod_n db_n \prod_m d\bar{b}_m \\ &= J \prod_n da_n \prod_m d\bar{a}_m \quad (J \text{ is a Jacobian}) \\ &= (\det C)^2 \mathcal{D}\psi \mathcal{D}\bar{\psi}. \end{aligned} \quad (2.10)$$

For an infinitesimal axial transformation ($\alpha \ll 1$), the Jacobian J becomes

$$\begin{aligned}
J &= (\det C)^2 \\
&= \left[\det \left(\int d^4x \phi_n^*(x) e^{i\alpha \gamma_5} \phi_m(x) \right) \right]^2 \\
&\simeq \left[\det \left(\delta_{n,m} + i\alpha \int d^4x \phi_n^*(x) \gamma_5 \phi_m(x) \right) \right]^2 \\
&= \left(\det \left[\exp \left(i\alpha \int d^4x \phi_n^*(x) \gamma_5 \phi_m(x) \right) \right] \right)^2 \\
&= \exp \left(2 \operatorname{tr} \left[i\alpha \int d^4x \phi_n^*(x) \gamma_5 \phi_m(x) \right] \right) \\
&= \exp (2i\alpha \operatorname{tr} D) \\
&\simeq 1 + 2i\alpha \operatorname{tr} D,
\end{aligned} \tag{2.11}$$

where

$$\operatorname{tr} D = \operatorname{tr} \left[\int d^4x \phi_n^*(x) \gamma_5 \phi_m(x) \right] = \sum_n \int d^4x \phi_n^*(x) \gamma_5 \phi_n(x). \tag{2.12}$$

Therefore, because of the Jacobian of the path integral measure coming from axial transformation, the Lagrangian of this quantum theory acquires an effective term given by

$$\delta \mathcal{L}_a = 2\alpha \sum_n \phi_n^*(x) \gamma_5 \phi_m(x). \tag{2.13}$$

In the plane wave state $\phi_n(x) = u_n(\mathbf{p}, s) e^{-ipx}$

$$\operatorname{tr} D = \sum_n \int d^4x e^{ipx}(x) u_n^\dagger(\mathbf{p}, s) \gamma_5 u_n(\mathbf{p}, s) e^{-ipx} = \underbrace{\delta^4(0)}_{\rightarrow \infty} \sum_n \underbrace{u_n^\dagger(\mathbf{p}, s) \gamma_5 u_n(\mathbf{p}, s)}_{\rightarrow 0}, \tag{2.14}$$

thereby $\operatorname{tr} D$ must be regularized to derive physical meaning.

In the work of Fujikawa (1979) [15], $\operatorname{tr} D$ is regularised by utilising Gaussian cutoff as

$$\operatorname{tr} D = \lim_{M \rightarrow \infty} \sum_n \int d^4x \phi_n^*(x) \gamma_5 \exp \left(-\frac{\lambda_n^2}{M^2} \right) \phi_n(x), \tag{2.15}$$

where λ_n is an eigenvalue of the Dirac operator $i\not{D}$, and M is a regulator mass.

Let $\operatorname{tr} D_M$ be

$$\operatorname{tr} D_M := \sum_n \int d^4x \left[\phi_n^*(x) \gamma_5 \exp \left(-\frac{\lambda_n^2}{M^2} \right) \phi_n(x) \right]. \tag{2.16}$$

Then, for special case when gauge coupling $g = 0$, we gain an eigenvalue $\not{k}$ by an eigenfuction $\chi(x) = e^{-ikx}$

$$\begin{aligned}
i\not{D}\chi(x) &= i\gamma^\mu \partial_\mu e^{-ikx} \\
&= \gamma^\mu k_\mu e^{-ikx} \\
&= \not{k}\chi(x).
\end{aligned} \tag{2.17}$$

Therefore $\text{tr } D_M$ is convergent:

$$\begin{aligned}
\text{tr } D_M &= \sum_n \int d^4x \left[\phi_n^*(x) \gamma_5 \exp\left(-\frac{k^2}{M^2}\right) \phi_n(x) \right] \\
&= \lim_{y \rightarrow x} \int d^4x \text{tr} \left[\gamma_5 \exp\left(-\frac{k^2}{M^2}\right) \right] \delta(x-y) \\
&= \lim_{y \rightarrow x} \int d^4x \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[\gamma_5 \exp\left(-\frac{k^2}{M^2}\right) \right] e^{ik(x-y)} \\
&= \lim_{y \rightarrow x} \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \text{tr} [\gamma_5] e^{iMk'(x-y)} \quad \left(k'^\mu = \frac{k^\mu}{M}\right) \\
&= \int d^4x M^4 \text{tr} [\gamma_5] \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \\
&= 0.
\end{aligned} \tag{2.18}$$

For the general case ($g \neq 0$: Non-zero gauge coupling), $\text{tr } D$ is expressed by using eigenfunction of $i\cancel{D}$ as $\phi_n(x)$

$$\text{tr } D = \lim_{M \rightarrow \infty} \sum_n \int d^4x \phi_n^*(x) \gamma_5 \exp\left(\frac{\cancel{D}^2}{M^2}\right) \phi_n(x). \tag{2.19}$$

Since matrix trace is unitary invariant, i.e. basis-change invariant, we calculate $\text{tr } D$ with plane-wave state $\phi_n(x) = e^{-ikx}$.

$$\begin{aligned}
\cancel{D}^2 &= \gamma^\mu \gamma^\nu D_\mu D_\nu \\
&= \gamma^\mu \gamma^\nu \left(\frac{1}{2} \{D_\mu, D_\nu\} + \frac{1}{2} [D_\mu, D_\nu] \right) \\
&= \frac{1}{4} (\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]) \{D_\mu, D_\nu\} + \frac{1}{2} \gamma^\mu \gamma^\nu (-igF_{\mu\nu}) \\
&= \frac{1}{2} g^{\mu\nu} \{D_\mu, D_\nu\} + \frac{1}{4} [\gamma^\mu, \gamma^\nu] (-igF_{\mu\nu}) \\
&= D^2 - \frac{g}{2} \sigma^{\mu\nu} F_{\mu\nu},
\end{aligned} \tag{2.20}$$

where $F_{\mu\nu}$ is the field strength, and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ is the generator of Lorentz algebra. Furthermore, we know that

$$\begin{aligned}
D^2 &= (\partial_\mu - igA_\mu) (\partial^\mu - igA^\mu) \\
&= \partial^2 - 2igA^\mu \partial_\mu - g^2 A^\mu A_\mu - ig(\partial \cdot A),
\end{aligned} \tag{2.21}$$

so we see that

$$\begin{aligned}
D^2 \phi_n(x) &= D^2 e^{-ikx} \\
&= [-k^2 - 2gA^\mu k_\mu - g^2 A^\mu A_\mu - ig(\partial \cdot A)] e^{-ikx} \\
&= [-(k + gA)^2 - ig(\partial \cdot A)] e^{-ikx}.
\end{aligned} \tag{2.22}$$

Thus

$$\exp\left(\frac{D^2}{M^2}\right)\phi_n(x) = \exp\left(-\frac{(k+gA)^2}{M^2} - \frac{ig(\partial \cdot A)}{M^2}\right)e^{-ikx}. \quad (2.23)$$

As a result,

$$\begin{aligned} \text{tr } D &= \lim_{M \rightarrow \infty} \sum_n \int d^4x \phi_n^*(x) \gamma_5 \exp\left(\frac{\not{D}}{M^2}\right) \phi_n(x) \\ &= \lim_{M \rightarrow \infty} \lim_{y \rightarrow x} \int d^4x \delta(x-y) \text{tr} \left[\gamma_5 \exp\left(-\frac{(k+gA)^2}{M^2} - \frac{ig(\partial \cdot A)}{M^2} - \frac{g}{2M^2} \sigma^{\mu\nu} F_{\mu\nu}\right) \right] \\ &= \lim_{M \rightarrow \infty} \lim_{y \rightarrow x} \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \text{tr} \left[\gamma_5 \exp\left(-\frac{(k+gA)^2}{M^2} - \frac{ig(\partial \cdot A)}{M^2} - \frac{g}{2M^2} \sigma^{\mu\nu} F_{\mu\nu}\right) \right] \\ &= \lim_{M \rightarrow \infty} \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \text{tr} \left[\gamma_5 \exp\left(-\frac{ig(\partial \cdot A)}{M^2} - \frac{g}{2M^2} \sigma^{\mu\nu} F_{\mu\nu}\right) \right] \left(k'^\mu = \frac{k^\mu + gA^\mu}{M} \right). \end{aligned} \quad (2.24)$$

Due to $\text{tr } \gamma_5 = 0$, we get

$$\begin{aligned} \text{tr } D &= \lim_{M \rightarrow \infty} \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \text{tr} \left[\gamma_5 \exp\left(-\frac{g}{2M^2} \sigma^{\mu\nu} F_{\mu\nu}\right) \right] \\ &= \lim_{M \rightarrow \infty} \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \text{tr} \left[\gamma_5 \exp\left(-\frac{ig}{2M^2} \gamma^\mu \gamma^\nu F_{\mu\nu}\right) \right] \\ &= \lim_{M \rightarrow \infty} \int d^4x M^4 \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \text{tr} \left[\gamma_5 \left(1 - \frac{ig}{2M^2} \gamma^\mu \gamma^\nu F_{\mu\nu} - \frac{g^2}{8M^4} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta F_{\mu\nu} F_{\alpha\beta} + \mathcal{O}(M^{-6}) \right) \right] \\ &= -\frac{g^2}{8} \int d^4x \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} \text{tr} \left[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \right] F_{\mu\nu} F_{\alpha\beta} \quad (\because \text{tr } \gamma_5 = 0, \text{tr } [\gamma_5 \gamma^\mu \gamma^\nu] = 0) \\ &= -\frac{g^2}{8} \int d^4x \left(\frac{i}{16\pi^2} \right) (4i\epsilon^{\mu\nu\alpha\beta}) F_{\mu\nu} F_{\alpha\beta} \quad \left(\because \int \frac{d^4k'}{(2\pi)^4} e^{-k'^2} = \frac{i}{16\pi^2}, \text{tr} [\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta] = 4i\epsilon^{\mu\nu\alpha\beta} \right) \\ &= \frac{g^2}{32\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \end{aligned} \quad (2.25)$$

By defining dual strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, we finally yield

$$\text{tr } D = \frac{g^2}{16\pi^2} \int d^4x F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (2.26)$$

The effective Lagrangian term $\delta\mathcal{L}_a$ originating from the infinitesimal axial transformation satisfies

$$\begin{aligned} \int d^4x \delta\mathcal{L}_a &= 2\alpha \text{tr } D \\ &\simeq \alpha \frac{g^2}{32\pi^2} \int d^4x F_{\mu\nu} \tilde{F}^{\mu\nu}, \end{aligned} \quad (2.27)$$

where α is an infinitesimal transformation parameter. Since the divergence of the axial vector current is gained from

$$\delta\mathcal{L}_a = \alpha (\partial_\mu J_A^\mu) = \alpha \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (2.28)$$

we consequently yield the axial anomaly equation

$$\partial_\mu J_A^\mu = \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (2.29)$$

Vafa and Witten showed that QCD vacuum energy

$$e^{-VE(\theta)} = \int \mathcal{D}A_\mu^a \det(\not{D} + M) e^{-\frac{1}{4g^2} \int d^4x \operatorname{tr} F_{\mu\nu} F_{\mu\nu}} e^{\frac{i\theta}{32\pi^2} \int d^4x \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}} \quad (2.30)$$

is minimised at $\theta = 0$ [16][17]. The general proof of this relation is given in Appendix. A.

3 Axions and Cosmology

Axions: Genesis

We followed the discussion in Kolb & Turner [18]. Axions in the cosmic thermal history begin with the spontaneous symmetry breaking (SSB) of Peccei-Quinn (PQ) symmetry, which occurs at the temperature $T \sim f_{PQ}$. Most axion models cause SSB when a complex scalar field Φ , which carries PQ charge, comes to have a non-zero vacuum expectation value (VEV) $\langle \Phi \rangle \neq 0$. When $T < f_{PQ}$, the potential $V(\Phi)$ is given by

$$V(\Phi) = \lambda \left(|\Phi|^2 - \frac{f_{PQ}^2}{2} \right)^2, \quad (3.1)$$

and $V(\Phi)$ reaches its minimum if $\langle \Phi \rangle = \frac{f_{PQ}}{\sqrt{2}} \neq 0$. Once the $U(1)$ global PQ symmetry is broken, $\arg \langle \Phi \rangle$ is no longer uniquely determined, that is $V(\Phi)$ is independent of $\bar{\Theta} := \arg \langle \Phi \rangle$. This degree of freedom $\bar{\Theta}$ from SSB of PQ symmetry turns out to be a massless scalar, which is a Nambu-Goldstone boson [1]. The scalar $a = \left(\frac{f_{PQ}}{N} \right) \bar{\Theta}$ is called an **axion**, where N is a domain-wall number. The QCD phase transition temperature is around 150 MeV [19]. In the high temperature situation satisfying $T \gg \Lambda_{QCD} = \mathcal{O}(1\text{MeV})$, axions are still massless. However in the low-temperature region, axions acquire mass because of QCD instanton effects. QCD instantons are essentially large objects, with a size of $\mathcal{O}(\Lambda_{QCD}^{-1})$. Therefore, the effect of QCD instantons is suppressed at high temperatures, which explains the mass gain mechanism. According to Gross, Pisarski, Yaffe (1981)[20], the density coefficient of finite-temperature instanton with winding number ν satisfies

$$C_\nu(\Lambda') = \left(\frac{\Lambda}{\Lambda'} \right)^{\frac{11\nu}{3}} C_\nu(\Lambda), \quad (3.2)$$

where Λ is the Pauli-Villars renormalization scale, and Λ' is another renormalization scale. Consequently, the temperature dependence of the axion mass is given by

$$m_a(T) \simeq 0.1 m_a(T=0) \left(\frac{\Lambda_{QCD}}{T} \right)^{3.7} \quad (\text{when } T \gg \Lambda_{QCD}). \quad (3.3)$$

When we allow rough discussion, axions are massless when $T > f_{PQ}$ unless T is not comparable to Λ_{QCD} .

In the derivation of the mass of the axion

$$a = \left(\frac{f_{PQ}}{N} \right) \bar{\Theta}, \quad (3.4)$$

there is a helpful method for the model with $N = 1$. The instantons are considered to have the effect of tilting the potential $V(a)$ by a slight angle of order $\mathcal{O}\left(\frac{m_a^2}{f_{PQ}^2}\right)$. The resulting

tilted potential is periodic in $\bar{\Theta} = \frac{a}{f_{PQ}}$ and takes a minimum in the CP-symmetric $\bar{\Theta} = 0$ state

$$V(a) = m_a^2 f_{PQ}^2 \left[1 - \cos\left(\frac{a}{f_{PQ}}\right) \right] \simeq \frac{1}{2} m_a^2 a^2 + \dots, \quad (3.5)$$

which gives the mass to the axion.

On the contrary, in the model with $N > 1$, the axion mass cannot be explained solely by the tilt of the wine-bottle potential. Because of the remaining discrete $\mathbb{Z}_N$ symmetry, there are N degenerate field values

$$\bar{\Theta} = 2\pi n \quad (n = 0, 1, \dots, N-1) \quad (3.6)$$

that take the minimum of CP-symmetric potentials:

$$V(a) = m_a^2 \left(\frac{f_{PQ}}{N} \right)^2 \left[1 - \cos\left(\frac{a}{f_{PQ}/N}\right) \right]. \quad (3.7)$$

Such an isolated vacuum state is degenerate and cannot be continuously shifted, which may result in an axion domain wall separating adjacent minuscule regions. In the cosmological context, domain walls are one of the topological defects that form in the early universe and are two-dimensional objects with areal extents. Domain walls are formed between different vacua when discrete symmetry is broken, and the vacuum manifold has disconnected components. According to cosmological observations, domain walls do not exist and must be circumvented by some means. There are two ways to achieve this; one is to set $N = 1$. The other is to consider that the Universe was inflated after or during the SSB of the PQ symmetry and that $\bar{\Theta}$ became constant in the observed range.

Relic axions from non-thermal process

Relic axions from the early universe arise from thermal and non-thermal processes. For the low-mass axion particles, the primary production process is highly non-thermal and is related to the relaxation of $\bar{\Theta}$ angle. Experimentally, current $\bar{\Theta}$ is measured to be the CP-conserving angle: $\bar{\Theta} = 0$ [21]. Despite this, axion dynamics does not specify the value of $\bar{\Theta}$ in principle because the axion mass from the instanton effect largely depends upon the temperature rather than $\bar{\Theta}$ and axion mass even becomes zero in sufficiently high temperature. Hence it is natural to deduce that the initial value $\bar{\Theta}_1$ is given randomly, so there is no reason to expect $\bar{\Theta}_1 = 0$. In the following discussion, we assume that the axion field is initially misaligned to the potential minimum $\bar{\Theta}_1 = 0$. When $T \sim \Lambda_{QCD}$, axion acquires mass. After mass of the axion is comparable to Hubble's parameter, the axion starts to roll down toward $\bar{\Theta} = 0$ and passes through it. This means that axion fields continue to oscillate around the potential minimum. This field oscillation corresponds to the condensate state of the axion's zero-momentum state, and it does not decay like the inflaton field in the slow roll. Time

evolution of $\bar{\Theta}$ angle is derived from the behaviour of a scalar field theory displaced from the minimum of the potential $V(\bar{\Theta})$. The potential is expanded around $\bar{\Theta} = 0$ as

$$\begin{aligned} V(\bar{\Theta}) &= m_a^2 \left(\frac{f_{PQ}}{N} \right)^2 \left[1 - \cos \left(\frac{a}{f_{PQ}/N} \right) \right] \\ &\simeq m_a^2 \left(\frac{f_{PQ}}{N} \right)^2 \frac{\bar{\Theta}^2}{2} \end{aligned} \quad (3.8)$$

When we assume $\bar{\Theta}$ takes a spatially uniform value, the Lagrangian of $\bar{\Theta}$ dynamics becomes

$$\mathcal{L} = \left(\frac{f_{PQ}}{N} \right)^2 \left[\frac{1}{2} \dot{\bar{\Theta}}^2 - \frac{1}{2} m_a^2 \bar{\Theta}^2 \right] \quad (3.9)$$

The action is given by

$$S = \int dt \int d^3x a^3(t) \mathcal{L}, \quad (3.10)$$

where $a(t)$ is the cosmic scale factor and Hubble's parameter is defined through $a(t)$ as

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \quad (3.11)$$

The equation of motion is gained from the variation of this action S

$$\begin{aligned} \delta S = 0 &\Rightarrow -a^3(t) m_a^2(T) \bar{\Theta}(t) - \frac{\partial}{\partial t} \left[a^3(t) \dot{\bar{\Theta}}(t) \right] = 0 \\ &\Rightarrow a^3(t) \ddot{\bar{\Theta}}(t) + 3a^2(t) \dot{a}(t) \dot{\bar{\Theta}}(t) + a^3(t) m_a^2(T) \bar{\Theta}(t) = 0 \\ &\Rightarrow \ddot{\bar{\Theta}}(t) + 3H(t) \dot{\bar{\Theta}}(t) + m_a^2(T) \bar{\Theta}(t) = 0. \end{aligned} \quad (3.12)$$

When $T \gg \Lambda_{QCD}$, $m_a \simeq 0$. Assuming $\dot{\bar{\Theta}}_1 \ll H$ for the initial value, we find

$$\bar{\Theta} = \bar{\Theta}_1 = \text{const.} \quad (3.13)$$

at high temperatures. $\bar{\Theta}$ starts to oscillate with frequency $m_a(T)$ if $m_a(T) \gtrsim 3H(t)$. After coherent oscillation of $\bar{\Theta}$ began, we can perform following replacement

$$\dot{\bar{\Theta}}^2 \rightarrow \langle \dot{\bar{\Theta}}^2 \rangle = \rho_a := \frac{1}{2} \left(\frac{f_{PQ}}{N} \right)^2 \left[\dot{\bar{\Theta}}^2 + m_a^2(T) \bar{\Theta}^2 \right], \quad (3.14)$$

where ρ_a is the energy density of the axion field. Multiplying $\dot{\bar{\Theta}}^2$ on the both sides of (3.12), we have

$$\begin{aligned} \dot{\rho}_a(t) &\simeq \left[\frac{\dot{m}_a(T)}{m_a(T)} - 3H(t) \right] \rho_a(t) \\ \Rightarrow \rho_a(t) &\propto \frac{m_a(T)}{a^3(t)}. \end{aligned} \quad (3.15)$$

This confirms that the coherent oscillation of the axion field behaves as a non-relativistic matter once axion mass settles to the zero-temperature value when $T \ll \Lambda_{QCD}$. The fact that the axion number density decreases as

$$n_a(t) = \frac{\rho_a}{m_a} \propto a^{-3}(t) \quad (3.16)$$

means the number of axions in the unit comoving volume is conserved while the axion mass changes with time. The axion field oscillation becomes at the temperature $T = T_1$, which fulfills $m_a(T_1) = 3H(T_1)$. The initial axion energy density is

$$\rho_a(T_1) \simeq \frac{1}{2} \left(\frac{f_{PQ}}{N} \right)^2 m_a^2(T_1) \bar{\Theta}_1^2. \quad (3.17)$$

And the initial axion number density is given by

$$n_a(T_1) \simeq \frac{1}{2} \left(\frac{f_{PQ}}{N} \right)^2 m_a(T_1) \bar{\Theta}_1^2. \quad (3.18)$$

We showed that the cosmological abundance of axions is proportional to the square of the initial misalignment angle $\bar{\Theta}_1$. From our assumption, $\bar{\Theta}_1$ takes random values in the range $[-\pi, \pi]$ in different patches of the observable universe. The average misalignment angle is calculated by

$$\langle \bar{\Theta}_1^2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\bar{\Theta}_1 (\bar{\Theta}_1)^2 = \frac{1}{2\pi} \frac{2\pi^3}{3} = \frac{\pi^2}{3} \quad (3.19)$$

$$\Rightarrow |\langle \bar{\Theta}_1 \rangle| = \frac{\pi}{\sqrt{3}} \quad (3.20)$$

Thus, the average axion abundance from the misalignment mechanism can be estimated by replacing $\bar{\Theta}_1 \rightarrow \frac{\pi}{\sqrt{3}}$.

4 Condensed State of the Galactic Axions

In this section, we investigate the Bose-Einstein Condensate (BEC) state of the axion system in a galactic halo. At first, we review the BEC of the free boson system for simplicity and confirm that sufficiently non-relativistic axions are in the Bose-Einstein Condensation (BEC) state. Secondly, we look closely at the condensed form of the axion system trapped in the galaxy's gravitational potential using the Gross-Pitaevski (GP) equation.

BEC of free boson system

The grand canonical ensemble describes thermal and chemical equilibrium systems with heat and particle reservoirs. The possibility P_i that N_i bosons are found in a state with energy ε_i is given by

$$P_i(N_i) = \frac{1}{\Xi} e^{\beta N_i(\mu - \varepsilon_i)}, \quad (4.1)$$

where μ , β are the reservoir's chemical potential and inverse temperature (i.e. $\frac{1}{k_B T}$), respectively. The grand partition function Ξ is defined as

$$\Xi = \sum_i e^{\beta N_i(\mu - \varepsilon_i)} \quad (4.2)$$

An occupation number of state i with an energy ε_i is calculated by

$$\begin{aligned} \langle N_i \rangle &= \sum_{N_i=0}^{\infty} N_i P_i(N_i) \\ &= \frac{1}{\Xi} \sum_{N_i=0}^{\infty} N_i e^{\beta N_i(\mu - \varepsilon_i)} \end{aligned} \quad (4.3)$$

We are interested in the convergent value of this infinite series. Hence, from d'Alembert's method,

$$\begin{aligned} \lim_{N_i \rightarrow \infty} \left| \frac{(N_i + 1) e^{\beta(N_i+1)(\mu - \varepsilon_i)}}{N_i e^{\beta N_i(\mu - \varepsilon_i)}} \right| &= \lim_{N_i \rightarrow \infty} \left| \left(1 + \frac{1}{N_i} \right) e^{\beta(\mu - \varepsilon_i)} \right| \\ &= e^{\beta(\mu - \varepsilon_i)}, \end{aligned} \quad (4.4)$$

and we conclude that $\mu < \varepsilon_i$ is required for absolute convergence. From this condition, we get the Bose-Einstein distribution

$$f_B(\varepsilon_i; \beta, \mu) = \langle N_i \rangle = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \quad (4.5)$$

Summing up this distribution in terms of state indices i , we gain the total number of bosons

$$\begin{aligned}
N &= \sum_i \langle N_i \rangle \\
&= \frac{1}{e^{-\beta\mu} - 1} + \sum_{i \neq 0} \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \\
&=: N_0 + N_{\text{excited}},
\end{aligned} \tag{4.6}$$

where N_0 and N_{excited} stand for the occupation number of the ground state and excited states, respectively, and without loss of generality, we can fix the ground state energy ε_0 to be zero. Taking continuum limit $\left(\varepsilon_i \rightarrow \varepsilon := \frac{\mathbf{k}^2}{2m}\right)$, N_{excited} reads

$$\begin{aligned}
N_{\text{excited}} &= V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} \\
&= V \frac{m^{3/2}}{\sqrt{2\pi^2}} \int_0^\infty d\varepsilon \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon - \mu)} - 1} \\
&= V \left(\frac{2\pi}{mk_B T} \right)^{-3/2} \frac{2}{\sqrt{\pi}} \int_0^\infty dx \frac{x^{1/2}}{z^{-1}e^x - 1} \quad (x = \beta\varepsilon, \quad z = e^{\beta\mu}) \\
&= \frac{V}{\lambda_{\text{th}}^3} g_{3/2}(z) \quad \left(\lambda_{\text{th}} := \sqrt{\frac{2\pi}{mk_B T}}, \quad g_{3/2}(z) : \text{Polylogarithm function} \right)
\end{aligned} \tag{4.7}$$

Because of the restriction on z ($0 < z < 1$), we find an upper bound of the number density of excited particles:

$$n_{\text{excited}} = \frac{N_{\text{excited}}}{V} < \frac{1}{\lambda_{\text{th}}^3} g_{3/2}(1) = \frac{1}{\lambda_{\text{th}}^3} \zeta(3/2), \tag{4.8}$$

For a fixed temperature T , there is a critical number density of bosons

$$n_C = \frac{1}{\lambda_{\text{th}}^3} \zeta(3/2), \tag{4.9}$$

and for a fixed number density n , there is a critical temperature

$$T_C = \frac{2\pi}{mk_B} \left(\frac{n}{\zeta(3/2)} \right)^{2/3}. \tag{4.10}$$

Above these critical values, all excited states are occupied, and all excess bosons must go to the ground state. This is called Bose-Einstein condensate. Under sufficient conditions, axions can also turn into BEC as it is a Nambu-Goldstone boson from SSB of $U(1)$ PQ symmetry.

Mean field approximation in the BEC system

Here, we review the mean field approximation in the BEC system. In particular, we derive Gross-Pitaevskii (GP) equation. The GP equation is the equation of motion for the BEC

order parameter. Let us consider spin 0 boson system with mass m . We start with the N body time-dependent Schrödinger equation:

$$\hat{H}\psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t), \quad (4.11)$$

where $\psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ is N body wave function, and $\hat{H}$ is the Hamiltonian. Here, the coordinate representation of $\hat{H}$ is given by

$$\hat{H} = \sum_{i=1}^N \left[-\frac{\hbar^2}{2m} \nabla_i^2 + V(\mathbf{r}_i, t) \right] + \frac{1}{2} \sum_{i \neq j}^N U_I(\mathbf{r}_i - \mathbf{r}_j), \quad (4.12)$$

where $V(\mathbf{r}_i, t)$ is the external field potential, and $U_I(\mathbf{r}_i - \mathbf{r}_j)$ is the two body self-interaction term. In the GP theory, it is assumed that whole particles occupy some single-particle state

$$\psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = \prod_{i=1}^N \phi(\mathbf{r}_i, t), \quad (4.13)$$

where the single-particle state wave function $\phi(\mathbf{r}_i, t)$ is normalised

$$\int d^3\mathbf{r} |\phi(\mathbf{r}, t)|^2 = 1. \quad (4.14)$$

Substituting (4.13) into (4.11), we get the energy eigenvalue

$$\begin{aligned} E(t) &= \int d^3\mathbf{r}_1 \cdots \int d^3\mathbf{r}_N \psi^*(\mathbf{r}_1, \dots, \mathbf{r}_N, t) \hat{H} \psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t) \\ &\simeq -N \int d^3\mathbf{r} \frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r}, t)|^2 \\ &\quad + N \int d^3\mathbf{r} V(\mathbf{r}, t) |\phi(\mathbf{r}, t)|^2 + \frac{N^2}{2} \int d^3\mathbf{r} \int d^3\mathbf{r}' U_I(\mathbf{r} - \mathbf{r}') |\phi(\mathbf{r}, t)|^2 |\phi(\mathbf{r}', t)|^2. \end{aligned} \quad (4.15)$$

Hence the condensate wave function

$$\Psi(\mathbf{r}, t) = \sqrt{N} \phi(\mathbf{r}, t) \quad (4.16)$$

turns out to be the BEC order parameter. From (4.15), the Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = -\frac{\hbar^2}{2m} |\nabla \Psi(\mathbf{r}, t)|^2 + V(\mathbf{r}, t) |\Psi(\mathbf{r}, t)|^2 + \frac{1}{2} \int d^3\mathbf{r}' U_I(\mathbf{r} - \mathbf{r}') |\Psi(\mathbf{r}, t)|^2 |\Psi(\mathbf{r}', t)|^2. \quad (4.17)$$

We know the Lagrangian density $\mathcal{L}$ reads

$$\mathcal{L} = \frac{i\hbar}{2} \left[\Psi^*(\mathbf{r}, t) \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) - \Psi(\mathbf{r}, t) \frac{\partial}{\partial t} \Psi^*(\mathbf{r}, t) \right] - \mathcal{H}, \quad (4.18)$$

since the Lagrangian density is the Legendre transformation of the Hamiltonian density. Consequently, the Euler-Lagrange equation for $\Psi(\mathbf{r}, t)$

$$\frac{\partial \mathcal{L}}{\partial \Psi^*(\mathbf{r}, t)} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial (\partial_t \Psi^*(\mathbf{r}, t))} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial (\nabla \Psi^*(\mathbf{r}, t))} \right) = 0 \quad (4.19)$$

leads to the time-dependent GP equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + V(\mathbf{r}, t) \Psi(\mathbf{r}, t) + \int d^3 \mathbf{r}' U_I(\mathbf{r} - \mathbf{r}') |\Psi(\mathbf{r}', t)|^2 \Psi(\mathbf{r}, t). \quad (4.20)$$

Semiclassical analogy of GP equation

Following discussion is based on the review by Dalfovo et al. (1999)[22], Chavanis (2011)[23] and Chavanis (2019)[24]. We normalise $\Psi(\mathbf{r}, t)$ again to satisfy

$$\int d^3 \mathbf{r} |\Psi(\mathbf{r}, t)|^2 = 1. \quad (4.21)$$

When we assume that the total potential consists of long-range potential with self-gravity and short-range collisional potential, then GP equation (4.20) becomes

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) &= -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + m \Phi_{\text{total}}(\mathbf{r}, t) \Psi(\mathbf{r}, t), \\ \text{where } \Phi_{\text{total}} &= \int d^3 \mathbf{r}' u(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}', t) \\ &:= \int d^3 \mathbf{r}' \left(\frac{-G_N}{|\mathbf{r} - \mathbf{r}'|} + g \delta(\mathbf{r} - \mathbf{r}') \right) \rho(\mathbf{r}', t). \end{aligned} \quad (4.22)$$

The coupling constant g corresponds to the type of self-interaction. When $g < 0$, it is attractive while it is repulsive when $g > 0$. The galactic dark matter density profile $\rho(\mathbf{r}, t)$ is determined by

$$\rho(\mathbf{r}, t) = M_{\text{gal}} |\Psi(\mathbf{r}, t)|^2 \quad (M_{\text{gal}} = Nm : \text{Galactic mass}) \quad (4.23)$$

Because gravitational potential $\Phi(\mathbf{r}, t)$ satisfies the Poisson equation

$$\begin{aligned} \nabla^2 \Phi(\mathbf{r}, t) &= 4\pi G_N \rho(\mathbf{r}, t) \\ &= 4\pi G_N M_{\text{gal}} |\Psi(\mathbf{r}, t)|^2, \end{aligned} \quad (4.24)$$

the total potential Φ_{total} becomes

$$\Phi_{\text{total}}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + g\rho(\mathbf{r}, t) =: \Phi + h(\rho), \quad (4.25)$$

where $h(\rho)$ is the short-range collisional interaction potential and is proportional to the coupling constant g . Therefore we shall focus on the following set of equations

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + m [\Phi + h(\rho)] \Psi(\mathbf{r}, t), \\ \nabla^2 \Phi(\mathbf{r}, t) = 4\pi G_N \rho(\mathbf{r}, t) = 4\pi G_N M_{\text{gal}} |\Psi(\mathbf{r}, t)|^2. \end{cases} \quad (4.26)$$

Since the wave function of the form $\Psi(\mathbf{r}, t) = A(\mathbf{r}, t) e^{iS(\mathbf{r}, t)/\hbar}$ ($A, S \in \mathbb{R}$) fulfills

$$\begin{aligned}\nabla^2 \Psi(\mathbf{r}, t) &= \Psi(\mathbf{r}, t) \left[\frac{2i}{\hbar} (\nabla S(\mathbf{r}, t)) \cdot \frac{\nabla A(\mathbf{r}, t)}{A(\mathbf{r}, t)} + \frac{\nabla^2 A(\mathbf{r}, t)}{A(\mathbf{r}, t)} - \frac{1}{\hbar^2} (\nabla S(\mathbf{r}, t))^2 + \frac{i}{\hbar} \nabla^2 S(\mathbf{r}, t) \right], \\ i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) &= \Psi(\mathbf{r}, t) \left[\frac{i\hbar}{A(\mathbf{r}, t)} \frac{\partial A(\mathbf{r}, t)}{\partial t} - \frac{\partial S(\mathbf{r}, t)}{\partial t} \right],\end{aligned}$$

GP equation(4.26) becomes

$$\begin{aligned}- \left[\frac{\partial S(\mathbf{r}, t)}{\partial t} + \frac{(\nabla S(\mathbf{r}, t))^2}{2m} + m[\Phi + h(\rho)] - \frac{\hbar^2}{2m} \frac{\nabla^2 A(\mathbf{r}, t)}{A(\mathbf{r}, t)} \right] \\ + \frac{i\hbar}{mA(\mathbf{r}, t)} \left[m \frac{\partial A(\mathbf{r}, t)}{\partial t} + (\nabla A(\mathbf{r}, t)) \cdot (\nabla S(\mathbf{r}, t)) + \frac{1}{2} A(\mathbf{r}, t) \nabla^2 S(\mathbf{r}, t) \right] = 0\end{aligned}\quad (4.27)$$

Separating the real and the imaginary parts results in

$$\begin{cases} \frac{\partial S(\mathbf{r}, t)}{\partial t} + \frac{(\nabla S(\mathbf{r}, t))^2}{2m} + m[\Phi + h(\rho)] - \frac{\hbar^2}{2m} \frac{\nabla^2 A(\mathbf{r}, t)}{A(\mathbf{r}, t)} = 0, \\ m \frac{\partial A(\mathbf{r}, t)}{\partial t} + (\nabla A(\mathbf{r}, t)) \cdot (\nabla S(\mathbf{r}, t)) + \frac{1}{2} A(\mathbf{r}, t) \nabla^2 S(\mathbf{r}, t) = 0. \end{cases}\quad (4.28)$$

The first equation of (4.28) is equivalent to the classical Hamilton-Jacobi equation when you take the classical limit $\hbar \rightarrow 0$. Hence the term $Q(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 A(\mathbf{r}, t)}{A(\mathbf{r}, t)}$ reflects purely quantum effect. With $A(\mathbf{r}, t)$, the density profile can be written as

$$\rho(\mathbf{r}, t) = M_{\text{gal}} |\Psi(\mathbf{r}, t)|^2 = M_{\text{gal}} A^2(\mathbf{r}, t). \quad (4.29)$$

Since the mass current is given by

$$\begin{aligned}\mathbf{j}(\mathbf{r}, t) &= -\frac{i\hbar M_{\text{gal}}}{2m} [\Psi^*(\mathbf{r}, t) (\nabla \Psi(\mathbf{r}, t)) - (\nabla \Psi^*(\mathbf{r}, t)) \Psi(\mathbf{r}, t)] \\ &= M_{\text{gal}} A^2(\mathbf{r}, t) \frac{\nabla S(\mathbf{r}, t)}{m} \\ &= \rho(\mathbf{r}, t) \frac{\nabla S(\mathbf{r}, t)}{m},\end{aligned}\quad (4.30)$$

you can introduce velocity field $\mathbf{u}(\mathbf{r}, t)$ as

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{m} \nabla S(\mathbf{r}, t). \quad (4.31)$$

From this definition, the velocity field satisfies no-curl condition $\nabla \times \mathbf{u}(\mathbf{r}, t) = 0$. When you multiply the second equation of (4.28) by $\frac{2M_{\text{gal}} A(\mathbf{r}, t)}{m}$, you get the continuity equation of this condensate dark matter

$$\begin{aligned}\frac{\partial}{\partial t} (M_{\text{gal}} A^2(\mathbf{r}, t)) + \nabla \cdot \left(M_{\text{gal}} A^2(\mathbf{r}, t) \frac{\nabla S(\mathbf{r}, t)}{m} \right) &= 0 \\ \Rightarrow \frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot (\rho(\mathbf{r}, t) \mathbf{u}(\mathbf{r}, t)) &= 0.\end{aligned}\quad (4.32)$$

As the purely quantum term $Q(\mathbf{r}, t)$ can be rewritten as

$$\begin{aligned}
Q(\mathbf{r}, t) &= -\frac{\hbar^2}{2m} \frac{\nabla^2 A(\mathbf{r}, t)}{A(\mathbf{r}, t)} \\
&= -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(\mathbf{r}, t)}}{\sqrt{\rho(\mathbf{r}, t)}} \\
&= -\frac{\hbar^2}{4m} \left[\frac{\nabla^2 \rho(\mathbf{r}, t)}{\rho(\mathbf{r}, t)} - \frac{1}{2} \frac{(\nabla \rho(\mathbf{r}, t))^2}{\rho^2(\mathbf{r}, t)} \right],
\end{aligned} \tag{4.33}$$

you gain from (4.28),

$$\frac{1}{m} \frac{\partial S(\mathbf{r}, t)}{\partial t} + \frac{(\nabla S(\mathbf{r}, t))^2}{2m^2} + [\Phi + h(\rho)] + \frac{1}{m} Q(\mathbf{r}, t) = 0. \tag{4.34}$$

Taking the spatial gradient of (4.34), we find

$$\begin{aligned}
&\frac{1}{m} \frac{\partial \nabla S(\mathbf{r}, t)}{\partial t} + \nabla \frac{(\nabla S(\mathbf{r}, t))^2}{2m^2} + \nabla [\Phi + h(\rho)] + \frac{1}{m} \nabla Q(\mathbf{r}, t) \\
&= \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + [\mathbf{u}(\mathbf{r}, t) \cdot \nabla] \mathbf{u}(\mathbf{r}, t) + \nabla h(\rho) + \nabla \Phi(\mathbf{r}, t) + \frac{1}{m} \nabla Q(\mathbf{r}, t) \\
&= 0,
\end{aligned} \tag{4.35}$$

where we have used the known relation

$$\nabla \left[\frac{(\nabla S(\mathbf{r}, t))^2}{2m^2} \right] = \nabla \left(\frac{\mathbf{u}(\mathbf{r}, t)^2}{2} \right) = [\mathbf{u}(\mathbf{r}, t) \cdot \nabla] \mathbf{u}(\mathbf{r}, t). \tag{4.36}$$

This analysis confirms that the self-gravitating GP system is analogous to the hydrodynamic system satisfying the following three key relations:

$$\begin{cases} \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + [\mathbf{u}(\mathbf{r}, t) \cdot \nabla] \mathbf{u}(\mathbf{r}, t) = -\nabla h(\rho) - \nabla \Phi(\mathbf{r}, t) - \frac{1}{m} \nabla Q(\mathbf{r}, t), \\ \frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot (\rho(\mathbf{r}, t) \mathbf{u}(\mathbf{r}, t)) = 0, \\ \nabla^2 \Phi(\mathbf{r}, t) = 4\pi G_N \rho(\mathbf{r}, t). \end{cases} \tag{4.37}$$

Density Profile of Galactic Axion Halo

The time-independent GP equation realises hydrostatic equilibrium $\left(\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} = \mathbf{u}(\mathbf{r}, t) = 0 \right)$:

$$-\nabla h(\rho) - \nabla \Phi(\mathbf{r}) - \frac{1}{m} \nabla Q(\mathbf{r}) = 0. \tag{4.38}$$

Taking the divergence of (4.38), we get

$$\begin{aligned}
&-\nabla^2 h(\rho) - \frac{1}{m} \nabla^2 Q(\mathbf{r}) = \nabla^2 \Phi(\mathbf{r}) \\
&\Rightarrow -g \nabla^2 \rho(\mathbf{r}) + \frac{\hbar^2}{2m^2} \nabla^2 \left(\frac{\nabla^2 \sqrt{\rho(\mathbf{r})}}{\sqrt{\rho(\mathbf{r})}} \right) = 4\pi G_N \rho(\mathbf{r}).
\end{aligned} \tag{4.39}$$

When the short-range interaction is fairly negligible ($g = 0$), we gain

$$\frac{\hbar^2}{2m^2} \nabla^2 \left(\frac{\nabla^2 \sqrt{\rho(\mathbf{r})}}{\sqrt{\rho(\mathbf{r})}} \right) = 4\pi G_N \rho(\mathbf{r}). \quad (4.40)$$

The numerical calculation of (4.40) is performed in [25] and in [26].

5 Axion Electromagnetism

Basics in electromagnetism

The Maxwell equations:

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \partial_0 \mathbf{E} = \mathbf{j}, \quad (5.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \partial_0 \mathbf{B} = 0. \quad (5.2)$$

With $\vec{x} = (x^\mu)_{\mu=0,\dots,3} = (t, \mathbf{x})$, $\vec{A} = (A^\mu)_{\mu=0,\dots,3} = (\phi, \mathbf{A})$, and $\vec{j} = (j^\mu)_{\mu=0,\dots,3} = (\rho, \mathbf{j})$, we define electromagnetic field strength tensor

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (5.3)$$

and its dual pseudo-tensor

$$\tilde{F}^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (5.4)$$

We employ the metric signature $(-, +, +, +)$. We get

$$E_i = F^{0i}, \quad B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}. \quad (5.5)$$

The equation of motion for the electromagnetic field is

$$\partial_\nu F^{\mu\nu} = j^\mu, \quad (5.6)$$

and for the particle,

$$\frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu. \quad (5.7)$$

In the absence of charged matter, calculating the equation of motion for each component leads to the equivalent expressions of 2 parts of the Maxwell equations in a vacuum (5.1).

For the 0-th component, EoM becomes $\partial_\mu F^{\mu 0} = 0$

$$\nabla \cdot \mathbf{E}(\mathbf{x}, t) = 0, \quad (5.8)$$

and from the 1st to the 3rd component, EoM becomes $\partial_\mu F^{\mu i} = 0$

$$\nabla \times \mathbf{B}(\mathbf{x}, t) - \partial_0 \mathbf{E}(\mathbf{x}, t) = 0. \quad (5.9)$$

The field strength tensor $F_{\mu\nu}$ follows Bianchi's identity [27]

$$\epsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = 0 \Leftrightarrow \partial_\nu \tilde{F}^{\mu\nu} = 0, \quad (5.10)$$

And this relation gives the rest of Maxwell's equations (5.2).

Modified-Maxwell equations

Here, we review the form of the Maxwell equation in a vacuum in the presence of an axion [28]. Our starting point is given by following Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} := -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}g_{a\gamma\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (5.11)$$

where, in the interaction term, $a = a(\mathbf{x}, t)$ represents the axion field, and $g_{a\gamma\gamma}$ is coupling constant of an axion to 2 photons as shown in the Fig. 2.

The first kinetic term of $F_{\mu\nu}$ results in the same equation of motion, while the second term changes this by an additional interaction effect. The Euler-Lagrange equation for the second term $\mathcal{L}_{int} := -\frac{1}{4}g_{a\gamma\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu}$ is given by

$$\frac{\partial(\mathcal{L}_0 + \mathcal{L}_{int})}{\partial A_\lambda} - \partial_\sigma \left(\frac{\partial(\mathcal{L}_0 + \mathcal{L}_{int})}{\partial(\partial_\sigma A_\lambda)} \right) = 0 \quad (5.12)$$

You can see that $\frac{\partial(\mathcal{L}_0 + \mathcal{L}_{int})}{\partial A_\lambda} = 0$, and the second term results in the following form

$$\begin{aligned} 0 &= \partial_\sigma F^{\lambda\sigma} + \partial_\sigma \left(\frac{\partial \mathcal{L}_{int}}{\partial(\partial_\sigma A_\lambda)} \right) \\ &= \partial_\sigma F^{\lambda\sigma} + \partial_\sigma \left(\frac{\partial}{\partial(\partial_\sigma A_\lambda)} \left(-\frac{1}{8}g_{a\gamma\gamma}aF_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \right) \right) \\ &= \partial_\sigma F^{\lambda\sigma} + \partial_\sigma \left(-\frac{1}{8}g_{a\gamma\gamma}a \frac{\partial F_{\mu\nu}}{\partial(\partial_\sigma A_\lambda)} \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} - \frac{1}{8}g_{a\gamma\gamma}aF_{\mu\nu}\epsilon^{\mu\nu\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\sigma A_\lambda)} \right) \\ &= \partial_\sigma F^{\lambda\sigma} + g_{a\gamma\gamma} \left(\partial_\sigma \tilde{F}^{\lambda\sigma} \right) a + g_{a\gamma\gamma} (\partial_\sigma a) \tilde{F}^{\lambda\sigma} \\ &= \partial_\sigma F^{\lambda\sigma} + g_{a\gamma\gamma} (\partial_\sigma a) \tilde{F}^{\lambda\sigma} \end{aligned} \quad (5.13)$$

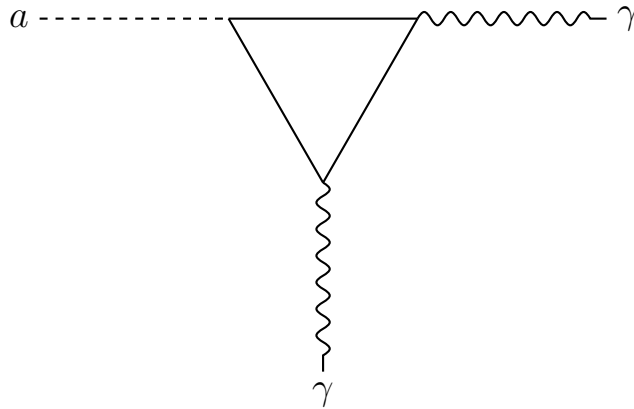


Figure 2. The Feynman diagram of the axion interaction with two photons introduced in Eq.(5.11).

To gain this, we used Bianchi's identity: $\partial_\nu \tilde{F}^{\mu\nu} = 0$, and

$$\begin{aligned}\frac{\partial F_{\mu\nu}}{\partial (\partial_\sigma A_\lambda)} &= \left(\delta_\mu^\sigma \delta_\nu^\lambda - \delta_\nu^\sigma \delta_\mu^\lambda \right) \\ \frac{\partial F_{\alpha\beta}}{\partial (\partial_\sigma A_\lambda)} &= \left(\delta_\alpha^\sigma \delta_\beta^\lambda - \delta_\beta^\sigma \delta_\alpha^\lambda \right).\end{aligned}\quad (5.14)$$

Finally, you get the equation of motion for the electromagnetic field with axion field coupling

$$\partial_\nu F^{\mu\nu} + g_{a\gamma\gamma} (\partial_\nu a) \tilde{F}^{\mu\nu} = 0. \quad (5.15)$$

In the same manner, as the previous section, you can construct a modified-Maxwell equation. For the 0-th component, EoM becomes $\partial_\nu F^{0\nu} + g_{a\gamma\gamma} (\partial_\nu a) \tilde{F}^{0\nu} = 0$

$$\nabla \cdot \mathbf{E}(\mathbf{x}, t) = -g_{a\gamma\gamma} (\nabla a(\mathbf{x}, t)) \cdot \mathbf{B}(\mathbf{x}, t), \quad (5.16)$$

and from the 1st to the 3rd component, EoM becomes $\partial_\nu F^{i\nu} + g_{a\gamma\gamma} (\partial_\nu a) \tilde{F}^{i\nu} = 0$ ($i = 1, 2, 3$)

$$\nabla \times \mathbf{B}(\mathbf{x}, t) - \partial_0 \mathbf{E}(\mathbf{x}, t) = +g_{a\gamma\gamma} \{ (\partial_0 a(\mathbf{x}, t)) \mathbf{B}(\mathbf{x}, t) + (\nabla a(\mathbf{x}, t)) \times \mathbf{E}(\mathbf{x}, t) \}. \quad (5.17)$$

Let us consider an axion travelling along the x-axis,

$$a(\mathbf{x}, t) = a_0 \cos(\omega t - kx),$$

in a constant magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$, ($B_0 \neq 0$).

If axion does not create any electric fields, modified-Maxwell equations become

$$\begin{aligned}\nabla \cdot \mathbf{E}(\mathbf{x}, t) &= -g_{a\gamma\gamma} (\nabla a(\mathbf{x}, t)) \cdot \mathbf{B}(\mathbf{x}, t) \\ &= -g_{a\gamma\gamma} [-ka_0 \sin(\omega t - kx) \mathbf{e}_x] \cdot (B_0 \mathbf{e}_z) \\ &= 0\end{aligned}\quad (5.18)$$

$$\begin{aligned}\nabla \times \mathbf{B}(\mathbf{x}, t) &= (\partial_2 B_0) \mathbf{e}_x - (\partial_1 B_0) \mathbf{e}_y \\ &= \partial_0 \mathbf{E}(\mathbf{x}, t) \\ &\quad + g_{a\gamma\gamma} [\{-\omega a_0 \sin(\omega t - kx)\} (B_0 \mathbf{e}_z) + \{ka_0 \sin(\omega t - kx) \mathbf{e}_x\} \times \mathbf{E}(\mathbf{x}, t)] \\ &= -g_{a\gamma\gamma} \omega a_0 \sin(\omega t - kx) (B_0 \mathbf{e}_z) \neq 0,\end{aligned}\quad (5.19)$$

but this contradicts with the fact that $\nabla \times \mathbf{B}(\mathbf{x}, t) = 0$. Therefore, you can conclude that under a constant magnetic field, an axion induces a constant electric field

$$\mathbf{E}_{axion}(\mathbf{x}, t) = \frac{\omega}{k} B_0 \mathbf{e}_y \quad (5.20)$$

to satisfy

$$\nabla \times \mathbf{B}(\mathbf{x}, t) = g_{a\gamma\gamma} [\{-\omega a_0 \sin(\omega t - kx)\} (B_0 \mathbf{e}_z) + \{ka_0 \sin(\omega t - kx) \mathbf{e}_x\} \times \mathbf{E}(\mathbf{x}, t)] = 0 \quad (5.21)$$

In this case, we can also evaluate the corresponding Poynting vector $\mathbf{S}(\mathbf{x}, t)$ as

$$\begin{aligned}\mathbf{S}(\mathbf{x}, t) &= \mathbf{E}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t) \\ &= \left(\frac{\omega}{k} B_0 \mathbf{e}_y \right) \times (B_0 \mathbf{e}_z) \\ &= \frac{\omega}{k} B_0^2 \mathbf{e}_x.\end{aligned}\quad (5.22)$$

Axion-induced Electromagnetic wave effect

Suppose an axion is light and non-relativistic [29]:

$$m_a \ll \omega, v \ll c,$$

, where ω is the angular frequency of the electromagnetic wave. Maxwell equations turn into the electromagnetic wave equations

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \mathbf{E}(\mathbf{x}, t) = g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t)) \nabla \times \mathbf{E}(\mathbf{x}, t) \quad (5.23)$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \mathbf{B}(\mathbf{x}, t) = g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t)) \nabla \times \mathbf{B}(\mathbf{x}, t) \quad (5.24)$$

Here, we omitted the spatial gradient of the axion field because the de Broglie length of the axion $\frac{1}{m_a v}$ is sufficiently larger than the typical length of the gravitational wave detector we consider later.

Proof for $\mathbf{B}(\mathbf{x}, t)$.

$$\nabla \times (\nabla \times \mathbf{B}(\mathbf{x}, t)) = -\nabla^2 \mathbf{B}(\mathbf{x}, t) \quad (5.25)$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}(\mathbf{x}, t)) &= \nabla \times [\partial_0 \mathbf{E}(\mathbf{x}, t) + g_{a\gamma\gamma} \{(\partial_0 a(\mathbf{x}, t)) \mathbf{B}(\mathbf{x}, t) + (\nabla a(\mathbf{x}, t)) \times \mathbf{E}(\mathbf{x}, t)\}] \\ &= \partial_0 (\nabla \times \mathbf{E}(\mathbf{x}, t)) + g_{a\gamma\gamma} \nabla \times [(\partial_0 a(\mathbf{x}, t)) \mathbf{B}(\mathbf{x}, t)] + g_{a\gamma\gamma} \nabla \times [(\nabla a(\mathbf{x}, t)) \times \mathbf{E}(\mathbf{x}, t)] \\ &= -\partial_0^2 \mathbf{B}(\mathbf{x}, t) + g_{a\gamma\gamma} \left[(\nabla (\partial_0 a(\mathbf{x}, t))) \times \mathbf{B}(\mathbf{x}, t) + (\partial_0 a(\mathbf{x}, t)) \nabla \times \mathbf{B}(\mathbf{x}, t) \right] \\ &\quad + g_{a\gamma\gamma} \left[(\mathbf{E}(\mathbf{x}, t) \cdot \nabla) (\nabla a(\mathbf{x}, t)) - (\nabla^2 a(\mathbf{x}, t)) \mathbf{E}(\mathbf{x}, t) \right. \\ &\quad \left. + (\nabla a(\mathbf{x}, t)) (\nabla \cdot \mathbf{E}(\mathbf{x}, t)) - [(\nabla a(\mathbf{x}, t)) \cdot \nabla] \mathbf{E}(\mathbf{x}, t) \right] \\ &= -\partial_0^2 \mathbf{B}(\mathbf{x}, t) \\ &\quad + g_{a\gamma\gamma} \left[(\nabla (\partial_0 a(\mathbf{x}, t))) \times \mathbf{B}(\mathbf{x}, t) \right. \\ &\quad \left. + (\partial_0 a(\mathbf{x}, t)) (\partial_0 \mathbf{E}(\mathbf{x}, t) + g_{a\gamma\gamma} \{(\partial_0 a(\mathbf{x}, t)) \mathbf{B}(\mathbf{x}, t) + (\nabla a(\mathbf{x}, t)) \times \mathbf{E}(\mathbf{x}, t)\}) \right] \\ &\quad + g_{a\gamma\gamma} \left[(\mathbf{E}(\mathbf{x}, t) \cdot \nabla) (\nabla a(\mathbf{x}, t)) - (\nabla^2 a(\mathbf{x}, t)) \mathbf{E}(\mathbf{x}, t) \right. \\ &\quad \left. + (\nabla a(\mathbf{x}, t)) (-g_{a\gamma\gamma} (\nabla a(\mathbf{x}, t)) \cdot \mathbf{B}(\mathbf{x}, t)) - [(\nabla a(\mathbf{x}, t)) \cdot \nabla] \mathbf{E}(\mathbf{x}, t) \right] \\ &\simeq -\partial_0^2 \mathbf{B}(\mathbf{x}, t) + g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t)) \nabla \times \mathbf{B}(\mathbf{x}, t). \end{aligned} \quad (5.26)$$

Hence you yield

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \mathbf{B}(\mathbf{x}, t) = g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t)) \nabla \times \mathbf{B}(\mathbf{x}, t) \quad (5.27)$$

□

Dispersion relation

Here, we consider the dispersion relation of Axion-induced electromagnetic waves. Assuming a plane-wave solution of (5.23)

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{\pm i \mathbf{k} \cdot \mathbf{x}} = \mathbf{E}_0 e^{\mp i \omega_L t \pm i \mathbf{k} \cdot \mathbf{x}}, \quad (5.28)$$

the dispersion relation becomes

$$-\omega_L^2 + \mathbf{k}^2 = \mp g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t)) |\mathbf{k}|. \quad (5.29)$$

From this relation, you can evaluate the phase velocity of left and right circular polarized light as

$$\begin{aligned} v_{phase} &:= \frac{\omega_L}{|\mathbf{k}|} \\ &= \sqrt{1 \pm \frac{g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t))}{|\mathbf{k}|}} \\ &\simeq 1 \pm \frac{g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t))}{2 |\mathbf{k}|}. \end{aligned} \quad (5.30)$$

Therefore, the right circular polarized light has more significant phase velocity than the left by $\frac{g_{a\gamma\gamma} (\partial_0 a(\mathbf{x}, t))}{|\mathbf{k}|}$. We will investigate the detectability of this difference in phase velocity by using gravitational wave-detecting laser interferometry.

6 Gravitational Wave Detectors

In this section, we focus on the signal detection of the gravitational wave (GW). We currently have two kinds of laser interferometers to detect and magnify the GW effects: the Michelson type and the Fabry-Perot type. In the first half of this section, we present how GW signals interact with these two instruments. The rest of this section concerns the significant noise sources for GW detectors. Our discussion is based on the textbook of Kawamura (2018)[30] and Maggiore (2007)[31].

Michelson and Fabry-Perot

First, we describe the principle of gravitational wave detection using laser interferometry. Consider a Michelson laser interferometer consisting of a suspended mirror and beamsplitter, as shown in Fig. 3

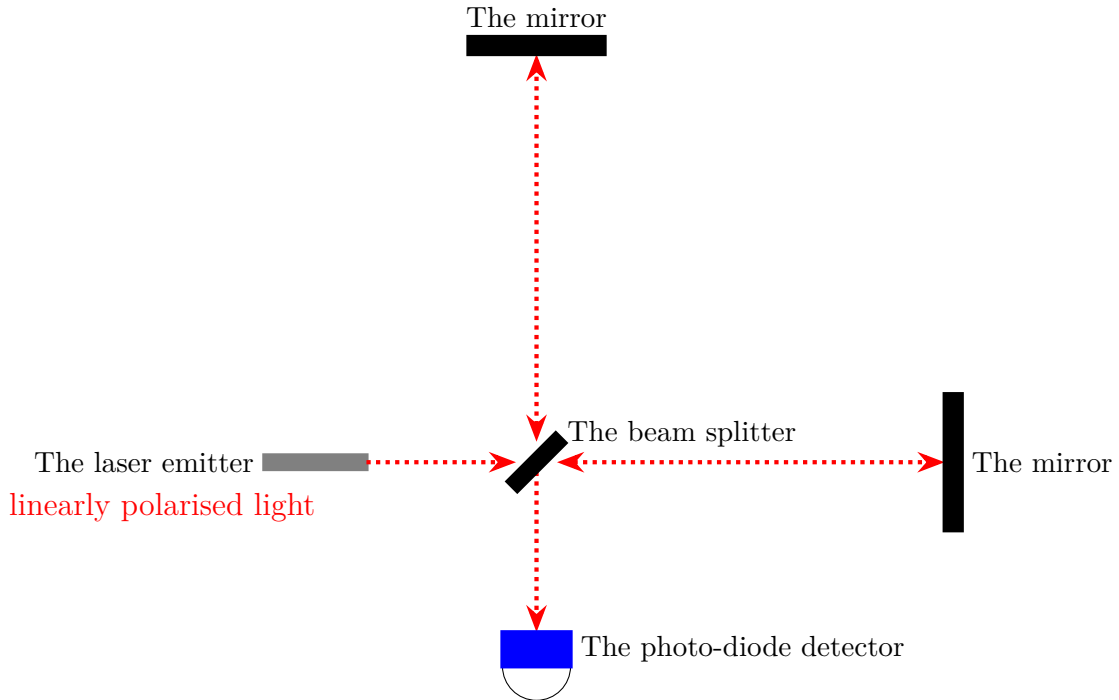


Figure 3. The Michelson laser interferometer consisting of a suspended mirror and beamsplitter

The light emitted from the laser is transmitted by the beam splitter, with half of the light transmitted and the other half reflected. Each beam is reflected by a mirror at the far end and returns to the beam splitter. Assume that the two beams are properly aligned, such as with mirrors, so that they overlap. Then the two beams interfere with each other, producing interference beams that are directed toward the detection port and the laser. These interference beams are bright or dark depending on the optical path lengths of the two arms. When the difference in the optical path length of the round trip changes by the wavelength

of the light, the interfering light responds with a periodic change between light and dark. Suppose that a gravitational wave with polarization along the arm of the interferometer comes from a direction perpendicular to the horizontal plane in which the interferometer is placed. The optical path lengths of the two arms would change differentially, resulting in a change in the brightness of the interference light. This difference is measurable and is an effective means of detecting gravitational waves.

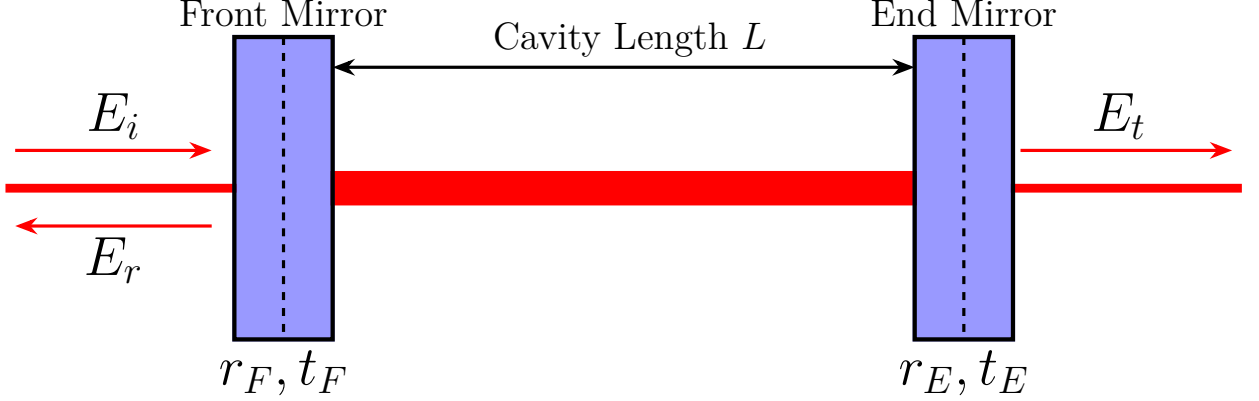


Figure 4. The setting of Fabry-Perot type cavity resonator.

Secondly, we move on to the Fabry-Perot type cavity resonator. A Fabry-Perot resonator is a device that can send light back and forth repeatedly by facing two mirrors, as shown in Fig.4. The following is a brief summary of the properties of the Fabry-Perot resonator. Consider the oscillating electric field with amplitude E_i entering through the front mirror of the Fabry-Perot resonator with cavity length L . Let the amplitude reflectance and transmittance of the front mirror be r_F, t_F , respectively, and similarly, let those of the end mirror be r_E, t_E . Considering the round trip of light within the cavity, the amplitude of the reflected light is calculated by the following infinite series

$$\begin{aligned}
E_r &= -E_i r_F - E_i t_F^2 r_E e^{-i\phi} - E_i t_F^2 r_E^2 r_F e^{-2i\phi} - E_i t_F^2 r_E^3 r_F e^{-3i\phi} + \dots \\
&= -E_i \left(r_F + t_F^2 r_E e^{-i\phi} \sum_{n=0}^{\infty} \left(r_F r_E e^{-i\phi} \right)^n \right) \\
&= -E_i \left(r_F + \frac{t_F^2 r_E e^{-i\phi}}{1 - r_F r_E e^{-i\phi}} \right), \tag{6.1}
\end{aligned}$$

where ϕ is the phase change of the light as it travels back and forth within the resonator

$$\phi = \frac{2L\omega}{c}, \tag{6.2}$$

with the laser frequency ω . In the same manner, the transmitting light amplitude is also calculated by

$$\begin{aligned}
E_t &= E_i t_F t_E e^{-\frac{i\phi}{2}} + E_i t_F r_F t_E r_E e^{-\frac{3i\phi}{2}} + E_i t_F r_F^2 t_E r_E^2 e^{-\frac{5i\phi}{2}} + \dots \\
&= E_i t_F t_E e^{-\frac{i\phi}{2}} \sum_{n=0}^{\infty} \left(r_F r_E e^{-i\phi} \right)^n \\
&= E_i \frac{t_F t_E e^{-\frac{i\phi}{2}}}{1 - r_F r_E e^{-i\phi}}.
\end{aligned} \tag{6.3}$$

Therefore effective reflectance and transmittance are derived by

$$\begin{cases} r_{\text{eff}}(\phi) = -r_F - \frac{t_F^2 r_E e^{-i\phi}}{1 - r_F r_E e^{-i\phi}}, \\ t_{\text{eff}}(\phi) = \frac{t_F t_E e^{-\frac{i\phi}{2}}}{1 - r_F r_E e^{-i\phi}}, \end{cases} \tag{6.4}$$

and the power of those lights are estimated by

$$P_r \propto |E_r|^2 = \left[r_F^2 + \frac{t_F^2 r_F r_E (2 \cos \phi - 2 r_F r_E) + t_F^2 r_E^2}{(1 - r_F r_E)^2 + 4 r_F r_E \sin^2(\phi/2)} \right] |E_i|^2, \tag{6.5}$$

$$P_t \propto |E_t|^2 = \frac{(t_F t_E)^2}{(1 - r_F r_E)^2 + 4 r_F r_E \sin^2(\phi/2)} |E_i|^2 \tag{6.6}$$

When the power of transmitting light (6.6) is maximised, the power of light in the FP resonator is also maximised. We call this *the resonant state* of the FP resonator. The resonant condition is $\phi = 2\pi m$ ($m \in \mathbb{Z}$) and from the definition of phase change ϕ , we rewrite this condition in terms of the cavity length L and the laser wavelength λ as

$$2L = m\lambda \quad (\text{with } m \in \mathbb{Z}). \tag{6.7}$$

The free spectral range (FSR) is defined as the interval in optical frequency between two consecutive maxima of transmitted light intensity in the interferometer. When we fix the cavity length L , the FSR of laser angular frequency ω_{FSR} satisfies

$$\Delta\phi = \frac{2L\omega_{\text{FSR}}}{c} = 2\pi, \tag{6.8}$$

and we define the resonance frequency ν_{FSR} by

$$\nu_{\text{FSR}} = \frac{\omega_{\text{FSR}}}{2\pi} = \frac{c}{2L}. \tag{6.9}$$

The full width at half maximum of the laser frequency ν_{FWHM} is given by

$$\nu_{\text{FWHM}} \simeq \frac{c(1 - r_F r_E)}{2\pi L \sqrt{r_F r_E}}. \tag{6.10}$$

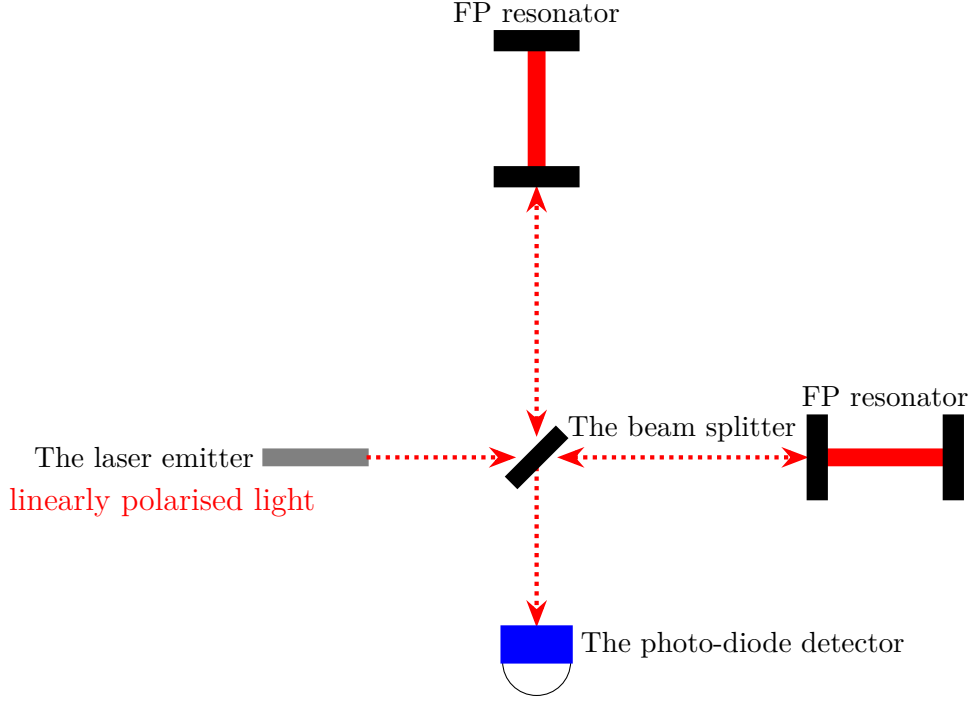


Figure 5. The setting of the Fabry-Perot laser interferometer

The ratio of these two quantities represents the sharpness of the resonance of the FP resonator

$$\mathcal{F} = \frac{\nu_{\text{FSR}}}{\nu_{\text{FWHM}}} \simeq \frac{\pi \sqrt{r_F r_E}}{1 - r_F r_E}, \quad (6.11)$$

and this quantity $\mathcal{F}$ is called *Finesse*. The Finesse is an important parameter to represent the performance of this device. The Fabry-Perot type laser interferometer shown in the Fig.5 is a gravitational wave interferometer using an FP resonator. In the following, we explain how the phase differences from the GWs are magnified by those 2 types of laser interferometry and discuss their noise sources.

Detection of the GW signals

At first, we consider the sine-Gaussian model for the burst GW signal.

$$h(t) = h_0 \sin(2\pi f_0 t) \exp \left[- \left(\frac{t}{\tau_g} \right)^2 \right], \quad (6.12)$$

with the frequency of GW f_0 , the initial amplitude h_0 , and the decaying time-scale τ_g . For the GW signal observation, the *Root-Sum-Squared* amplitude (rss amplitude) is the major means

$$h_{\text{rss}} := \sqrt{\int_{-\infty}^{\infty} dt h^2(t)} = h_0 \sqrt{\tau_g \sqrt{\frac{\pi}{8}} \left(1 - e^{-2\pi^2 f_0^2 \tau_g^2} \right)} \text{ Hz}^{-1/2}. \quad (6.13)$$

The emission energy of the burst GW is given by

$$\Delta E_{\text{rad}} \simeq 1.0 \times 10^{-2} M_{\odot} c^2 \left(\frac{r}{8 \text{ kpc}} \right) \left(\frac{h_{\text{rss}}}{10^{-19} \text{ Hz}^{-1/2}} \right) \left(\frac{f_0}{1 \text{ kHz}} \right), \quad (6.14)$$

where r is the size of the burst source. Suppose a burst GW signal with

$$r = 14 \text{ Mpc}, \quad h_{\text{rss}} = 6 \times 10^{-23} \text{ Hz}^{-1/2}, \quad \tau_g = 1 \text{ ms}, \quad f_0 = 1 \text{ kHz} \quad (6.15)$$

is observed. If the typical emission energy of a burst gravity wave is known to be $1.0 \times 10^{-2} M_{\odot} c^2$, the initial amplitude of the gravity wave can be back-calculated from Eq. (6.13) as $h_0 \simeq 2.0 \times 10^{-21}$.

Secondly consider a GW with the amplitude $h_0 = 2.0 \times 10^{-21}$ propagating from the vertical direction of the horizontal plane of the device. In the Michelson interferometer with arm length $l_x + l_y = 4 \text{ km}$, laser wavelength $\lambda_L = 1 \mu\text{m}$, the phase change $\Delta\phi_{\text{Mich}}$ due to this GW is given by

$$\begin{aligned} \Delta\phi_{\text{Mich}} &= 2\pi \frac{\Delta L}{\lambda_L} \\ &\leq 2\pi \frac{h_0 (l_x + l_y)}{\lambda_L} \\ &\simeq 5.0 \times 10^{-11} \text{ rad}, \end{aligned} \quad (6.16)$$

where ΔL is the effective optical path length which can be expanded and contracted by GW. While in the Fabry-Perot interferometer with finesse $\mathcal{F} = 200$, since the laser light is trapped in the cavity for a time proportional to the finesse $\mathcal{F}$, the phase change $\Delta\phi_{\text{FP}}$ due to this burst GW is given by

$$\begin{aligned} \Delta\phi_{\text{FP}} &= 2\pi \frac{\Delta L}{\lambda_L} \\ &\leq 2\pi \frac{\mathcal{F} \times h_0 (l_x + l_y)}{\lambda_L} \\ &\simeq 1.0 \times 10^{-8} \text{ rad}. \end{aligned} \quad (6.17)$$

Thus, the FP interferometer magnifies the GW signal more than the Michaelson interferometer, making it easier to collect the signal data without attenuation of the laser intensity. The next concern is that the noise interference during the observation. The three most fundamental noises that determine the sensitivity of an interferometer are seismic(vibration of the ground) noise, thermal noise, and quantum noise. In addition to these noises, there are many other noise sources in interferometers. To increase the sensitivity of the interferometer and detect gravitational waves, all these noises must be reduced as much as possible. We explain how each noise source appears as noise in the output of the interferometer.

Here, we mainly focus on quantum noise as the DECIGO observation is a space-based project. Quantum noise is due to the fact that the laser is a collection of photons, and there is shot

noise and radiation pressure noise. Shot noise is caused by the number fluctuation of photons in the laser. Suppose we observe the N_γ photons during the time-interval T . Then the average power P of the laser with frequency ω_L is evaluated by

$$\bar{P} = \frac{1}{T} N_\gamma \hbar \omega_L \quad (6.18)$$

Since the photo-diode detector counts discrete events in the fixed time interval, the photon number count obeys the Poisson distribution

$$p(N; \bar{N}) = \frac{1}{N!} \bar{N}^N e^{-\bar{N}}. \quad (6.19)$$

The large N limit of this distribution coincides with the Gaussian distribution with the standard deviation $\sqrt{\bar{N}}$. The power fluctuation corresponding to the photon number fluctuation $\Delta N_\gamma = \sqrt{\bar{N}_\gamma}$ is given by

$$\begin{aligned} \Delta P_{\text{shot}} &= \frac{1}{T} \sqrt{\bar{N}_\gamma} \hbar \omega_L \\ &= \sqrt{\frac{\hbar \omega_L}{T}} \bar{P}. \end{aligned} \quad (6.20)$$

Radiation pressure noise is caused by the photo-pressure fluctuation acting on the mirrors. Consider an incident photon striking a beamsplitter mirror perpendicularly and being reflected in the opposite direction. Then the magnitude of momentum transfer from the photon to the mirror is $2|\mathbf{p}|$. The force acting on the mirror F from the laser with power P is yielded by

$$F = \frac{2P}{c} \quad (6.21)$$

since the energy of a photon is generally given by $E_\gamma = \frac{|p|}{c}$. Hence the fluctuation of photon force onto the mirror ΔF is evaluated as

$$\Delta F = \frac{2}{c} \Delta P = 2 \sqrt{\frac{\hbar \omega_L}{c^2 T}} P. \quad (6.22)$$

7 Axion Interferometry

The circularly polarised light enables an exact correspondence between the effects of axions and GWs. From DeRocco & Hook (2018)[28], the results of the GW interferometry can be imported directly into axion interferometry. As we have seen in Sec. 4 that the cosmological axions are in BEC state and behave as a classical field due to their large occupation number. We apply classical approximation of the axion field

$$a(t, z) = a_0 \sin(m_a t + k_a z) \quad \left(\text{with the coherence time } \tau = \frac{2\pi}{m_a v^2} \sim \frac{10^{-6}}{m_a} \right). \quad (7.1)$$

The setting of the FP type axion interferometer is shown in Fig. 6.

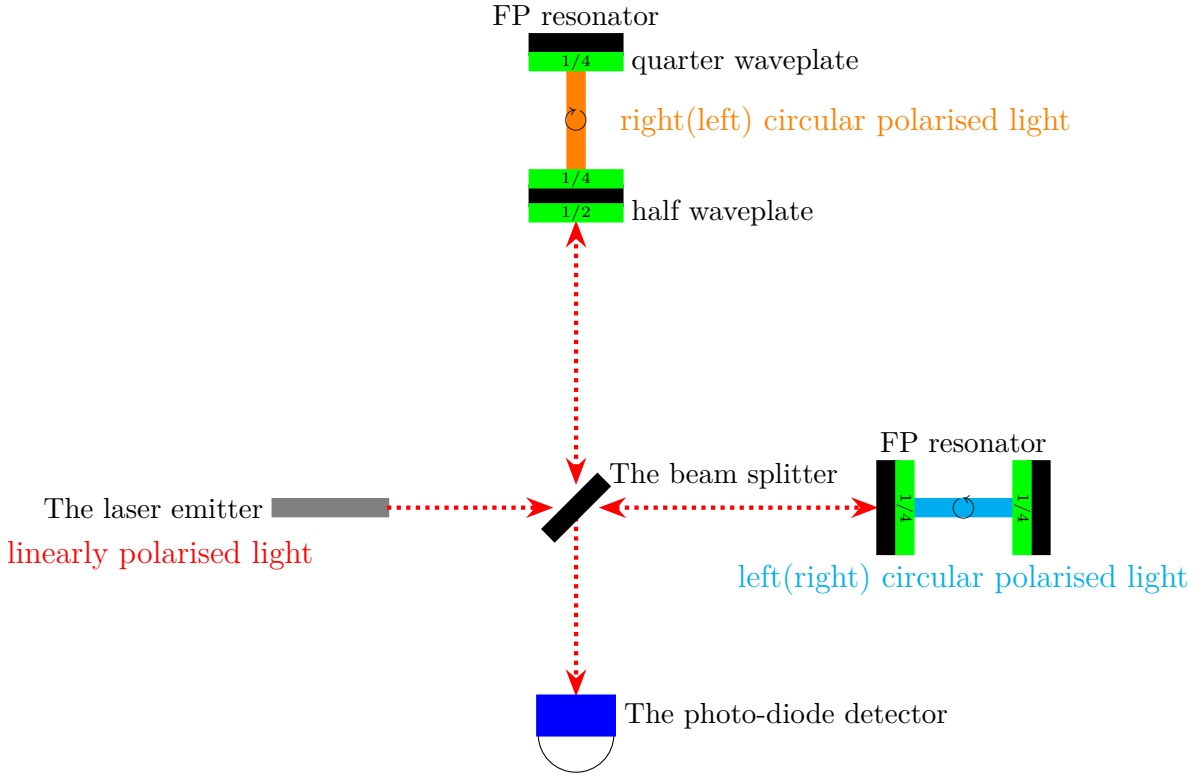


Figure 6. The setting of the Fabry-Perot laser interferometer with additional quarter/half waveplates for cosmic axion search.

For cosmic axion search, we added four quarter waveplates and one half waveplate to the normal Fabry-Perot resonators. The quarter waveplates are introduced to convert a linearly polarised light into a circularly polarised state. The half waveplate is introduced to shift the polarisation angle of the incoming linearly polarised light by $\pi/2$. This results in the opposite circular polarisation to the other FP cavity. Thereby the two arms of the interferometer receive opposite sign effects from the background axion, causing interference when recombining the beams at the detector. From the phase velocity of the electromagnetic wave (5.30), the

effective optical path length of the left-right polarization $L_{\odot,\odot}$ in an interferometer on the $z = 0$ plane is given by

$$L_{\odot,\odot} = \int_{t_0}^{t_0+\tau} dt \left(1 \pm \frac{g_{a\gamma\gamma} m_a a_0}{2\omega_L} \cos(m_a t) \right), \quad (7.2)$$

while for GWs, the standard formula for the path length is

$$L_{x,y} = \int_{t_0}^{t_0+\tau} dt \left(1 \pm \frac{1}{2} h_0 \cos(\omega_{\text{GW}} t) \right). \quad (7.3)$$

Comparing Eq. (7.2) to Eq. (7.3), we see the exact correspondence

$$h_0 \leftrightarrow \frac{g_{a\gamma\gamma} m_a a_0}{\omega_L} = \frac{g_{a\gamma\gamma} \sqrt{2\rho_a}}{\omega_L}, \quad \omega_{\text{GW}} \leftrightarrow m_a. \quad (7.4)$$

where we have used the relation (3.14)

$$\rho_a \simeq \frac{1}{2} m_a^2 a_0^2. \quad (7.5)$$

In the following calculations, the axion energy density ρ_a is approximated by the local DM energy density $\rho_{\text{DM}} \sim 0.3 \text{ GeV}/\text{cm}^3$ [10][11][12], which is a complete assumption. From [28], the shot noise and the radiation pressure noise signals are given by

$$S_{\text{shot}}^{1/2} = \frac{1}{4L} \sqrt{\frac{\lambda}{\pi P_0}} \sqrt{1 + r_F^2 - 2r_F \cos(2m_a L)} \quad (7.6)$$

$$S_{\text{rad}} = \frac{16\mathcal{F}}{MLm_a^2} \sqrt{\frac{P_0}{\pi\lambda} \frac{m_a L}{\sin(m_a L)}} \frac{1 - r_F^2}{2\sqrt{1 + r_F^2 - 2r_F \cos 2m_a L}} \quad (7.7)$$

The sensitivity curve is obtained by comparing the axion signal to the noise signal. This characteristic is called the signal-to-noise ratio (S/N) [31]. Let the total signal be of the form

$$h(t) = h_0 \sin[2\pi f_0 t + \phi(t)] + B_n(t), \quad (7.8)$$

where the noise $B_n(t)$ has the power spectral density S_n with $\langle S_n \rangle = 0$. The signal has its coherence time τ and the phase function $\phi(t)$ is constant within this coherent time interval. The spectrum of this total signal within the observation time interval T is given by

$$\begin{aligned} P(f) &= \frac{1}{\sqrt{T}} \int_0^T dt h(t) e^{2\pi i f t} \\ &= P_0(f) + P_n(f). \end{aligned} \quad (7.9)$$

The spectral energy is decomposed into the pure signal part and the noise part:

$$|P(f)|^2 = |P_0(f)|^2 + |P_n(f)|^2 \quad \left(\text{with } |P_n(f)|^2 = S_n^2 \right). \quad (7.10)$$

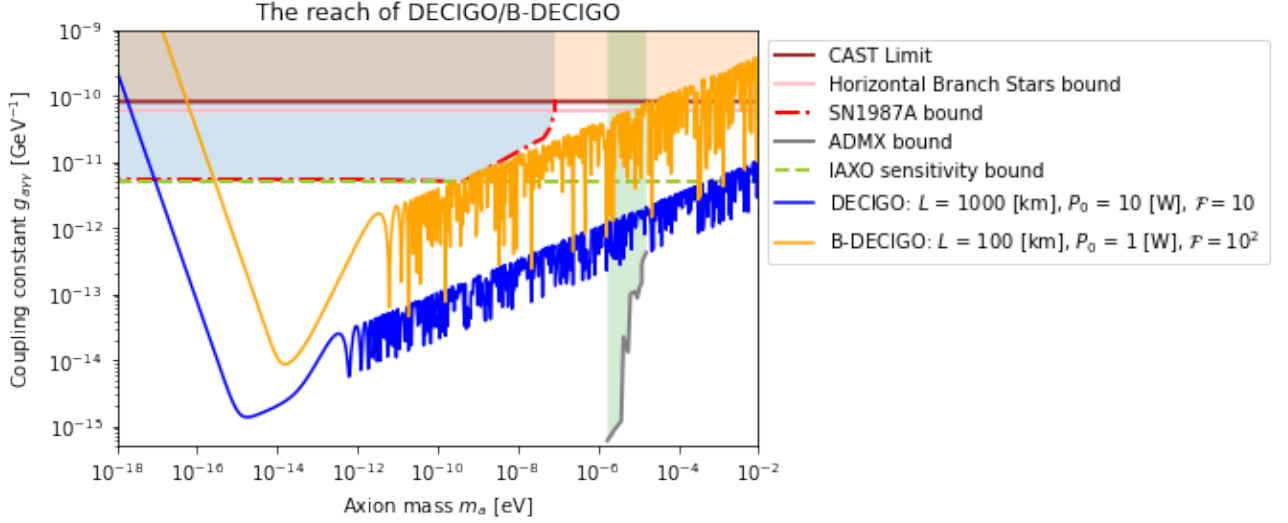


Figure 7. Axion parameter regions that can be explored with DECIGO, B-DECIGO

When the observational time T is shorter than the coherence time τ , the signal becomes a single sine wave and the spectrum becomes a delta function with the centre of $f = f_0$. Then since $|P_0(f_0)|^2 = h_0^2 T$, the S/N of this case is evaluated as

$$(S/N)^2 = \frac{h_0^2 T}{S_n(f_0)}. \quad (7.11)$$

When the observational time T is longer than the coherence time τ , the signal takes a complex shape and it is hard to apply the former method. Therefore we first divide total time T into T/τ parts of coherent time interval τ . Then we apply the former method at every time packet and finally take the sum of (T/τ) S/Ns. The S/N of this case is evaluated as

$$\begin{aligned} (S/N)^2 &= \frac{(T/\tau)}{\sqrt{T/\tau}} \frac{h_0^2 \tau}{S_n(f_0)} \\ &= \frac{h_0^2}{S_n(f_0)} \sqrt{T\tau}. \end{aligned} \quad (7.12)$$

From this relation, the S/N is given by replacing the amplitude h_0 to $\frac{g_{a\gamma\gamma}\sqrt{2\rho_a}}{\omega_L}$:

$$S/N = \frac{1}{\sqrt{S_{\text{shot}} + S_{\text{rad}}}} \frac{g_{a\gamma\gamma}\sqrt{2\rho_{\text{DM}}}}{\omega_L} (T\tau)^{1/4} \quad (7.13)$$

When $S/N \geq 1$, the signal overcome the noise and it can be detected. We defined the sensitivity curves to satisfy the equality condition and calculated them for the DECIGO and the B-DECIGO as shown in Fig 7. The plot was made assuming 1000km and 100km long arms for DECIGO and B-DECIGO respectively with 100kg mirrors for both. The Finesse

of DECIGO is set to be $\mathcal{F} = 10$, while for B-DECIGO it is set to be $\mathcal{F} = 100$. The initial power of DECIGO laser source is 10W and that of B-DECIGO is 1W. Bounds placed by CAST[32] and ADMX[33], and constraints coming from the production of axions in supernova and subsequent conversions into photons in the interstellar medium are shown[34][35]. The reach of the other proposed experiments IAXO[36], ALPS II[37] is also shown in this figure. Firstly, we see the fluctuation slope in the heavier mass region. This is because the shot noise S_{shot} becomes dominant and the coupling $g_{a\gamma\gamma}$ is of order $\mathcal{O}(m_a^{1/4})$. This fluctuation is important when the mass m_a is comparable to the inverse of the arm length L . In fact $\hbar c/L = 197\text{MeV fm}/1000\text{km} \sim 10^{-13}$. Although it is expected that the radiation pressure noise S_{rad} causes a dominant effect on the mass-coupling relation in the lighter mass region, the figure does not follow the expected order of $\mathcal{O}(m_a^{-3/4})$. Secondly, we find that the longer the cavity length is, the more sensitive the interferometer is to lighter axion candidates. And it is confirmed that the longer the cavity length, the wider the mass range the interferometer can search.

The analysis was performed by the following code on GitHub:

https://github.com/kyphys-creator/IPython_projects/tree/master/BSc_Dissertation_Numerics.

8 Conclusion and Discussion

The purpose of this study was to investigate the mass and coupling sensitivity of future gravitational wave laser interferometers, and to this end, we set out to assess the S/N of DECIGO with additional quarter/half wave retarders. We confirmed that the most sensitive mass varied with arm length, allowing for a wider range of ALP candidates to be searched as a result of longer arms. The major problem we see from this result is that the detectable range is too vast to distinguish the weight and the coupling of the detected axion particles. Besides, The generalizability of these findings is limited because of the assumption that the detector is placed in the galactic halo of the BEC state non-relativistic ALPs. It is recommended that further research be undertaken on the high-temperature behaviour of instantons and axion mass in order to validate the large occupation number of ALPs. Some studies show that the ALPs cannot cool themselves by emitting light because of the weak photon interaction of ALPs[38][39][40] and it is also interesting to explore the axion laser scattering phenomenon for a more realistic theoretical estimate.

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Appendix.

A The general proof of the Vafa-Witten relation (2.30)

Let $\mathcal{L}$ be a Lagrangian. We define the general Lagrangian $\hat{\mathcal{L}}(\lambda)$ ($\lambda \in \mathbb{R}$) by

$$\hat{\mathcal{L}}(\lambda) = \mathcal{L} - \lambda X, \quad (\text{A.1})$$

where X is a Hermitian and parity non-conserving operator. Since $\lambda \in \mathbb{R}$, we find that $\mathcal{L}$ is also Hermitian. Let $E(\lambda)$ be the vacuum energy of the theory with the Lagrangian $\hat{\mathcal{L}}(\lambda)$. Taking the lowest order of λ , you get

$$E(\lambda) = E(0) + \lambda \langle X \rangle|_{\lambda=0}, \quad (\text{A.2})$$

where $\langle X \rangle|_{\lambda=0}$ be the vacuum expectation value of the operator X when $\lambda = 0$. Since $\langle X \rangle|_{\lambda=0}$ is parity odd when $\langle X \rangle|_{\lambda=0} \neq 0$, theory always chooses negative part of $\lambda \langle X \rangle|_{\lambda=0}$. Then we obtain the following proposition.

$$\langle X \rangle|_{\lambda=0} \neq 0 \Rightarrow E(\lambda) < E(0) \quad \text{when } \lambda \neq 0, \lambda \ll 0. \quad (\text{A.3})$$

We prove $\langle X \rangle|_{\lambda=0} = 0$ by using the contraposition of this.

Proof. Consider the Euclidean path integral representation of $E(\lambda)$. The ground state energy within the Euclidean volume V is given by

$$e^{-VE(\lambda)} = \int \mathcal{D}A_\mu^a \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[- \int d^4x (\mathcal{L} + i\lambda X) \right] \quad (\text{A.4})$$

Performing the fermion integrals, you yield

$$e^{-VE(\lambda)} = \int \mathcal{D}A_\mu^a \det(\not{D} + M) \exp \left[- \int d^4x \left(\frac{1}{4g^2} \text{tr} F_{\mu\nu} F_{\mu\nu} + i\lambda X \right) \right], \quad (\text{A.5})$$

where M is the fermion mass matrix. According to Vafa & Witten (1984) [16], $\det(\not{D} + M)$ is shown to be positive in the vector-like theory. Let $i\not{D} = i \sum_{\alpha=1}^4 \gamma^\alpha D_\alpha$ be the Euclidean Dirac operator with Hermitian operator γ^α satisfying $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$. Suppose there is a non-zero eigenvalue of the Dirac operator, say ε ,

$$i\not{D}\psi = \varepsilon\psi, \quad (\text{A.6})$$

then it is clear that $-\varepsilon$ is also the eigenvalue of the Dirac operator:

$$\begin{aligned} i\not{D}(\gamma_5\psi) &= -\gamma_5(i\not{D}\psi) \\ &= -\varepsilon(\gamma_5\psi). \end{aligned} \quad (\text{A.7})$$

In the vector-like theory like QCD, the Euclidean Dirac Lagrangian is given by $\bar{\psi} (\not{D} + M) \psi$. Then the Fermion determinant is evaluated by

$$\begin{aligned}
\det (\not{D} + M) &= \prod_{\varepsilon} (M - i\varepsilon) \\
&= \prod_{\varepsilon > 0} (M - i\varepsilon) (M + i\varepsilon) \\
&= \prod_{\varepsilon > 0} (M^2 + \varepsilon^2) \\
&> 0.
\end{aligned} \tag{A.8}$$

Then it is shown that $\det (\not{D} + M)$ is positive and real. From this fact, the integrand of the path integral (A.5) is positive except for the phase factor $\exp \left[-i\lambda \int d^4x X \right]$. Because $-1 \leq (\text{Phase Factor}) \leq 1$, we see that $e^{-VE(0)} > e^{-VE(\lambda)}$. Hence $E(0) \leq E(\lambda)$ shows that there is no λ fulfilling $E(0) > E(\lambda)$. The contraposition of (A.3) proves $\langle X \rangle|_{\lambda=0} = 0$. The same argument is applied to the θ dependence of QCD vacuum energy

$$e^{-VE(\theta)} = \int \mathcal{D}A_{\mu}^a \det (\not{D} + M) \exp \left[- \int d^4x \left(\frac{1}{4g^2} \text{tr} F_{\mu\nu} F_{\mu\nu} - \frac{i\theta}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}_{\mu\nu} \right) \right], \tag{A.9}$$

and see that $E(\theta)$ is minimised when $\theta = 0$. $\square$

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